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REMARKS ON THE BOUNDS OF GRAPH ENERGY IN TERMS OF VERTEX COVER NUMBER OR MATCHING NUMBER

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Abstract. We give a novel upper bound on graph energy in terms of the vertex cover number, and present a complete characterization of the graphs whose energy equals twice their matching number.

Keywords: graph energy; vertex cover number; matching number; bound

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1. Introduction

We consider finite, undirected, and simple graphs throughout this paper. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. By $n(G)$ and $m(G)$ we always denote the numbers of vertices and edges in G, respectively. We also denote by $N_G(v)$ the neighborhood of the vertex v in G. A vertex v is called *isolated* if $|N_G(v)| = 0$. We refer to the quantity $|N_G(v)|$ as the degree of v in G, and write $\Delta(G)$ for the maximum vertex degree of G. As usual, let K_n, $K_{p,q}$ $(p+q=n)$, C_n, and P_n denote the complete graph, the complete bipartite graph, the cycle, and the path with n vertices, respectively.

The vertex-disjoint union of two graphs G and H is denoted by $G \cup H$, and the vertex-disjoint union of k copies of G is written as kG. For a subset U of $V(G)$, we denote by $G - U$ the graph obtained from G by deleting the vertices in U together with all edges incident to them; in particular, if $U = \{v\}$, then we always write $G - v$ instead of $G - \{v\}$. For an induced subgraph F of G, we denote by $G - F$ the induced subgraph of G with the vertex set $V(G) - V(F)$, which is also called the

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complement of F in G. For a subset Ω of $E(G)$, we denote by $G-\Omega$ the spanning subgraph of G obtained by deleting all edges in Ω from G. If $G-\Omega$ is the union of two complementary induced subgraphs, then Ω is called a *cut set* of G. For an edge e with endpoints x and y, we sometimes denote it by xy. We also denote by $[e]$ or $[xy]$ the subgraph of G induced by $\{x, y\}$. Notice that $G-[e]$ or $G-[xy]$ is the subgraph of G obtained by deleting e as well as its two endpoints x and y from G.

A matching of a graph G is a set of edges with no shared endpoints. The maximum number of edges in a matching of G is called the *matching number* of G and is denoted by $\beta(G)$; the corresponding matching is called a *maximum matching* of G. If a matching covers all vertices in G, then it is called a *perfect matching* of G. A vertex cover of a graph G is a set of vertices that contains at least one endpoint of every edge in G. The minimum number of vertices in a vertex cover of G is called the *vertex cover number* of G and is denoted by $\tau(G)$; the corresponding vertex cover is called a *minimum vertex cover* of G.

The adjacency matrix of a graph G (of order n), denoted by $A(G)$, is the $n\times n$ matrix (a_{ij}) in which $a_{ij}=1$ if the vertices v_i and v_j are adjacent, and $a_{ij}=0$ otherwise. The eigenvalues of $A(G)$, denoted by $\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G)$, are referred to as the eigenvalues of the graph G, which form the spectrum of G. The energy $\mathcal{E}(G)$ of G is defined to be the sum of the absolute values of all its eigenvalues, namely,

$$\mathcal{E}(G)=\sum_{i=1}^{n}|\lambda_i(G)|.$$

The motivation for this definition comes from chemistry, where the first results on graph energy were reported as early as the 1940s, see [6]. However, in the last two decades research on graph energy has much intensified, and a large number of results on graph energy have been obtained; for details see a recent book [13] by Li, Shi and Gutman.

Here, we are mainly concerned with the bounds on graph energy. One of early and classical upper bounds on graph energy was due to McClelland (see [14]), who showed that if G is a graph with n vertices and m edges, then

$$\mathcal{E}(G)\leqslant\sqrt{2mn} \tag{1.1}$$

with equality if and only if G is either an empty graph or a regular graph of degree one. After the bound (1.1), various upper bounds on $\mathcal{E}(G)$ were obtained; for example, Wang and Ma in [17] proved that if G is a graph with vertex cover number τ and maximum vertex degree Δ, then

$$\mathcal{E}(G)\leqslant 2\tau\sqrt{\Delta} \tag{1.2}$$

with equality if and only if G is the vertex-disjoint union of τ copies of $K_{1,\Delta}$ together with some possible isolated vertices. For some other results on this aspect, one can refer to the papers [4], [12], [15], [16], [19], [20] and the book [13].

