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THE BOUNDEDNESS OF TWO CLASSES OF INTEGRAL OPERATORS

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Abstract. The aim of this paper is to characterize the $L^p - L^q$ boundedness of two classes of integral operators from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$ in terms of the parameters a, b, c, p, q and α, β, where $\mathcal{U}$ is the Siegel upper half-space. The results in the presented paper generalize a corresponding result given in C. Liu, Y. Liu, P. Hu, L. Zhou (2019).

Keywords: integral operator; Siegel upper half-space; weighted L^p space; boundedness

MSC 2020: 47B38, 47G10

1. Introduction and preliminaries

Let $\mathbb{C}^n$ denote the n-dimensional complex Euclidean space, where n is a positive integer. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, the conjugate of z, denoted by $\overline{z}$, is defined by $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. For z and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we use the following notations:

$$\langle z, w\rangle := z_1\overline{w}_1 + \ldots + z_n\overline{w}_n \quad \text{and} \quad |z| := \sqrt{\langle z, z\rangle} = \biggl(\sum_{k=1}^n |z_k|^2\biggr)^{1/2},$$
$$z = (z', z_n), \quad \text{where } z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1},\ z_n \in \mathbb{C}^1,$$

and

$$\varrho(z, w) := \frac{\mathrm{i}}{2}(\overline{w}_n - z_n) - \langle z', w'\rangle, \quad \varrho(z) := \varrho(z, z) = \operatorname{Im} z_n - |z'|^2.$$

It is obvious that $\varrho(z, w) = \overline{\varrho(w, z)}$. The Siegel upper half space in $\mathbb{C}^n$ is the set

$$\mathcal{U} := \{z \in \mathbb{C}^n \colon \operatorname{Im} z_n > |z'|^2\} = \{z \in \mathbb{C}^n \colon \varrho(z) > 0\}.$$

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This domain is biholomorphically equivalent to the unit ball of $\mathbb{C}^n$, and its boundary $b\mathcal{U} := \{z \in \mathbb{C}^n \colon \operatorname{Im} z_n = |z'|^2\}$ is the standard representation of the Heisenberg group $\mathbb{H}^{n-1}$. The Bergman projection P on $\mathcal{U}$ is defined by

$$Pf(z) = \frac{n!}{4\pi^n} \int_{\mathcal{U}} \frac{f(w)}{\varrho(z,w)^{n+1}} \,\mathrm{d}V(w), \quad z \in \mathcal{U}.$$

Fix real parameter α and define $\mathrm{d}V_\alpha(z)$ by

$$\mathrm{d}V_\alpha(z) := \varrho(z)^\alpha \,\mathrm{d}V(z),$$

where $\mathrm{d}V$ is the Lebesgue measure on $\mathbb{C}^n$. As usual, for $p > 0$, the space $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ consists of all Lebesgue measurable functions f on $\mathcal{U}$ for which

$$\|f\|_{p,\alpha} := \left(\int_{\mathcal{U}} |f(w)|^p \,\mathrm{d}V_\alpha(w) \right)^{1/p} < \infty.$$

The boundedness of the two classes of Bergman type integral operators has been studied by many authors. In particular, their L^p boundedness is of considerable interest, see [1], [2], [3], [12]. The properties of several subclasses of biholomorphic mappings were investigated in [5], [6], [7], [8], [9]. In [11], Zhou obtained the L^p norm of $T_{a,b.c}$ by applying the sharp Forelli-Rudin estimates. In [1], Furdui focuses on the L^p boundedness of $T_{a,b,c}$ on Fock space. Most recently, Liu et al. in [4] introduced the following integral operators:

$$T_{a,b,c}f(z) := \varrho(z)^a \int_{\mathcal{U}} \frac{\varrho(w)^b}{\varrho(z,w)^c} f(w) \,\mathrm{d}V(w)$$

and

$$S_{a,b,c}f(z) := \varrho(z)^a \int_{\mathcal{U}} \frac{\varrho(w)^b}{|\varrho(z,w)|^c} f(w) \,\mathrm{d}V(w),$$

where a, b and c are real parameters. These operators are modeled on the Bergman projection $Pf(z) = (n!/4\pi^n) T_{0,0,n+1} f(z)$. They characterized the L^p boundedness of $T_{a,b,c}$ and $S_{a,b,c}$ as follows.

Theorem A ([4]). *Suppose $\alpha \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$. Then the following conditions are equivalent:*

(i) *The operator $T = T_{a,b,c}$ is bounded on $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$.*
(ii) *The operator $S = S_{a,b,c}$ is bounded on $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$.*
(iii) *The parameters satisfy the conditions*

$$\begin{cases} -pa < \alpha + 1 < p(b+1), \\ c = n + 1 + a + b. \end{cases}$$

When $p = \infty$, these conditions should be interpreted as

$$\begin{cases} a > 0, \quad b > -1, \\ c = n + 1 + a + b. \end{cases}$$

Zhao extended the Schur's test and studied the $L^p - L^q$ boundedness of a class of integral operators on the unit ball of $\mathbb{C}^n$ in [10]. Motivated by the works of Zhao (see [10]) and Liu et al. (see [4]), our aim is to extend Theorem A to the $L^p - L^q$ boundedness of integral operators $T_{a,b,c}$ and $S_{a,b,c}$ over the Siegel upper half-space $\mathcal{U}$. We prove the following results.

