# SQUAREFREE MONOMIAL IDEALS WITH MAXIMAL DEPTH 

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$-module. We say $M$ has maximal depth if there is an associated prime $\mathfrak{p}$ of $M$ such that depth $M=\operatorname{dim} R / \mathfrak{p}$. In this paper we study squarefree monomial ideals which have maximal depth. Edge ideals of cycle graphs, transversal polymatroidal ideals and high powers of connected bipartite graphs with this property are classified.

Keywords: maximal depth; cycle graph; line graph; whisker graph; transversal polymatroidal ideal; power of edge ideal

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## Introduction

Let $K$ be a field and ( $R, \mathfrak{m}$ ) be a Noetherian local ring, or a standard graded $K$-algebra with graded maximal ideal $\mathfrak{m}$. Let $M$ be a finitely generated $R$-module. A basic fact in commutative algebra says that

$$
\operatorname{depth} M \leqslant \min \{\operatorname{dim} R / \mathfrak{p}: \mathfrak{p} \in \operatorname{Ass}(M)\} .
$$

We set $\operatorname{mdepth}_{R} M=\min \{\operatorname{dim} R / \mathfrak{p}: \mathfrak{p} \in \operatorname{Ass}(M)\}$. For simplicity, we write mdepth $M$ instead of $\operatorname{mdepth}_{R} M$. We say $M$ has maximal depth if the equality holds, i.e. depth $M=\operatorname{mdepth} M$. In other words, there is an associated prime $\mathfrak{p}$ of $M$ such that depth $M=\operatorname{dim} R / \mathfrak{p}$. In this paper, we study squarefree monomial ideals with maximal depth.

Let $I \subset S=K\left[x_{1}, \ldots, x_{n}\right]$ be a squarefree monomial ideal. We say $I$ has maximal depth if $S / I$ has maximal depth. We observe that $I$ has maximal depth is equivalent to saying that reg $I^{\vee}$ is the maximum degree of the generators of $I^{\vee}$. This fact motivates us to work on squarefree monomial ideals with maximal depth. Here $I^{\vee}$
is the Alexander dual of $I$ and reg $M$ denotes the regularity of a finitely generated graded $S$-module $M$.

Several authors have been working on this topic and some known results in this regards are as follows: If $I \subset S$ is a generic monomial ideal, then it has maximal depth, see [11], Theorem 2.2. If a monomial ideal $I$ has maximal depth, then so does its polarization, see [5]. Algebraic properties and some classifications of modules with maximal depth are given in [12].

In [8], the depth of the line graph $L_{n}$ is explicitly computed. In Section 2, we compute the depth of the line graph $L_{n}$ in a different way. Our proof relies on the fact that trees, and line graphs in particular, have maximal depth, see Proposition 2.1.

In [8], the depth of the cycle graph $C_{n}$ of length $n$ is also computed. This number is independent of the characteristic of the chosen field. By using this result, we classify all cycle graphs $C_{n}$ which have maximal depth. In fact, $C_{n}$ has maximal depth if and only if $n \equiv 0(\bmod 3)$ or $n \equiv 2(\bmod 3)$, see Proposition 2.3.

Adding a whisker to $C_{n}$ at a vertex $x_{1}$ means adding a new vertex $x_{n+1}$ and the edge $\left\{x_{1}, x_{n+1}\right\}$ to $C_{n}$. We denote by $C_{n} \cup W\left(x_{1}\right)$ the graph obtained from $C_{n}$ by adding a whisker at $x_{1}$. By using Proposition 2.1, we show that $C_{n}$ and $C_{n} \cup W\left(x_{1}\right)$ have the same depth as well as $C_{n} \cup W\left(x_{1}\right)$ has maximal depth.

In Section 3, we consider the transversal polymatroidal ideals. A transversal polymatroidal ideal is an ideal $I$ of the form $I=\mathfrak{p}_{F_{1}} \ldots \mathfrak{p}_{F_{r}}$, where $F_{1}, \ldots, F_{r}$ is a collection of nonempty subsets of $[n]$ with $r \geqslant 1$. Here for a nonempty subset $F$ of $[n]$ we denote by $\mathfrak{p}_{F}$ the monomial prime ideal $\left(\left\{x_{i}: i \in F\right\}\right)$. The depth of a transversal polymatroidal ideal is explicitly given in [7]. By applying this result, we classify all transversal polymatroidal ideals which have maximal depth. In fact, we prove the following: Let $I \subset S$ be a transversal polymatroidal ideal. Then $I$ has maximal depth if and only if $I$ is a product of monomial prime ideals such that at most one of the factors is not principal. In the following, we also classify ideals of Veronese type which have maximal depth.