As for the lower bounds on graph energy, Caporossi et al. in [4] showed that for any graph G with m edges,

$$\mathcal{E}(G) \geqslant 2\sqrt{m} \tag{1.3}$$

with equality if and only if G is a complete bipartite graph plus arbitrarily many isolated vertices. For some other lower bounds on graph energy one may refer to [1], [2], [3], [8], [11], [13], [14], [16] and the references cited therein. Recently, Wang and Ma in [17] proved that if G is a bipartite graph, then

$$\mathcal{E}(G) \geqslant 2\beta(G) \tag{1.4}$$

with equality if and only if G is the vertex-disjoint union of some complete bipartite graphs with perfect matchings and some possible isolated vertices. Wong et al. in [18] further extended the lower bound (1.4) to general graphs and gave a partial characterization of the graphs which attain the bound, i.e., if all cycles (if any) of G are pairwise vertex-disjoint, then $\mathcal{E}(G) = 2\beta(G)$ if and only if G is the disjoint union of some copies of K_2 and some copies of C_4 together with some isolated vertices.

In this note, we continue the study of upper and lower bounds on graph energy in terms of the vertex cover number or matching number. We shall give a novel upper bound on $\mathcal{E}(G)$ in terms of the vertex cover number, which improves the bound (1.2). We also present a complete characterization of the graphs satisfying $\mathcal{E}(G) = 2\beta(G)$.

2. Lemmas and results

We first list some necessary lemmas. Let $r(B)$ denote the rank of a matrix B. The *rank of a graph* G, denoted by $r(G)$, is defined to be the rank of its adjacency matrix $A(G)$, that is, $r(G) = r(A(G))$.

Lemma 1 ([10]). *Let $X_1, X_2, \ldots, X_t$ be matrices of the same size, and let $X = X_1 + X_2 + \ldots + X_t$. Then $r(X) \leqslant r(X_1) + r(X_2) + \ldots + r(X_t)$.*

Lemma 2 ([5]). *Let $G_1, G_2, \ldots, G_t$ be graphs and $G = G_1 \cup G_2 \cup \ldots \cup G_t$. Then $r(G) = r(G_1) + r(G_2) + \ldots + r(G_t)$.*

Lemma 3 ([5]). *If G is a graph of order $n \geqslant 2$, then $r(G) = 2$ if and only if G is a complete bipartite graph.*

Lemma 4 ([9]). *If Ω is a cut set of a graph G, then $\mathcal{E}(G-\Omega) \leqslant \mathcal{E}(G)$.*

Lemma 5 ([9]). *If F is an induced subgraph of a graph G, then $\mathcal{E}(F) \leqslant \mathcal{E}(G)$ with equality if and only if every edge of G is that of F.*

Now we are ready to give the main results of this note.

Proposition 6. *If G is a graph with $r(G)=r$ and $\tau(G)=\tau$, then $r \leqslant 2\tau$.*

P r o o f. Without loss of generality, suppose that $\{v_1, v_2, \ldots, v_\tau\}$ is a minimum vertex cover of the graph G. Let $G_1, G_2, \ldots, G_\tau$ be spanning subgraphs of G such that

$$E(G_1) = \{v_1 v_k \colon\ v_k \in N_G(v_1)\}$$

and

$$E(G_i) = \{v_i v_k \colon\ v_k \in (N_G(v_i) - \{v_1, \ldots, v_{i-1}\})\}, \quad i = 2, \ldots, \tau.$$