Theorem 1.1. *Suppose $1 < p \leqslant q < \infty$ and $-1 < \alpha, \beta < \infty$. If a, b and c are real numbers, then the following conditions are equivalent:*

(i) *The operator $T_{a,b,c}$ is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.*

(ii) *The operator $S_{a,b,c}$ is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.*

(iii) *The parameters satisfy the conditions*

$$\begin{cases} -aq < \beta + 1, \\ \alpha + 1 < p(b + 1), \\ c = n + 1 + a + b + \lambda, \end{cases}$$

where

$$\lambda = \frac{n + 1 + \beta}{q} - \frac{n + 1 + \alpha}{p}.$$

Theorem 1.2. *Suppose $1 = p \leqslant q < \infty$ and $-1 < \alpha, \beta < \infty$. If a, b and c are real numbers such that $c > 0$ and $b > \alpha$, then the following conditions are equivalent:*

(i) *The operator $T_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.*

(ii) *The operator $S_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.*

(iii) *The parameters satisfy the condition*

$$\begin{cases} -aq < \beta + 1, \\ c = n + 1 + a + b + \lambda, \end{cases}$$

where

$$\lambda = \frac{n + 1 + \beta}{q} - (n + 1 + \alpha).$$

Theorem 1.3. *Suppose $1 < p \leqslant q = \infty$. If a, b and c are real numbers such that $a > 0$ and $b > \beta$, then the following conditions are equivalent:*

(i) *The operator $T_{a,b,c}$ is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$.*

(ii) *The operator $S_{a,b,c}$ is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$.*

(iii) *The parameters satisfy the conditions*

$$\begin{cases} \dfrac{\alpha+1}{p} < b+1, \\ c = n+1+a+b+\lambda, \end{cases}$$

where

$$\lambda = -\frac{n+1+\alpha}{p}.$$

For the case of $1 = p < q = \infty$, we cannot obtain a similar result. We pose a conjecture as follows.

Conjecture 1.4. *Suppose $1 = p < q = \infty$. If a, b and c are real numbers such that $b > \beta$, then the following conditions are equivalent:*

(i) *The operator $T_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$.*

(ii) *The operator $S_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$.*

(iii) *The parameters satisfy the conditions*

$$\begin{cases} a > 0, \quad b > \alpha, \\ c = n+1+a+b+\lambda, \end{cases}$$

where

$$\lambda = -n-1-\alpha.$$

When $a = n+1$, $c = 2n+2$, $b = 0$, we obtain the sufficient conditions for the boundedness of the Berezin transform from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.

Corollary 1.5. *Suppose $1 < p \leqslant q < \infty$ and $-1 < \alpha, \beta < \infty$. If the parameters satisfy the conditions*

$$\begin{cases} -(n+1)q < \beta+1, \\ \alpha+1 < p, \\ \dfrac{n+1+\beta}{q} = \dfrac{n+1+\alpha}{p}, \end{cases}$$

then Berezin transform

$$Bf(z) = \frac{n!}{4\pi^n} \int_{\mathcal{U}} \frac{\varrho(z)^{n+1}}{|\varrho(z,w)|^{2n+2}} f(w)\, \mathrm{d}V(w)$$

is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.

Corollary 1.6. *Suppose $1 = p \leqslant q < \infty$ and $-1 < \alpha, \beta < \infty$. If the parameters satisfy the conditions*

$$\begin{cases} -(n+1)q < \beta + 1, \\ \alpha < 0, \\ \dfrac{n+1+\beta}{q} = n+1+\alpha, \end{cases}$$

then Berezin transform

$$Bf(z) = \frac{n!}{4\pi^n} \int_{\mathcal{U}} \frac{\varrho(z)^{n+1}}{|\varrho(z,w)|^{2n+2}} f(w)\,\mathrm{d}V(w)$$

is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.

In order to establish our main results, we recall the following lemmas.

