ON GROUPS OF AUTOMORPHISMS OF NILPOTENT p-GROUPS OF FINITE RANK

Tao Xu, Handan, Heguo Liu, Wuhan

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Abstract. Let α and β be automorphisms of a nilpotent p-group G of finite rank. Suppose that $\langle(\alpha\beta(g))(\beta\alpha(g))^{-1}\colon g \in G\rangle$ is a finite cyclic subgroup of G, then, exclusively, one of the following statements holds for G and Γ, where Γ is the group generated by α and β.
(i) G is finite, then Γ is an extension of a p-group by an abelian group.
(ii) G is infinite, then Γ is soluble and abelian-by-finite.

Keywords: automorphism; nilpotent group; finite rank

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1. Introduction

A well-known result of Hirsch (see [7]) asserts that a polycyclic group G is nilpotent if and only if all finite images of G are nilpotent. Hirsch's result has been extended to wider classes of groups: finitely generated hyper-abelian groups (see [5]), residually finite soluble minimax groups (see [4]) and linear groups over a finitely generated integral domain (see [8]).

In particular, Dardano, Eick and Menth in [1] proved that if the automorphism subgroup Γ of a residually finite soluble minimax group G induces a nilpotent group of automorphisms on every finite characteristic quotient of G, then Γ is nilpotent, which improves Hirsch's theorem.

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Recall that a group has finite rank r if every finitely generated subgroup can be generated by r elements and if r is the least positive integer with this property. It is natural to ask what the structure of the group generated by automorphisms is if the automorphisms of nilpotent groups of finite rank have some given property.

Liu and Zhang in [3] determined the structure of the group Γ generated by two p-automorphisms α and β acting on a finite rank nilpotent p-group G such that the subgroup I generated by the set $\{(\alpha\beta(g))(\beta\alpha(g))^{-1}\colon\ g \in G\}$ is finite cyclic. In this paper the same question is treated under the hypothesis that both α and β are automorphisms.

Theorem 1.1. *Let α and β be automorphisms of a nilpotent p-group G of finite rank. Assume that $I = \langle(\alpha\beta(g))(\beta\alpha(g))^{-1}\colon\ g \in G\rangle$ is a finite cyclic subgroup of G, then, exclusively, one of the following holds for G and Γ, where Γ is the group generated by α and β.*

(i) *G is finite, then Γ is an extension of a p-group by an abelian group.*
(ii) *G is infinite, then Γ is soluble and abelian-by-finite.*

Let G be a group. We shall write $\operatorname{Aut} G$ for the automorphism group of G and ζG for its centre. Assume that $\alpha \in \operatorname{Aut} G$ and H is an α-invariant subgroup of G. Denote the order of α by $|\alpha|$ and the restriction of α to H by $\alpha|_H$. Other notations and terminologies used throughout this paper are consistent with those in [6].

2. Proof of Theorem

Lemma 2.1. *Let α and β be automorphisms of a finite vector space A. If $\operatorname{Im}(\alpha\beta - \beta\alpha)$ is a cyclic group, then the group generated by α and β is an extension of a p-group by an abelian group.*

Proof. Suppose that $|\alpha| = m$, $|\beta| = n$, and A is a r-dimensional vector space over the finite field K. We may assume that F is an algebraic closure of K. It is easy to see that $\alpha, \beta \in \mathrm{GL}(r, K) \leqslant \mathrm{GL}(r, F)$. Since $\operatorname{Im}(\alpha\beta - \beta\alpha)$ is finite cyclic, $\operatorname{rank}(\alpha\beta - \beta\alpha) \leqslant 1$. By [2], α and β can be put in upper triangular form simultaneously. In other words, there exists $\gamma \in \mathrm{GL}(r, F)$ such that

$$\gamma^{-1}\alpha\gamma = \begin{pmatrix} a_1 & & & \\ & a_2 & * & \\ & & \ddots & \\ 0 & & & a_r \end{pmatrix}, \quad \gamma^{-1}\beta\gamma = \begin{pmatrix} b_1 & & & \\ & b_2 & * & \\ & & \ddots & \\ 0 & & & b_r \end{pmatrix},$$

where $a_i^m = 1$, $b_i^n = 1$ for all $i \in \{1, 2, \ldots, r\}$.

Since F is a finite field, the group of unitriangular matrices $U(r, F)$ is a finite p-group. Noticing

$$T(r, F) = (\underbrace{F^* \times F^* \times \ldots \times F^*}_{r}) \ltimes U(r, F),$$

where $T(r, F)$ is a group of upper triangular matrices over F, we have that the group generated by $\gamma^{-1}\alpha\gamma$ and $\gamma^{-1}\beta\gamma$ is an extension of a finite p-group by an abelian group. Hence the group generated by α and β is an extension of a finite p-group by an abelian group. $\square$

Lemma 2.2. *Let G and Γ be as in Theorem* 1.1 *and suppose that N is a normal Γ-invariant proper subgroup with G/N a finite vector space, then N and $\Gamma|_N = \Gamma/C_\Gamma(N)$ satisfy the premises of Theorem* 1.1. *If $\Gamma/C_\Gamma(N)$ satisfies the conclusion of* (i) *or* (ii), *respectively, then correspondingly so does Γ.*

Proof. The epimorphism, say φ, from Γ to $\Gamma/C_\Gamma(N)$ is given by sending $\gamma \in \Gamma$ to the restriction on N. Clearly $\varphi(\alpha)$ and $\varphi(\beta)$ satisfy the premise of Theorem 1.1.