It is easy to see that $E(G_i) \cap E(G_j) = \emptyset$ for any $i \neq j$, and

$$E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_\tau).$$

Therefore, we obtain

$$A(G) = A(G_1) + A(G_2) + \ldots + A(G_\tau).$$

Consequently, by Lemma 1, we have

$$r(G) \leqslant r(G_1) + r(G_2) + \ldots + r(G_\tau). \tag{2.1}$$

On the other hand, we can easily see that

$$G_i \cong K_{1,m_i} \cup (n - m_i - 1)K_1,$$

where $i = 1, 2, \ldots, \tau$ and $m_i = |E(G_i)|$. Moreover, by Lemmas 2 and 3, we have

$$r(G_i) = r(K_{1,m_i}) = 2, \quad i = 1, 2, \ldots, \tau,$$

which, together with (2.1), yields the desired result, completing the proof. □

Proposition 7. *If G is a graph with $m \geqslant 1$ edges and $r(G) = r$, then $\mathcal{E}(G) \leqslant \sqrt{2mr}$ with equality if and only if r is even and, G is the vertex-disjoint union of $r/2$ complete bipartite graphs K_{p_i,q_i} with $p_i q_i = 2m/r$ $(i = 1, \ldots, r/2)$ and some possible isolated vertices.*

P r o o f. Without loss of generality, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the nonzero eigenvalues of G such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r$. It is well known that $\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_r^2 = 2m$. Thus, by the definition of $\mathcal{E}(G)$ and the Cauchy-Schwarz inequality, we have

$$\mathcal{E}(G) = \sum_{i=1}^{r} |\lambda_i| \leqslant \sqrt{r \sum_{i=1}^{r} \lambda_i^2} = \sqrt{2mr}$$

with equality if and only if $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \sqrt{2m/r}$.

To complete the proof, we now just need to prove that $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \sqrt{2m/r}$ if and only if r is even and G has $r/2$ nontrivial connected components, which are complete bipartite graphs K_{p_i,q_i} with $p_iq_i = 2m/r$ $(i = 1, \ldots, r/2)$.

Suppose first that $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \sqrt{2m/r}$. Then for $k = 1, 2, \ldots, r$, λ_k is equal to $\sqrt{2m/r}$ or $-\sqrt{2m/r}$. Bearing in mind that $\lambda_1 + \lambda_2 + \ldots + \lambda_r = 0$, one may conclude that r must be even and

$$\lambda_1 = \ldots = \lambda_{r/2} = \sqrt{2m/r} \quad \text{and} \quad \lambda_{r/2+1} = \ldots = \lambda_r = -\sqrt{2m/r}.$$

Moreover, the Perron-Frobenius theorem tells us that the largest eigenvalue of a connected graph is simple (see, e.g. [7], Theorem 1.3.6, page 15). Therefore, each nontrivial connected component C of G has only one positive eigenvalue $\sqrt{2m/r}$ and one negative eigenvalue $-\sqrt{2m/r}$, which, together with Lemma 3, yields that C is a complete bipartite graph $K_{p,q}$ with $pq = 2m/r$. Also, it is easy to see that G has $r/2$ such nontrivial connected components.

Conversely, if r is even and G has $r/2$ nontrivial connected components, which are complete bipartite graphs K_{p_i,q_i} with $p_iq_i = 2m/r$ $(i = 1, \ldots, r/2)$, then it is clear that $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \sqrt{2m/r}$.

This completes the proof. □

Propositions 6 and 7 yield an upper bound on graph energy in terms of the vertex cover number.

Theorem 8. *If G is a graph with $m \geqslant 1$ edges and $\tau(G) = \tau$, then*

$$\mathcal{E}(G) \leqslant 2\sqrt{m\tau} \tag{2.2}$$

with equality if and only if G is the vertex-disjoint union of τ copies of $K_{1,m/\tau}$ together with some possible isolated vertices.

P r o o f. Set $r(G) = r$ for convenience. From Propositions 6 and 7, it follows that

$$\mathcal{E}(G) \leqslant \sqrt{2mr} \leqslant 2\sqrt{m\tau}, \tag{2.3}$$

as desired.

Moreover, if G is the vertex-disjoint union of τ copies of $K_{1,m/\tau}$ together with some possible isolated vertices, then $\mathcal{E}(G) = \tau\mathcal{E}(K_{1,m/\tau}) = 2\sqrt{m\tau}$, i.e., the equality holds in (2.2).