Lemma 1.7 ([4]). *Let $s, t \in \mathbb{R}$. Then we have*

$$\int_{\mathcal{U}} \frac{\varrho(w)^t}{|\varrho(z,w)|^s}\,\mathrm{d}V(w) = \begin{cases} \dfrac{C_1(n,s,t)}{\varrho(z)^{s-t-n-1}} & \text{if } t > -1 \text{ and } s - t > n+1, \\ \infty & \text{otherwise} \end{cases} \tag{1.1}$$

for all $z \in \mathcal{U}$, where

$$C_1(n,s,t) = \frac{4\pi^n \Gamma(1+t)\Gamma(s-t-n-1)}{\Gamma^2(s/2)}.$$

Lemma 1.8 ([4]). *Suppose that $r, s > 0$, $t > -1$ and $r + s - t > n + 1$. Then*

$$\int_{\mathcal{U}} \frac{\varrho(w)^t}{\varrho(z,w)^r \varrho(w,u)^s}\,\mathrm{d}V(w) = \frac{C_2(n,r,s,t)}{\varrho(z,u)^{r+s-t-n-1}} \tag{1.2}$$

holds for all $z, u \in \mathcal{U}$, where

$$C_2(n,r,s,t) = \frac{4\pi^n \Gamma(1+t)\Gamma(r+s-t-n-1)}{\Gamma(r)\Gamma(s)}.$$

Lemma 1.9 ([10]). *Let μ and v be positive measures on the space X and let $K(x,y)$ be a nonnegative measurable function on $X \times X$. Let T be the integral operator with kernel K, defined as*

$$Tf(x) = \int_X f(y) K(x,y)\,\mathrm{d}\mu(y).$$

Suppose $1 < p \leqslant q < \infty$ *and* $1/p + 1/p' = 1$. *Let* γ *and* δ *be two real numbers such that* $\gamma + \delta = 1$. *If there exist positive functions* h_1 *and* h_2 *with two positive constants* C_1 *and* C_2 *such that*

$$\int_X h_1(y)^{p'} K(x,y)^{\gamma p'} \,\mathrm{d}\mu(y) \leqslant C_1 h_2(x)^{p'}$$

for almost all $x \in X$, *and*

$$\int_X h_2(x)^q K(x,y)^{\delta q} \,\mathrm{d}\upsilon(x) \leqslant C_2 h_1(y)^q$$

for almost all $y \in Y$, *then* T *is bounded from* $L^p(X, \mathrm{d}\mu)$ *into* $L^q(X, \mathrm{d}\upsilon)$, *and the norm of this operator does not exceed* $C_1^{1/p'} C_2^{1/q}$.

2. Sufficiency for boundedness of $S_{a,b,c}$

In this section we obtain sufficient conditions for the boundedness of the operator $S_{a,b,c}$ from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$. In the next section we shall show that these conditions are also necessary.

Lemma 2.1. *Suppose* a, b *and* c *are three real numbers. Let* $1 < p \leqslant q < \infty$ *and* $-1 < \alpha, \beta < \infty$. *If the parameters satisfy the conditions*

$$\begin{cases} -aq < \beta + 1, \\ \alpha + 1 < p(b+1), \\ c = n + 1 + a + b + \lambda, \end{cases}$$

where

$$\lambda = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p},$$

then the operator $S_{a,b,c}$ *is bounded from* $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ *to* $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.

Proof. Let $\tau = c - a - b + \alpha$, then it is easy to see that

$$\tau = n + 1 + \alpha + \lambda = \frac{n+1+\beta}{q} + \frac{n+1+\alpha}{p'},$$

where $1/p + 1/p' = 1$. From the last expression we see that $\tau > 0$. Since $\beta + 1 > -aq$, we get

$$-\frac{\tau(\beta+1)}{q} < a\tau,$$

which is equivalent to

$$-\frac{\tau(\beta+1)}{q}-\frac{a(\beta+n+1)}{q}<\frac{a(\alpha+n+1)}{p'}. \tag{2.1}$$

It is easy to see that $\alpha+1<p(b+1)$ is equivalent to

$$(b-\alpha)+\frac{1+\alpha}{p'}>0.$$

Hence

$$(b-\alpha)\tau+\frac{(1+\alpha)\tau}{p'}>0$$

or

$$-\frac{(\alpha+1)\tau}{p'}-\frac{(b-\alpha)(\alpha+n+1)}{p'}<\frac{(b-\alpha)(\beta+n+1)}{q}. \tag{2.2}$$

By (2.1) and (2.2), we can find two real numbers r and s such that

$$-\frac{\tau(\beta+1)}{q}-\frac{a(\beta+n+1)}{q}<r\tau+a(r-s)<\frac{a(\alpha+n+1)}{p'} \tag{2.3}$$

and

$$-\frac{(\alpha+1)\tau}{p'}-\frac{(b-\alpha)(\alpha+n+1)}{p'}<s\tau+(b-\alpha)(s-r)<\frac{(b-\alpha)(\beta+n+1)}{q}. \tag{2.4}$$

Let

$$\gamma=\frac{(\alpha+n+1)/p'+(s-r)}{\tau},\quad \delta=\frac{(\beta+n+1)/q+(r-s)}{\tau}. \tag{2.5}$$

Then

$$\gamma+\delta=\frac{1}{\tau}\Big(\frac{\alpha+n+1}{p'}+\frac{\beta+n+1}{q}\Big)=1.$$