In the final section, we consider $G$ to be a connected bipartite graph and $I$ its edge ideal. We show $I^{k}$ has maximal depth for $k \gg 0$ if and only if $G$ is a star graph.

## 1. Preliminaries

Let $K$ be a field and ( $R, \mathfrak{m}$ ) be a Noetherian local ring, or a standard graded $K$-algebra with graded maximal ideal $\mathfrak{m}$. It is a classical fact that if $M \neq 0$ is an $R$-module, then

$$
\operatorname{depth} M \leqslant \min \{\operatorname{dim} R / \mathfrak{p}: \mathfrak{p} \in \operatorname{Ass}(M)\},
$$

see [2]. We set $\operatorname{mdepth}_{R} M=\min \{\operatorname{dim} R / \mathfrak{p}: \mathfrak{p} \in \operatorname{Ass}(M)\}$. For simplicity, we write
mdepth $M$ instead of $\operatorname{mdepth}_{R} M$. Thus depth $M \leqslant \operatorname{mdepth} M \leqslant \operatorname{dim} M$. Observe that $\operatorname{depth}(M)=0$ if and only if $\operatorname{mdepth}(M)=0$. Thus, if mdepth $M=1$, then depth $M=1$.

Definition 1.1. We say $M$ has maximal depth if the equality holds, i.e.

$$
\text { depth } M=\operatorname{mdepth} M .
$$

In other words, there is an associated prime $\mathfrak{p}$ of $M$ such that depth $M=\operatorname{dim} R / \mathfrak{p}$.
Some examples of modules with maximal depth property are as follows:
$\triangleright$ Cohen-Macaulay modules have maximal depth because $\operatorname{depth} M=\operatorname{dim} R / \mathfrak{p}$ for every associated prime of $M$, see [13], Proposition 2.3.13.
$\triangleright$ Sequentially Cohen-Macaulay modules have maximal depth, see [12], Proposition 1.4, see also [13], Theorem 6.4.23, where the ring $R$ is a polynomial ring.
$\triangleright$ If $M$ is unmixed, then $M$ has maximal depth if and only if $M$ is Cohen-Macaulay.
Let $I \subset S=K\left[x_{1}, \ldots, x_{n}\right]$ be a squarefree monomial ideal. Then $I=\bigcap_{j=1}^{m} \mathfrak{p}_{j}$, where each of the $\mathfrak{p}_{j}$ is a monomial prime ideal of $I$. The ideal $I^{\vee}$, which is minimally generated by the monomials $u_{j}=\prod_{x_{i} \in \mathfrak{p}_{j}} x_{i}$, is called the Alexander dual of $I$. As usual we denote by reg $M$ the regularity of a finitely generated graded $S$-module $M$. We quote the following facts which for example can be found in [6].

Theorem 1.2 (Terai). $\operatorname{reg} I^{\vee}=\operatorname{pd} S / I$.

Theorem 1.3 (Auslander-Buchsbam formula). Let $M$ be a finitely generated $R$-module with pd $M<\infty$. Then

$$
\operatorname{pd} M+\operatorname{depth} M=\operatorname{depth} R .
$$

The big height of an ideal $J \subset S$, denoted by bight $J$, is the maximum height of the minimal primes of $J$. The following simple fact motivates us to work on squarefree monomial ideals with maximal depth. We say $I \subset S$ has maximal depth if $S / I$ has maximal depth.

Proposition 1.4. Let $I \subset S$ be a squarefree monomial ideal. Then $I$ has maximal depth if and only if reg $I^{\vee}$ is the maximum degree of the generators of $I^{\vee}$.