Conversely, if the equality holds in (2.2), then by (2.3), we have $\mathcal{E}(G) = \sqrt{2mr}$ and $r = 2\tau$, which imply that G is the vertex-disjoint union of τ complete bipartite graphs K_{p_i,q_i} with $p_iq_i = m/\tau$ $(i = 1,\ldots,\tau)$ and some possible isolated vertices (by Proposition 7). For $i = 1,\ldots,\tau$, without loss of generality, suppose that $p_i \leqslant q_i$. Then we have $\tau(K_{p_i,q_i}) = p_i \geqslant 1$ and hence,

$$\tau = \tau(G) = \tau(K_{p_1,q_1}) + \ldots + \tau(K_{p_\tau,q_\tau}) = p_1 + \ldots + p_\tau,$$

which yields that $p_1 = \ldots = p_\tau = 1$ and $q_1 = \ldots = q_\tau = m/\tau$. Consequently, we may conclude that G is the vertex-disjoint union of τ copies of $K_{1,m/\tau}$ together with some possible isolated vertices.

The proof of Theorem 8 is completed. □

From the definition of vertex cover, it is easily seen that the number of edges in a graph G does not exceed the sum of degrees of the vertices in any of its vertex covers. Therefore, we have $m \leqslant \tau\Delta$, which implies that the upper bound (2.2) is better than the upper bound (1.2).

The next result gives a lower bound on graph energy in terms of the matching number.

Theorem 9. *For any graph G,*

$$\mathcal{E}(G) \geqslant 2\beta(G) \tag{2.4}$$

with equality if and only if G is the vertex-disjoint union of some complete bipartite graphs with perfect matchings and some possible isolated vertices.

Proof. It should be mentioned that the inequality (2.4) has been proven in [18]. But for completeness, we reprove this inequality here. In addition, a noncomplete characterization of the graphs for which the equality holds in (2.4) was also given in [18]. We here present a complete one by using the method of Wang and Ma which was used in the case of bipartite graphs (see [17]), with some necessary modification.

We first prove the inequality (2.4). Clearly, if $\beta(G) = 1$, the unique nontrivial connected component of G is K_3 or $K_{1,q}$ $(q \geqslant 1)$. Furthermore, it is easy to check that $\mathcal{E}(K_3) = 4 > 2\beta(K_3)$ or $\mathcal{E}(K_{1,q}) = 2\sqrt{q} \geqslant 2\beta(K_{1,q})$. Therefore, the inequality (2.4) holds for $\beta(G) = 1$.

If $\beta(G) \geqslant 2$, we may assume, for contradiction, that G is a graph with $\mathcal{E}(G) < 2\beta(G)$ and has the minimum possible matching number. The minimality of G yields that

$$\text{if there is a graph } G^* \text{ satisfying } \beta(G^*) < \beta(G), \text{ then } \mathcal{E}(G^*) \geqslant 2\beta(G^*). \tag{2.5}$$

Let M be a maximum matching of G with $e \in M$, and let Ω be the cut set such that $G - \Omega \cong [e] \cup (G - [e])$, that is, the set of edges which are incident to e. Since $M - \{e\}$ is a matching of $G - [e]$, we have $\beta(G - [e]) \geqslant \beta(G) - 1$. On the other hand, since the union of a maximum matching of $G - [e]$ and $\{e\}$ is a matching of G, we have $\beta(G - [e]) + 1 \leqslant \beta(G)$. Thus, it follows that $\beta(G - [e]) = \beta(G) - 1 < \beta(G)$ and consequently, by (2.5) we obtain $\mathcal{E}(G - [e]) \geqslant 2\beta(G - [e])$. However, by Lemma 4,

$$\mathcal{E}(G) \geqslant \mathcal{E}(G - \Omega) = \mathcal{E}([e]) + \mathcal{E}(G - [e]) \geqslant 2 + 2\beta(G - [e]) = 2\beta(G),$$

contradicting the assumption $\mathcal{E}(G) < 2\beta(G)$. This proves that the inequality (2.4) holds for $\beta(G) \geqslant 2$ as well, completing the proof of the inequality (2.4).