Now, (2.3) is equivalent to

$$-\frac{\beta+1}{q}-\frac{a}{\tau}\Big(\frac{\beta+n+1}{q}+r-s\Big)<r<\frac{a}{\tau}\Big(\frac{\alpha+n+1}{p'}+s-r\Big)$$

or

$$-\frac{\beta+1}{q}-a\delta<r<a\gamma \tag{2.6}$$

and (2.4) is equivalent to

$$-\frac{\alpha+1}{p'}-\frac{b-\alpha}{\tau}\Big(\frac{\alpha+n+1}{p'}+s-r\Big)<s<\frac{b-\alpha}{\tau}\Big(\frac{\beta+n+1}{q}+r-s\Big)$$

or

$$-\frac{\alpha+1}{p'}-(b-\alpha)\gamma<s<(b-\alpha)\delta. \tag{2.7}$$

For $z \in \mathcal{U}$, let $h_r(z) = \varrho(z)^r$ and $h_s(z) = \varrho(z)^s$. Write the operator S as

$$Sf(z) = \int_{\mathcal{U}} f(w) H(z,w) \, \mathrm{d}V_\alpha(w),$$

where

$$H(z,w) = \frac{\varrho(w)^{b-\alpha} \varrho(z)^a}{|\varrho(z,w)|^c}.$$

Then we apply Lemma 1.9 using the testing functions h_r and h_s and the parameters γ and δ defined in (2.5). When $1 < p \leqslant q < \infty$, we consider

$$\int_{\mathcal{U}} h_s(w)^{p'} H(z,w)^{\gamma p'} \, \mathrm{d}v_\alpha(w) = \varrho(z)^{a\gamma p'} \int_{\mathcal{U}} \frac{\varrho(w)^{sp' + (b-\alpha)\gamma p' + \alpha}}{|\varrho(z,w)|^{c\gamma p'}} \, \mathrm{d}V(w).$$

From the first inequality in (2.7) we know that

$$sp' + (b-\alpha)\gamma p' + \alpha > -1.$$

Notice that from (2.5) and the fact that $c - a - b + \alpha = \tau$, we have

$$(c - a - b + \alpha)\gamma = \tau\gamma = \frac{\alpha + n + 1}{p'} + s - r.$$

From the above equation, we obtain

$$c\gamma p' - (sp' + (b-\alpha)\gamma p' + \alpha) - n - 1 = a\gamma p' - rp',$$

and from the second inequality in (2.6) we have

$$a\gamma p' - rp' > 0.$$

Hence, we can apply Lemma 1.7 to get that

$$\int_{\mathcal{U}} h_s(w)^{p'} H(z,w)^{\gamma p'} \, \mathrm{d}v_\alpha(w) \leqslant C_3 \varrho(z)^{rp'} = C_3 \, h_r(z)^{p'}$$

for a constant $C_3 > 0$.

Similarly, using the first inequality in (2.6) and the second inequality in (2.7) we can obtain

$$\int_{\mathcal{U}} h_r(z)^q H(z,w)^{sq} \, \mathrm{d}v_\beta(z) \leqslant C_4 h_s(w)^q$$

for a constant $C_4 > 0$. Hence, by Lemma 1.9 we know that $S_{a,b,c}$ is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$. □

Lemma 2.2. *Let $1 = p \leqslant q < \infty$ and $-1 < \alpha < b$, $-1 < \beta < \infty$. If the parameters satisfy the condition*

$$\begin{cases} -aq < \beta + 1, \\ c = n + 1 + a + b + \lambda, \end{cases}$$

where

$$\lambda = \frac{n+1+\beta}{q} - (n+1+\alpha),$$

then the operator $S_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.

Proof. If $f \in L^1(\mathcal{U}, \mathrm{d}V_\alpha)$, then it is easy to see that

$$|S_{a,b,c}f(z)| \leqslant \varrho(z)^a \int_{\mathcal{U}} \frac{\varrho(w)^b}{|\varrho(z,w)|^c} |f(w)| \, \mathrm{d}V(w).$$

We first prove the result for the case $q > 1$. In this case, we apply Minkowski's inequality to get

$$\begin{aligned} \Big(\int_{\mathcal{U}} |S_{a,b,c}f(z)|^q \, \mathrm{d}V_\beta(z)\Big)^{1/q} &\leqslant \Big(\int_{\mathcal{U}} \varrho(z)^{aq} \Big(\int_{\mathcal{U}} \frac{\varrho(w)^b}{|\varrho(z,w)|^c} |f(w)| \, \mathrm{d}V(w)\Big)^q \mathrm{d}V_\beta(z)\Big)^{1/q} \\ &\leqslant \int_{\mathcal{U}} \Big(\int_{\mathcal{U}} \frac{\varrho(z)^{aq+\beta}\varrho(w)^{bq}}{|\varrho(z,w)|^{cq}} |f(w)|^q \, \mathrm{d}V(z)\Big)^{1/q} \mathrm{d}V(w) \\ &= \int_{\mathcal{U}} |f(w)| \varrho(w)^b \Big(\int_{\mathcal{U}} \frac{\varrho(z)^{aq+\beta}}{|\varrho(z,w)|^{cq}} \, \mathrm{d}V(z)\Big)^{1/q} \mathrm{d}V(w). \end{aligned}$$