Proof. Suppose $I$ has maximal depth. Hence

$$
\operatorname{reg} I^{\vee}=\operatorname{pd} S / I=n-\operatorname{depth} S / I=n-\operatorname{mdepth} S / I=\operatorname{bight} I
$$

Theorem 1.2 explains the first step in this sequence. Theorem 1.3 provides the second step. Our assumption implies the third step. The fourth step follows from that fact that when $I$ is squarefree, the associated primes are the same as minimal primes containing $I$. Notice that the bight $I$ is the maximum degree of the generators of $I^{\vee}$. Therefore the conclusion follows. Conversely, suppose $\operatorname{reg} I^{\vee}$ is the maximum degree of the generators of $I^{\vee}$. By the same reasons as above, we have

$$
\operatorname{depth} S / I=n-\operatorname{pd} S / I=n-\operatorname{reg} I^{\vee}=n-\operatorname{bight} I=\operatorname{mdepth} S / I,
$$

as desired.
We recall the following fact from [13], Lemma 2.3.8.

Lemma 1.5 (Depth Lemma). If $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ is a short exact sequence of $R$-modules, then:
(a) If $\operatorname{depth}(M)<\operatorname{depth}(L)$, then $\operatorname{depth}(N)=\operatorname{depth}(M)$.
(b) If $\operatorname{depth}(M)=\operatorname{depth}(L)$, then $\operatorname{depth}(N) \geqslant \operatorname{depth}(M)$.
(c) If $\operatorname{depth}(M)>\operatorname{depth}(L)$, then $\operatorname{depth}(N)=\operatorname{depth}(L)+1$.

## 2. Line and cycle graphs

Let $G$ be a graph. The vertex set of $G$ will be denoted by $V(G)$ and will be the set $[n]=\{1,2, \ldots, n\}$. We denote the set of edges of $G$ by $E(G)$. We consider the edge ideal $I(G)$ which is generated by all monomials $x_{i} x_{j}$ with $\{i, j\} \in E(G)$. A subset $C \subset[n]$ is called a vertex cover of $G$ if $C \cap\{i, j\} \neq \emptyset$ for all edges $\{i, j\}$ of $G$. A vertex cover $C$ is called minimal if $C$ is a vertex of $G$, and no proper subset of $C$ is a vertex cover of $G$. A minimal vertex cover of $G$ is called maximum if it has maximum cardinality among the minimal vertex covers of $G$. Thus, bight $I(G)$ is the cardinality of the maximum minimal vertex covers of $G$.

It is well known that the minimal vertex covers of $G$ are the sets of generators of the minimal primes of $I(G)$. In fact, a subset $C=\left\{i_{1}, \ldots, i_{r}\right\} \subset[n]$ is a minimal vertex cover of $G$ if and only if $\mathfrak{p}_{C}=\left(x_{i_{1}}, \ldots, x_{i_{r}}\right)$ is a minimal prime ideal of $I(G)$, see [6], Lemma 9.1.4.

The graph $G$ is called disconnected if $V(G)$ is the disjoint union of $W_{1}$ and $W_{2}$ and there is no edge $\{i, j\}$ of $G$ with $i \in W_{1}$ and $j \in W_{2}$. The graph $G$ is called connected if it is not disconnected. A graph which has no cycle and which is connected is called a tree.

For $n \geqslant 2$ we let $L_{n}$ denote the line graph on $n$ vertices. This is the graph with vertices $[n]$ and edges $\{j, j+1\}$ for all $j=1, \ldots, n-1$. Hence, its edge ideal is $I\left(L_{n}\right)=\left(x_{1} x_{2}, x_{2} x_{3}, \ldots, x_{n-1} x_{n}\right)$ in a polynomial ring with $n$ variables. In the following, we explicitly compute the depth of the line graph $L_{n}$. However, this is a known fact, see [8], Corollary 7.7.35, but here we prove it in a different way.

Notation. For any graph $G$ we write depth $G$ for the depth of $S / I(G)$.
Proposition 2.1. The depth of the line graph $L_{n}$ is independent of the characteristic of the chosen field and is

$$
\operatorname{depth} L_{n}= \begin{cases}\frac{n}{3} & \text { if } n \equiv 0(\bmod 3) \\ \frac{n+2}{3} & \text { if } n \equiv 1(\bmod 3) \\ \frac{n+1}{3} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

Proof. Notice that the line graph is a tree. Trees are sequentially CohenMacaulay, see [3]. As sequentially Cohen-Macaulay modules have maximal depth, all trees have maximal depth. In particular, $L_{n}$ has maximal depth for all $n$. Let $I=I\left(L_{n}\right)$ be the edge ideal of $L_{n}$ in a polynomial ring $S$ with $n$ variables. We consider the following cases:

Case 1: $n \equiv 0(\bmod 3)$. We claim that the set

$$
C=\{1,3,4,6,7,9,10, \ldots, n-3, n-2, n\}
$$

is a maximum minimal vertex cover of $L_{n}$. A minimal vertex cover of $L_{n}$ cannot contain 3 consecutive vertices because of minimality. This implies that if we divide the vertices of $L_{n}$ into blocks of 3 vertices, then each block can have at most 2 vertices in the cover. Therefore the cardinality of a minimal vertex cover can be at most $2 n / 3$. Thus

$$
\mathfrak{p}_{C}=\left(x_{1}, x_{3}, x_{4}, x_{6}, x_{7}, x_{9}, x_{10}, \ldots, x_{n-3}, x_{n-2}, x_{n}\right)
$$

is a minimal prime ideal of $I$ with maximum height and so bight $I=2 n / 3$. It follows that mdepth $L_{n}=n-2 n / 3=n / 3$ and hence depth $L_{n}=n / 3$.

Case 2: $n \equiv 1(\bmod 3)$. Hence $n-1 \equiv 0(\bmod 3)$. We claim that the set

$$
C=\{2,3,5,6,8,9, \ldots, n-2, n-1\}
$$

is a maximum minimal vertex cover of $L_{n}$. In fact, the vertices of $L_{n}$ can be divided into blocks with 3 vertices as well as one block with only 1 vertex. Then each block
of 3 vertices can have at most 2 vertices in the cover. The vertex in the block with one vertex need not be in the cover. Therefore the cardinality of a minimal vertex cover can be at most $2(n-1) / 3$. Hence

$$
\mathfrak{p}_{C}=\left(x_{2}, x_{3}, x_{5}, x_{6}, x_{8}, x_{9}, \ldots, x_{n-2}, x_{n-1}\right)
$$

is a minimal prime ideal of $I$ with maximum height and so bight $I=2(n-1) / 3$. Consequently, mdepth $L_{n}=n-2(n-1) / 3=(n+2) / 3$ and so depth $L_{n}=(n+2) / 3$.

Case 3: $n \equiv 2(\bmod 3)$. Hence, $n-2 \equiv 0(\bmod 3)$. The set

$$
C=\{2,3,5,6,8,9, \ldots, n-3, n-2, n\}
$$

is a maximum minimal vertex cover of $L_{n}$. Indeed, the vertices of $L_{n}$ can be divided into blocks with 3 vertices as well as only one block with 2 vertex. Then each block of 3 vertices can have at most 2 vertices and the block of 2 vertices can have at most 1 vertex in the cover. Therefore the cardinality of a minimal vertex cover can be at most $2(n-2) / 3+1=(2 n-1) / 3$. Hence

$$
\mathfrak{p}_{C}=\left(x_{2}, x_{3}, x_{5}, x_{6}, x_{8}, x_{9}, \ldots, x_{n-3}, x_{n-2}, x_{n}\right)
$$

is a minimal prime ideal of $I$ with maximum height and so bight $I=2(n-1) / 3$. Consequently, mdepth $L_{n}=n-(2 n-1) / 3=(n+1) / 3$ and so depth $L_{n}=(n+1) / 3$. We remark that the proof of the proposition does not depend on the characteristic of the field $K$.

Let $C_{n}$ be a cycle graph of length $n$. We recall the following result from [8], Corollary 7.6.30.

Fact 2.2. The depth of the cycle graph is independent of the characteristic of the chosen field and is

$$
\operatorname{depth} C_{n}= \begin{cases}\frac{n}{3} & \text { if } n \equiv 0(\bmod 3) \\ \frac{n-1}{3} & \text { if } n \equiv 1(\bmod 3) \\ \frac{n+1}{3} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

In the following, we classify all cycle graphs which have maximal depth.
Proposition 2.3. The cycle graph $C_{n}$ has maximal depth if and only if $n \equiv 0$ $(\bmod 3)$ or $n \equiv 2(\bmod 3)$.

Proof. Let $I=I\left(C_{n}\right)$ be the edge ideal of $C_{n}$ in a polynomial ring $S$ with $n$ variables. We need to consider the following three cases.