We next discuss the equality case of the inequality (2.4). We first suppose that G is the vertex-disjoint union of some complete bipartite graphs with perfect matchings and some possible isolated vertices. That is to say, every nontrivial connected component of G is a complete bipartite graph with a perfect matching, say $K_{p,p}$. Moreover, we have $\mathcal{E}(K_{p,p}) = 2p = 2\beta(K_{p,p})$ and hence, the equality holds in (2.4).

Conversely, if $\mathcal{E}(G) = 2\beta(G)$, we shall prove that each nontrivial connected component of G is a complete bipartite graph with a perfect matching. Without loss of generality, we can suppose that $G_1, \ldots, G_t$ $(t \geqslant 1)$ are all the nontrivial connected components of G. Noting that the inequality (2.4) yields that $\mathcal{E}(G_i) \geqslant 2\beta(G_i)$ for each $i \in \{1, \ldots, t\}$, we have

$$2\beta(G) = \mathcal{E}(G) = \mathcal{E}(G_1) + \ldots + \mathcal{E}(G_t) \geqslant 2\beta(G_1) + \ldots + 2\beta(G_t) = 2\beta(G),$$

which implies that

$$\mathcal{E}(G_i) = 2\beta(G_i), \quad i = 1, \ldots, t. \tag{2.6}$$

Claim 1. G_i $(i = 1, \ldots, t)$ has a perfect matching.

Proof of Claim 1. Indeed, if G_i has no perfect matching, then there exists a vertex $v_i \in G_i$ not covered by a maximum matching of G_i and hence, $\beta(G_i - v_i) = 2\beta(G_i)$. Furthermore, by Lemma 5 and the inequality (2.4), we obtain

$$\mathcal{E}(G_i) > \mathcal{E}(G_i - v_i) \geqslant 2\beta(G_i - v_i) = 2\beta(G_i),$$

which contradicts (2.6). Thus, Claim 1 holds. □

Claim 2. G_i $(i = 1, \ldots, t)$ is a complete bipartite graph.

P r o o f of Claim 2. Clearly, if $\beta(G_i) = 1$, noting that G_i has a perfect matching, we can easily see that G_i must be the complete bipartite graph $K_{1,1}$. □

If $\beta(G_i) = 2$, then by (1.3) and (2.6), we have $m(G_i) \leqslant (\mathcal{E}(G_i)/2)^2 = \beta(G_i)^2 = 4$, from which we may conclude that G_i must be P_4, $K_{2,2}$ or $K_{1,3}^{+}$, where $K_{1,3}^{+}$ is the graph obtained from $K_{1,3}$ by joining two of its vertices of degree one. Further, by a direct calculation, we have $\mathcal{E}(P_4) \approx 4.472 > 2\beta(P_4)$, $\mathcal{E}(K_{2,2}) = 4 = 2\beta(K_{2,2})$, and $\mathcal{E}(K_{1,3}^{+}) \approx 4.962 > 2\beta(K_{1,3}^{+})$, which yield that G_i must be the complete bipartite graph $K_{2,2}$.

If $\beta(G_i) \geqslant 3$, we may assume, for contradiction, that G_i is not a complete bipartite graph. We can further suppose, without loss of generality, that G_i has the minimum possible matching number. The minimality of G_i yields that

(2.7) if there is a connected graph G^{**} with a perfect matching such that

$$\mathcal{E}(G^{**}) = 2\beta(G^{**}) \quad \text{and} \quad \beta(G^{**}) < \beta(G_i),$$

then G^{**} must be a complete bipartite graph.

Now, let M_i be a perfect matching of G_i with $e_i \in M_i$, and let $H_i = G_i - [e_i]$. It is easy to see that $M_i - \{e_i\}$ is a perfect matching of H_i and hence,

$$\beta(H_i) = \beta(G_i) - 1 < \beta(G_i). \tag{2.8}$$

We will further prove that H_i is a complete bipartite graph. Indeed, by (2.7) we just need to show that H_i is a connected graph with $\mathcal{E}(H_i) = 2\beta(H_i)$.