Since $aq + \beta > -1$ and $b > \alpha$, simple computation yields that $cq - aq - \beta = n + 1 + q(b - \alpha) > n + 1$, thus, by Lemma 1.1, there exist a constant $C > 0$ such that

$$\int_{\mathcal{U}} \frac{\varrho(z)^{aq+\beta}}{|\varrho(z,w)|^{cq}} \, \mathrm{d}V(z) \leqslant \frac{C}{\varrho(w)^{q(b-\alpha)}}.$$

It follows that

$$\Big(\int_{\mathcal{U}} |S_{a,b,c}f(z)|^q \, \mathrm{d}V_\beta(z)\Big)^{1/q} \leqslant C^{1/q} \int_{\mathcal{U}} |f(w)| \, \mathrm{d}V_\alpha(w),$$

and so the operator $S_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$.

If $1 = p = q$ and $\alpha < b$, then for every $f \in L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ we can apply Fubini's theorem to obtain

$$\begin{aligned} \int_{\mathcal{U}} |S_{a,b,c}f(z)| \, \mathrm{d}V_\beta(z) &\leqslant \int_{\mathcal{U}} \Big(\varrho(z)^a \int_{\mathcal{U}} \frac{\varrho(w)^b}{|\varrho(z,w)|^c} |f(w)| \, \mathrm{d}V(w)\Big) \mathrm{d}V_\beta(z) \\ &= \int_{\mathcal{U}} \varrho(w)^b |f(w)| \, \mathrm{d}V(w) \int_{\mathcal{U}} \frac{\varrho(z)^{a+\beta}}{|\varrho(z,w)|^c} \, \mathrm{d}V(z). \end{aligned}$$

Since $a+\beta>-1$ and $b>\alpha$, by Lemma 1.7, it is easy to see that

$$\int_{\mathcal{U}} \frac{\varrho(z)^{a+\beta}}{|\varrho(z,w)|^c}\,\mathrm{d}V(z) \leqslant \frac{C'}{\varrho(w)^{b-\alpha}}.$$

Hence

$$\int_{\mathcal{U}} |S_{a,b,c}f(z)|^q\,\mathrm{d}V_\beta(z) \leqslant C' \int_{\mathcal{U}} |f(w)|\,\mathrm{d}V_\alpha(w),$$

and so the operator $S_{a,b,c}$ is bounded from $L^1(\mathcal{U},\mathrm{d}V_\alpha)$ to $L^1(\mathcal{U},\mathrm{d}V_\beta)$. □

Lemma 2.3. *Suppose a, b and c are three real numbers such that $a>0$. Let $1<p<q=\infty$ and $-1<\alpha$, $\beta<\infty$. If $\alpha+1<p(b+1)$ and $c=n+a+b+1+\lambda$, where $\lambda=-(n+1+\alpha)/p$, then the operator $S_{a,b,c}$ is bounded from $L^p(\mathcal{U},\mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$.*

Proof. Let $f\in L^p(\mathcal{U},\mathrm{d}V_\alpha)$. Then using Hölder's inequality, we obtain

$$\begin{aligned}|S_{a,b,c}f(z)| &\leqslant \varrho(z)^a \int_{\mathcal{U}} \frac{\varrho(w)^{b-\alpha}}{|\varrho(z,w)|^c}|f(w)|\,\mathrm{d}V_\alpha(w)\\ &\leqslant \varrho(z)^a\Big(\int_{\mathcal{U}}|f(w)|^p\,\mathrm{d}V_\alpha(w)\Big)^{1/p}\Big(\int_{\mathcal{U}}\frac{\varrho(w)^{(b-\alpha)p'}}{|\varrho(z,w)|^{cp'}}\,\mathrm{d}V_\alpha(w)\Big)^{1/p'},\end{aligned}$$

where $1/p+1/p'=1$.

Notice the fact that $c=n+a+b+1+\lambda$, $\lambda=-(n+1+\alpha)/p$ and $a>0$, simple computation leads to

$$ap' = cp' - (b-\alpha)p' - \alpha - n - 1 > 0.$$

Since $\alpha+1<p(b+1)$, we see that $(b-\alpha)p'+\alpha>-1$. Hence, it follows from Lemma 1.7 that there exists $C>0$ such that

$$\Big(\int_{\mathcal{U}}\frac{\varrho(w)^{(b-\alpha)p'}}{|\varrho(z,w)|^{cp'}}\,\mathrm{d}V_\alpha(w)\Big)^{1/p'} \leqslant C\varrho(z)^{-a},$$

this implies

$$\sup_{z\in\mathcal{U}} |S_{a,b,c}f(z)| \leqslant C\|f\|_{p,\alpha}.$$

The proof is complete. □

3. Necessity for boundedness of $T_{a,b,c}$

In this section we obtain necessary conditions for the boundedness of the operator $T_{a,b,c}$ from $L^p(\mathcal{U},\mathrm{d}V_\alpha)$ to $L^q(\mathcal{U},\mathrm{d}V_\beta)$.