Suppose first that N and $\Gamma/C_\Gamma(N)$ satisfy (i). Then G is finite and, by assumption, $\Gamma/C_\Gamma(N)$ contains a normal p-subgroup $P/C_\Gamma(N)$ such that Γ/P is abelian. By [6], Theorem 5.3.3, $C_\Gamma(N/\operatorname{Frat} N)$ is a p-group. Since $C_\Gamma(N) \leqslant C_\Gamma(N/\operatorname{Frat} N)$, $C_\Gamma(N)$ is a p-group. It follows that Γ is an extension of a p-group by an abelian group. Thus (i) holds for Γ.

Suppose next that N and $\Gamma/C_\Gamma(N)$ satisfy (ii). Then, by assumption, $\Gamma/C_\Gamma(N)$ is soluble and abelian-by-finite. Thus there is a normal series of Γ

$$\Gamma \geqslant A \geqslant C_\Gamma(N) \geqslant 1,$$

where $A/C_\Gamma(N)$ is abelian and Γ/A is finite and soluble. Since

$$C_\Gamma(N) \leqslant C_\Gamma(N/\operatorname{Frat} N)$$

and, as already observed during the proof of (i), the latter group is a finite p-group, it follows that Γ is soluble. Note that $A/C_\Gamma(N)$ is abelian and $C_\Gamma(N)$ is a finite p-group, and we have that the commutator subgroup of A must be finite. As A has finite index in the finitely generated group Γ we conclude that A is finitely generated. By [6], Exercise 7, ζA has a finite index in A. Since both Γ/A and $A/\zeta A$ are finite, Γ is an extension of the abelian group ζA by the finite soluble group $\Gamma/\zeta A$, establishing (ii) for Γ. $\square$

Lemma 2.3. *Let α and β be automorphisms of an abelian divisible p-group A of finite rank. If $\operatorname{Im}(\alpha\beta - \beta\alpha)$ is a finite cyclic group, then $\Gamma = \langle \alpha, \beta \rangle$ is abelian.*

P r o o f. The endomorphism $\alpha\beta - \beta\alpha$ maps the divisible group A onto a divisible subgroup of A and, as A is infinite, its image cannot be finite. It follows that α and β commute. Hence $\Gamma = \langle \alpha, \beta \rangle$ is abelian. $\square$

P r o o f of Theorem 1.1. Since G is a finite rank p-group, it satisfies the minimal condition on normal subgroups. By [6], Theorem 5.4.23, G is a finite extension of a divisible subgroup D. Thus G/D is a finite p-group and G is finite if and only if D is trivial. We need to consider the cases (i) G being finite and (ii) G being infinite.

(i) If G is elementary abelian, the result follows from Lemma 2.1. Otherwise, by Lemma 2.2, we may replace G by any maximal Γ-invariant subgroup N with G/N elementary abelian. Hence (i) holds for G and Γ if and only if (i) holds for N and $\Gamma/C_\Gamma(N)$. Now apply induction on $|G|$ until N itself becomes elementary abelian.

(ii) Now D is nontrivial. If $G = D$, then Lemma 2.3 implies the result. Otherwise G/D is a nontrivial finite p-group. Then G/D has a normal Γ-invariant subgroup N containing D such that G/N is elementary abelian. Using Lemma 2.2, we may replace G by N. Now apply induction on $|G/D|$ until N becomes identical to D. $\square$

3. Example

The following example validates the correctness of Lemma 2.1 and also illustrates that in Lemma 2.1 the group generated by α and β is not necessarily nilpotent.

Example 3.1. Let A be a 2-dimensional vector space over $\mathbb{Z}_3$, then the automorphism group of A is $\mathrm{GL}(2, \mathbb{Z}_3)$. In $\mathrm{GL}(2, \mathbb{Z}_3)$, we take

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Clearly $|\alpha| = 3$, $|\beta| = 2$, and

$$\alpha\beta - \beta\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \beta^{-1}\alpha\beta = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \alpha^{-1}.$$

It is easy to check that $\mathrm{Im}(\alpha\beta - \beta\alpha)$ is a group of order 3 and $\langle \alpha, \beta \rangle$ is a symmetric group of degree 3.

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Authors' addresses: Tao Xu, Department of Science, Hebei University of Engineering, No. 19 Taiji Road, Handan, 056038, Hebei, P. R. China, e-mail: `gtxutao@163.com`; Heguo Liu, Department of Mathematics, Hubei University, No. 368 Youyi Road, Wuhan, 430062, Hubei, P. R. China, e-mail: `ghliu@hubu.edu.cn`.