First, let Ω_i be the cut set such that $G_i - \Omega_i \cong [e_i] \cup H_i$. Noting that the inequality (2.4) gives that $\mathcal{E}(H_i) \geqslant 2\beta(H_i)$, and by (2.6), (2.8), and Lemma 4, we have

$$2\beta(G_i) = \mathcal{E}(G_i) \geqslant \mathcal{E}(G_i - \Omega_i) = \mathcal{E}([e_i]) + \mathcal{E}(H_i) \geqslant 2 + 2\beta(H_i) = 2\beta(G_i),$$

which yields that $\mathcal{E}(H_i) = 2\beta(H_i)$.

Second, we will show that H_i is a connected graph. Assume for contradiction that $H_i^{(1)}, H_i^{(2)}, \ldots, H_i^{(s)}$ $(s \geqslant 2)$ are all the connected components of H_i. Clearly, $H_i^{(j)}$ $(j = 1, 2, \ldots, s)$ has a perfect matching, and by the same arguments as we proved (2.6), we can obtain

$$\mathcal{E}(H_i^{(j)}) = 2\beta(H_i^{(j)}). \tag{2.9}$$

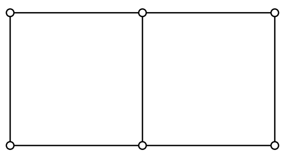
Also, let $K_i^{(j)} = G_i - H_i^{(j)}$ and let $\Omega_i^{(j)}$ be the cut set such that $G_i - \Omega_i^{(j)} \cong H_i^{(j)} \cup K_i^{(j)}$. It is easy to check that $K_i^{(j)}$ is a connected graph with a perfect matching and

$$\beta(K_i^{(j)}) = \beta(G_i) - \beta(H_i^{(j)}) < \beta(G_i). \tag{2.10}$$

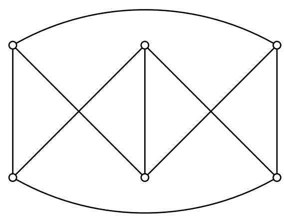
Thus, noting that the inequality (2.4) gives that $\mathcal{E}(K_i^{(j)}) \geqslant 2\beta(K_i^{(j)})$, and by (2.6), (2.9), (2.10), and Lemma 4, we have

$$2\beta(G_i) = \mathcal{E}(G_i) \geqslant \mathcal{E}(G_i - \Omega_i^{(j)}) = \mathcal{E}(H_i^{(j)}) + \mathcal{E}(K_i^{(j)}) \geqslant 2\beta(H_i^{(j)}) + 2\beta(K_i^{(j)}) = 2\beta(G_i),$$

which yields that $\mathcal{E}(K_i^{(j)}) = 2\beta(K_i^{(j)})$. Now, by combining the above arguments, we see that $K_i^{(j)}$ is exactly a graph satisfying the condition in (2.7) and hence, $K_i^{(j)}$ must be a complete bipartite graph. This also implies that $s = 2$. Further, we let $e_i' \in M_i \cap E(H_i^{(1)})$, $H_i' = G_i - [e_i']$, and Ω_i' be the cut set such that $G_i - \Omega_i' \cong [e_i'] \cup H_i'$. By the same arguments as done on H_i, we can derive that H_i' is a graph with a perfect matching such that $\mathcal{E}(H_i') = 2\beta(H_i')$ and $\beta(H_i') < \beta(G_i)$. Also, since both $K_i^{(1)}$ and $K_i^{(2)}$ are complete bipartite graphs, we can deduce that H_i' is connected. Thus, H_i' is also a graph satisfying the condition in (2.7) and hence, H_i' must be a complete bipartite graph. This, as well as the fact that $H_i^{(1)}$ has a perfect matching, implies that $H_i^{(1)} \cong K_{1,1}$. Similarly, we obtain $H_i^{(2)} \cong K_{1,1}$. Now, bearing in mind that $G_i - [e_i] = H_i \cong H_i^{(1)} \cup H_i^{(2)}$ and $G_i - H_i^{(j)} = K_i^{(j)}$ $(j = 1, 2)$ is a complete bipartite graph, we may conclude that $G_i - H_i^{(j)} \cong K_{2,2}$ $(j = 1, 2)$ and consequently, G_i must be isomorphic to the graph C_6^+ (see Figure 1 (a), where the middle vertical edge is e_i). However, a direct calculation shows that $\mathcal{E}(C_6^+) \approx 7.6568 > 2\beta(C_6^+)$, which contradicts (2.6). This proves that H_i is a connected graph.