Lemma 3.1. *Let $1 < p \leqslant q < \infty$ and $-1 < \alpha,\beta < \infty$. Suppose that $T_{a,b,c}$ is bounded from $L^p(\mathcal{U},\mathrm{d}V_\alpha)$ to $L^q(\mathcal{U},\mathrm{d}V_\beta)$. Then $c > 0$, $\alpha+1 < p(b+1)$ and $-aq < \beta+1$.*

Proof. Let p' and q' be the conjugate indices of p and q, respectively. Then $1/p+1/p'=1$ if $1<p<\infty$, and similarly, $1/q+1/q'=1$ if $1<q<\infty$. By duality, the boundedness of T from $L^p(\mathcal{U},\mathrm{d}V_\alpha)$ to $L^q(\mathcal{U},\mathrm{d}V_\beta)$ implies the boundedness of T^* from $L^{q'}(\mathcal{U},\mathrm{d}V_\beta)$ to $L^{p'}(\mathcal{U},\mathrm{d}V_\alpha)$. It is easy to see that

$$T^*_{a,b,c}f(z) = \varrho(z)^{b-\alpha}\int_{\mathcal{U}} \frac{\varrho(w)^{\beta+a}}{\varrho(z,w)^c} f(w)\,\mathrm{d}V(w),$$

where

$$L^p(\mathcal{U},\mathrm{d}V_\alpha)^* = L^{p'}(\mathcal{U},\mathrm{d}V_\alpha) \quad\text{and}\quad L^q(\mathcal{U},\mathrm{d}V_\beta)^* = L^{q'}(\mathcal{U},\mathrm{d}V_\beta).$$

We first know from [4] that $c>0$. Next, for $\eta>0$ we put

$$f_\eta(z) = \frac{\varrho(z)^t}{\varrho(z,\eta\mathrm{i})^s}, \quad z\in\mathcal{U},$$

where s, t are real parameters satisfying conditions

$$\begin{cases} s>0, \\ t > \max\Big\{-\dfrac{\beta+1}{q'}, -a-\beta-1\Big\}, \\ s-t > \max\Big\{\dfrac{n+1+\beta}{q'}, \beta+a-c+n+1\Big\}. \end{cases}$$

By Lemma 1.7, these conditions guarantee that $f_\eta \in L^{q'}(\mathcal{U},\mathrm{d}V_\beta)$ and

$$\|f_\eta(z)\|^{q'}_{q',\beta} = C_5\eta^{n+1+\beta-q'(s-t)}.$$

Also, in view of the above condition and that $c>0$, we can apply Lemma 1.8 to get

$$T^*_{a,b,c}f_\eta(z) = \varrho(z)^{b-\alpha}\int_{\mathcal{U}} \frac{\varrho(w)^{\beta+a+t}}{\varrho(z,w)^c\varrho(w,\eta\mathrm{i})^s}\,\mathrm{d}V(w) = C_6\frac{\varrho(z)^{b-\alpha}}{\varrho(z,\eta\mathrm{i})^{c+s-\beta-a-t-n-1}}.$$

It is easy to see that

$$\|T^*_{a,b,c}f_\eta(z)\|^{p'}_{p',\alpha} = C_6^{p'} \int_{\mathcal{U}} \frac{\varrho(z)^{(b-\alpha)p'+\alpha}}{|\varrho(z,\eta\mathrm{i})|^{p'(c+s-\beta-a-t-n-1)}}\,\mathrm{d}V(z).$$

Since $T^*_{a,b,c}f_\eta(z) \in L^{p'}(\mathcal{U}, \mathrm{d}V_\alpha)$, by Lemma 1.7 we have

$$(b-\alpha)p' + \alpha > -1,$$

that is,

$$\alpha + 1 < p(b+1).$$

Applying the above arguments to T, we conclude that

$$-1 < \beta + aq.$$

□

Lemma 3.2. *Let* $1 < p \leqslant q < \infty$, $c > 0$ *and* $-1 < \alpha, \beta + aq < \infty$. *Denote*

$$\lambda = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}.$$

If $T_{a,b,c}$ *is bounded from* $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ *to* $L^q(\mathcal{U}, \mathrm{d}V_\beta)$ *and*

$$\alpha + 1 < p(b+1),$$

then

$$c = n + a + b + 1 + \lambda.$$

P r o o f. For any $\xi \in \mathcal{U}$, let

$$f_\alpha(z) = \frac{\varrho(\xi)^{n+1+b-(n+1+\alpha)/p}}{\varrho(z,\xi)^{n+1+b}}.$$

By Lemma 1.7 we can easily see that there is a positive constant C_7 independent of ξ such that

$$\|f_\alpha(z)\|_{p,\alpha} \leqslant C_7.$$

Notice that $c > 0$ and $b > (\alpha+1)/p - 1 > -1$, by Lemma 1.8 we have

$$\begin{aligned} T_{a,b,c}f_\alpha(z) &= \varrho(z)^a \varrho(\xi)^{(n+1+b)-(n+1+\alpha)/p} \int_{\mathcal{U}} \frac{\varrho(w)^b}{\varrho(z,w)^c \varrho(w,\xi)^{n+1+b}}\,\mathrm{d}V(w) \\ &= \varrho(z)^a \varrho(\xi)^{(n+1+b)-(n+1+\alpha)/p} \varrho(z,\xi)^{-c}. \end{aligned}$$

Since $T_{a,b,c}f_\alpha \in L^q(\mathcal{U}, \mathrm{d}V_\beta)$, we know that there is a positive constant C_8 such that

$$\|T_{a,b,c}f_\alpha(z)\|^q_{q,\beta} = \varrho(\xi)^{q(n+1+b)-q(n+1+\alpha)/p} \int_{\mathcal{U}} \frac{\varrho(z)^{\beta+aq}}{|\varrho(z,\xi)|^{cq}}\,\mathrm{d}V(z) \leqslant C_8.$$

Hence

$$\int_{\mathcal{U}} \frac{\varrho(z)^{\beta+aq}}{|\varrho(z,\xi)|^{cq}}\,\mathrm{d}V(z) \leqslant \frac{C_8}{\varrho(\xi)^{q(n+1+b)-q(n+1+\alpha)/p}}.$$

By Lemma 1.7 we have

$$q(n+1+b) - \frac{q(n+1+\alpha)}{p} = cq - (\beta+aq) - (n+1),$$

that is,

$$c = n+a+b+1+\lambda.$$

□

Lemma 3.3. *Suppose $1 = p \leqslant q < \infty$ and $c > 0$, $b > \alpha > -1$. If $T_{a,b,c}$ is bounded from $L^1(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$, then the parameters satisfy the condition*

$$\begin{cases} -aq < \beta+1, \\ c = n+1+a+b+\lambda, \end{cases}$$

where

$$\lambda = \frac{n+1+\beta}{q} - (n+1+\alpha).$$

Proof. We first prove the condition $-aq < \beta+1$. In fact, for $\eta > 0$, we put

$$f_\eta(z) = \frac{\varrho(z)^t}{\varrho(z,\eta \mathrm{i})^s}, \quad z \in \mathcal{U},$$

where s, t are real parameters satisfying condition

$$\begin{cases} s > 0, \\ t > \max\Big\{-\dfrac{\alpha+1}{p}, -b-1\Big\}, \\ s-t > \max\Big\{\dfrac{n+1+\alpha}{p}, n+1+b-c\Big\}. \end{cases}$$

By Lemma 1.7 these conditions guarantee that $f_\eta \in L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ and

$$\|f_\eta(z)\|_{p,\alpha}^p = C\eta^{n+1+\alpha-p(s-t)}.$$

Also, in view of the above condition and that $c > 0$, we can apply Lemma 1.8 to get

$$T_{a,b,c}f_\eta(z) = \varrho(z)^a \int_{\mathcal{U}} \frac{\varrho(w)^{b+t}}{\varrho(z,w)^c \varrho(w,\eta \mathrm{i})^s}\,\mathrm{d}V(w) = C\frac{\varrho(z)^a}{\varrho(z,\eta \mathrm{i})^{c+s-b-t-n-1}}.$$

Since $T_{a,b,c}f_\eta \in L^q(\mathcal{U}, \mathrm{d}V_\beta)$, by Lemma 1.7 we have $-1 < \beta+aq$ or $-aq < \beta+1$. Now we prove that $c = n+1+a+b+\lambda$. In fact, for any $\xi \in \mathcal{U}$, let

$$f_\alpha(z) = \frac{\varrho(\xi)^{b-\alpha}}{\varrho(z,\xi)^{n+1+b}}.$$

By Lemma 1.7 we can easily see that there is a positive constant C independent of ξ such that

$$\|f_\alpha(z)\|_{1,\alpha} \leqslant C.$$

Notice that $c > 0$ and $b > \alpha > -1$, by Lemma 1.8 we have

$$T_{a,b,c} f_\alpha(z) = \varrho(z)^a \varrho(\xi)^{b-\alpha} \int_{\mathcal{U}} \frac{\varrho(w)^b}{\varrho(z,w)^c \varrho(w,\xi)^{n+1+b}} \, \mathrm{d}V(w) = \varrho(z)^a \varrho(\xi)^{b-\alpha} \varrho(z,\xi)^{-c}.$$

Since $T_{a,b,c} f_\alpha \in L^q(\mathcal{U}, \mathrm{d}V_\beta)$, we know that there is a positive constant C such that

$$\|T_{a,b,c} f_\alpha(z)\|_{q,\beta}^q = \varrho(\xi)^{q(b-\alpha)} \int_{\mathcal{U}} \frac{\varrho(z)^{\beta+aq}}{|\varrho(z,\xi)|^{cq}} \, \mathrm{d}V(z) \leqslant C.$$