(a) C_6^+



(b) $G^{\#}$

Figure 1. The graphs C_6^+ and $G^{\#}$

We can now complete this proof by showing that G_i must be a complete bipartite graph for $\beta(G_i) \geqslant 3$ (contradicting the previous assumption). Since H_i has proven to be a complete bipartite graph with a perfect matching, we can suppose, without loss of generality, that $\{x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(\beta_i - 1)}\}$ and $\{y_i^{(1)}, y_i^{(2)}, \ldots, y_i^{(\beta_i - 1)}\}$ are the two partite sets of H_i, and let $e_i = x_i^{(\beta_i)} y_i^{(\beta_i)}$, where $\beta_i = \beta(G_i) \geqslant 3$. Note that $\{x_i^{(1)} y_i^{(1)}, x_i^{(2)} y_i^{(2)}, \ldots, x_i^{(\beta_i)} y_i^{(\beta_i)}\}$ is a perfect matching of G_i, and by the same arguments as done on H_i, we can deduce that $G_i - [x_i^{(k)} y_i^{(k)}]$ $(k = 1, 2, \ldots, \beta_i - 1)$ is also a complete bipartite graph with a perfect matching.

If $\beta_i = 3$, then it is easy to check that $G_i - [x_i^{(l)} y_i^{(l)}]$ $(l = 1, 2, 3)$ is isomorphic to $K_{2,2}$. This implies that $x_i^{(3)}$ must be adjacent to exactly one vertex of each edge in $\{x_i^{(1)} y_i^{(1)}, x_i^{(2)} y_i^{(2)}\}$ and $y_i^{(3)}$ must be adjacent to the other two vertices that are not adjacent to $x_i^{(3)}$. Further, if $x_i^{(3)}$ is adjacent to the two vertices $x_i^{(1)}$ and $x_i^{(2)}$ (or $y_i^{(1)}$ and $y_i^{(2)}$), then $G_i \cong K_{3,3}$; otherwise, $G_i \cong G^{\#}$ (see Figure 1 (b)). Now, by a direct calculation, we have $\mathcal{E}(K_{3,3}) = 6 = 2\beta(K_{3,3})$ and $\mathcal{E}(G^{\#}) = 8 > 2\beta(G^{\#})$, which yield that G_i must be the complete bipartite graph $K_{3,3}$, as desired.

If $\beta_i \geqslant 4$, then we see that $H_i - [x_i^{(1)} y_i^{(1)}] - [x_i^{(2)} y_i^{(2)}]$ is a complete bipartite graph with the two partite sets being $\{x_i^{(3)}, \ldots, x_i^{(\beta_i - 1)}\}$ and $\{y_i^{(3)}, \ldots, y_i^{(\beta_i - 1)}\}$. Moreover, since $G_i - [x_i^{(1)} y_i^{(1)}]$ is a complete bipartite graph, $G_i - [x_i^{(1)} y_i^{(1)}] - [x_i^{(2)} y_i^{(2)}]$ is also a complete bipartite graph, whose partite sets are $\{x_i^{(3)}, \ldots, x_i^{(\beta_i - 1)}, x_i^{(\beta_i)}\}$ and $\{y_i^{(3)}, \ldots, y_i^{(\beta_i - 1)}, y_i^{(\beta_i)}\}$, or $\{x_i^{(3)}, \ldots, x_i^{(\beta_i - 1)}, y_i^{(\beta_i)}\}$ and $\{y_i^{(3)}, \ldots, y_i^{(\beta_i - 1)}, x_i^{(\beta_i)}\}$. This yields that G_i must be the complete bipartite graph K_{β_i, β_i} (whose partite sets are $\{x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots, x_i^{(\beta_i - 1)}, x_i^{(\beta_i)}\}$ and $\{x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots, y_i^{(\beta_i - 1)}, y_i^{(\beta_i)}\}$, or $\{x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots, x_i^{(\beta_i - 1)}, y_i^{(\beta_i)}\}$ and $\{x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots, y_i^{(\beta_i - 1)}, x_i^{(\beta_i)}\}$), as required.