Hence

$$\int_{\mathcal{U}} \frac{\varrho(z)^{\beta+aq}}{|\varrho(z,\xi)|^{cq}} \, \mathrm{d}v(z) \leqslant \frac{C_8}{\varrho(\xi)^{q(b-\alpha)}}.$$

By Lemma 1.7 we have $q(b-\alpha) = cq - (\beta + aq) - (n+1)$, which implies

$$c = n + 1 + a + b + \lambda$$

□

Lemma 3.4. *Suppose $1 < p < q = \infty$ and $b > \beta$. If the operator $T_{a,b,c}$ is bounded from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$, then the parameters satisfy the conditions*

$$\begin{cases} \alpha + 1 < p(b+1), \\ c = n + 1 + a + b + \lambda, \end{cases}$$

where

$$\lambda = -\frac{n+1+\alpha}{p}.$$

P r o o f. The necessity of condition $\alpha + 1 < p(b+1)$ is obtained as in Lemma 3.1. To see that $c = n + 1 + a + b + \lambda$ holds, observe that the boundedness of $T_{a,b,c}$ from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^\infty(\mathcal{U})$ is equivalent to the boundedness of the adjoint $T^*_{a,b,c}$ of $T_{a,b,c}$ from $L^1(\mathcal{U}, \mathrm{d}V_\beta)$ to $L^{p'}(\mathcal{U}, \mathrm{d}V_\alpha)$.

For any $\xi \in \mathcal{U}$,

$$f_\beta(z) = \frac{\varrho(\xi)^{b-\beta}}{\varrho(z,\xi)^{n+1+b}}.$$

Since $b > \beta$, using Lemma 1.7, there is a positive constant C independent of ξ such that

$$\|f_\beta(z)\|_{1,\beta} \leqslant C.$$

Note that $\alpha+1 < p(b+1)$ and $b > \beta$ imply that $c+b-\beta-a > 0$. By Lemma 1.8 we have

$$T^*_{a,b,c} f_\beta(z) = \varrho(z)^{b-\alpha}\varrho(\xi)^{b-\beta}\int_{\mathcal{U}} \frac{\varrho(w)^{\beta+a}}{\varrho(z,w)^c\varrho(w,\xi)^{n+1+b}}\,\mathrm{d}V(w)$$
$$= \varrho(z)^{b-\alpha}\varrho(\xi)^{b-\beta}\varrho(z,\xi)^{\beta+a-b-c}.$$

Since $T^*_{a,b,c} f_\beta \in L^{p'}(\mathcal{U}, \mathrm{d}V_\alpha)$, we see that there is a positive constant C_0 such that

$$\|T^*_{a,b,c} f_\beta(z)\|^{p'}_{p',\alpha} = \varrho(\xi)^{p'(b-\beta)}\int_{\mathcal{U}} \frac{\varrho(z)^{p'(b-\alpha)+\alpha}}{|\varrho(z,\xi)|^{(c+b-\beta-a)p'}}\,\mathrm{d}V(z) \leqslant C_0.$$

Hence

$$\int_{\mathcal{U}} \frac{\varrho(z)^{p'(b-\alpha)+\alpha}}{|\varrho(z,\xi)|^{(c+b-\beta-a)p'}}\,\mathrm{d}V(z) \leqslant \frac{C_0}{\varrho(\xi)^{p'(b-\beta)}}.$$

Again by Lemma 1.7, we have

$$p'(b-\beta) = p'(c+b-\beta-a) - p'(b-\alpha) - \alpha - n - 1,$$

that is,

$$c = n + a + b + 1 + \lambda.$$

□

4. Completing the proof of Theorems 1.1, 1.2 and 1.3

P r o o f of Theorems 1.1 and 1.2. We now put all the pieces together to prove the two main theorems.

It is obvious that the boundedness of $S_{a,b,c}$ from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$ implies the boundedness of $T_{a,b,c}$ from $L^p(\mathcal{U}, \mathrm{d}V_\alpha)$ to $L^q(\mathcal{U}, \mathrm{d}V_\beta)$. So (ii) implies (i) in Theorems 1.1 and 1.2.

That (i) implies (iii) in Theorem 1.1 follows from Lemmas 3.1 and 3.2. That (i) implies (iii) in Theorem 1.2 follows from Lemma 3.3.

It follows from Lemma 2.1 that (iii) implies (ii) in Theorem 1.1; and it follows from Lemma 2.2 that (iii) implies (ii) in Theorem 1.2. This completes the proof of Theorems 1.1 and 1.2. □

P r o o f of Theorem 1.3. According to Theorem A, we only need to prove the case $1 < p < q = \infty$.

The test that (ii) implies (i) in Theorem 1.3 is obvious. By Lemma 2.3, we conclude that (iii) implies (ii) in Theorem 1.3; and it follows from Lemma 3.4 that (i) implies (iii) in Theorem 1.3. This completes the proof of Theorem 1.3. □

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