The proof of Theorem 9 is thus completed. $\square$

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References

[1] *S. Akbari, E. Ghorbani, S. Zare*: Some relations between rank, chromatic number and energy of graphs. Discrete Math. *309* (2009), 601–605. zbl MR doi

[2] *E. Andrade, M. Robbiano, B. San Martín*: A lower bound for the energy of symmetric matrices and graphs. Linear Algebra Appl. *513* (2017), 264–275. zbl MR doi

[3] *Ş. B. Bozkurt Altındağ, D. Bozkurt*: Lower bounds for the energy of (bipartite) graphs. MATCH Commun. Math. Comput. Chem. *77* (2017), 9–14. MR

[4] *G. Caporossi, D. Cvetković, I. Gutman, P. Hansen*: Variable neighborhood search for extremal graphs 2. Finding graphs with extremal energy. J. Chem. Inf. Comput. Sci. *39* (1999), 984–996. doi

[5] *B. Cheng, B. Liu*: On the nullity of graphs. Electron. J. Linear Algebra *16* (2007), 60–67. zbl MR doi

[6] *C. A. Coulson*: On the calculation of the energy in unsaturated hydrocarbon molecules. Proc. Camb. Philos. Soc. *36* (1940), 201–203. doi

[7] *D. Cvetković, P. Rowlinson, S. Simić*: An Introduction to the Theory of Graph Spectra. London Mathematical Society Student Texts 75. Cambridge University Press, Cambridge, 2010. zbl MR doi

[8] *K. C. Das, S. A. Mojallal, I. Gutman*: Improving McClelland's lower bound for energy. MATCH Commun. Math. Comput. Chem. *70* (2013), 663–668. zbl MR

[9] *J. Day, W. So*: Graph energy change due to edge deletion. Linear Algebra Appl. *428* (2008), 2070–2078. zbl MR doi

[10] *R. A. Horn, C. R. Johnson*: Matrix Analysis. Cambridge University Press, Cambridge, 1985. zbl MR doi

[11] *A. Jahanbani*: Some new lower bounds for energy of graphs. Appl. Math. Comput. *296* (2017), 233–238. zbl MR doi

[12] *J. H. Koolen, V. Moulton*: Maximal energy graphs. Adv. Appl. Math. *26* (2001), 47–52. zbl MR doi

[13] *X. Li, Y. Shi, I. Gutman*: Graph Energy. Springer, New York, 2012. zbl MR doi

[14] *B. J. McClelland*: Properties of the latent roots of a matrix: The estimation of π-electron energies. J. Chem. Phys. *54* (1971), 640–643. doi

[15] *I. Milovanović, E. Milovanović, I. Gutman*: Upper bounds for some graph energies. Appl. Math. Comput. *289* (2016), 435–443. zbl MR doi

[16] *J. Rada, A. Tineo*: Upper and lower bounds for the energy of bipartite graphs. J. Math. Anal. Appl. *289* (2004), 446–455. zbl MR doi

[17] *L. Wang, X. Ma*: Bounds of graph energy in terms of vertex cover number. Linear Algebra Appl. *517* (2017), 207–216. zbl MR doi

[18] *D. Wong, X. Wang, R. Chu*: Lower bounds of graph energy in terms of matching number. Linear Algebra Appl. *549* (2018), 276–286. zbl MR doi

[19] *A. Yu, M. Lu, F. Tian*: New upper bounds for the energy of graphs. MATCH Commun. Math. Comput. Chem. *53* (2005), 441–448. zbl MR

[20] *B. Zhou*: Energy of a graph. MATCH Commun. Math. Comput. Chem. *51* (2004), 111–118. zbl MR

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