Case 1: $n \equiv 0(\bmod 3)$. For the maximum minimal vertex covers of cycles one can use line graphs. A similar argument as in the proof of Proposition 2.3 shows that the cardinality of a minimal vertex cover of $C_{n}$ in this case can be at most $2 n / 3$. The set

$$
C=\{i, i+1, i+3, i+4, i+6, i+7, \ldots, n-i-1, n-i\}
$$

is a maximum minimal vertex cover of $C_{n}$ for all $i$. Thus,

$$
\mathfrak{p}_{C}=\left(x_{i}, x_{i+1}, x_{i+3}, x_{i+4}, x_{i+6}, x_{i+7}, \ldots, x_{n-i-1}, x_{n-i}\right)
$$

is a minimal prime ideal of $I$ with maximum height and so bight $I=2 n / 3$. Hence mdepth $C_{n}=n-2 n / 3=n / 3=$ depth $C_{n}$. Fact 2.2 provides the last equality. Therefore $C_{n}$ has maximal depth.

Case 2: $n \equiv 2(\bmod 3)$. A similar argument as in the proof of Proposition 2.3 shows that the cardinality of a minimal vertex cover of $C_{n}$ in this case can be at most $2(n-2) / 3+1=(2 n-1) / 3$. One observes that the set

$$
C=\{i, i+1, i+3, i+4, \ldots, n+i-5, n+i-4, n+i-2\}
$$

is a maximum minimal vertex cover of $C_{n}$ for all $i$. Hence

$$
\mathfrak{p}_{C}=\left(x_{i}, x_{i+1}, x_{i+3}, x_{i+4}, \ldots, x_{n+i-5}, x_{n+i-4}, x_{n+i-2}\right)
$$

is a minimal prime ideal of $I$ with maximum height and so bight $I=(2 n-1) / 3$. Consequently,

$$
\text { mdepth } C_{n}=n-(2 n-1) / 3=(n+1) / 3=\operatorname{depth} C_{n}
$$

Fact 2.2 explains the last equality.
Case 3: $n \equiv 1(\bmod 3)$. In this case, one has that the cardinality of a minimal vertex cover of $C_{n}$ can be at most $2(n-1) / 3$ and the set

$$
C=\{i, i+2, i+3, i+5, i+6, \ldots, n+i-5, n+i-4, n+i-2\}
$$

is a maximum minimal vertex cover of $C_{n}$ for all $i$. Hence

$$
\mathfrak{p}_{C}=\left(x_{i}, x_{i+2}, x_{i+3}, x_{i+5}, x_{i+6}, \ldots, x_{n+i-5}, x_{n+i-4}, x_{n+i-2}\right)
$$

is a minimal prime ideal of $I$ with maximum height and so bight $I=2(n-1) / 3$. Thus mdepth $C_{n}=n-2(n-1) / 3=(n+2) / 3$. Fact 2.2 provides depth $C_{n}=(n-1) / 3$. Thus, $C_{n}$ has no maximal depth in this case.

Adding a whisker to $C_{n}$ at a vertex $x_{1}$ means adding a new vertex $x_{n+1}$ and the edge $\left\{x_{1}, x_{n+1}\right\}$ to $C_{n}$. We denote by $C_{n} \cup W\left(x_{1}\right)$ the graph obtained from $C_{n}$ by adding a whisker at $x_{1}$. Thus $I\left(C_{n} \cup W\left(x_{1}\right)\right)=I\left(C_{n}\right)+\left(x_{1} x_{n+1}\right)$. In the following, by using Proposition 2.1, we show that $C_{n}$ and $C_{n} \cup W\left(x_{1}\right)$ have the same depth as well as $C_{n} \cup W\left(x_{1}\right)$ has maximal depth.

Proposition 2.4. The following statements hold:

$$
\operatorname{depth} C_{n}=\operatorname{depth} C_{n} \cup W\left(x_{1}\right)
$$

and $C_{n} \cup W\left(x_{1}\right)$ has maximal depth.
Proof. We set $I\left(C_{n}\right)=J$ and $I\left(C_{n} \cup W\left(x_{1}\right)\right)=I$. Consider the exact sequence

$$
\begin{equation*}
0 \rightarrow R /\left(I: x_{n+1}\right) \rightarrow R / I \rightarrow R /\left(I+\left(x_{n+1}\right)\right) \rightarrow 0 \tag{2.1}
\end{equation*}
$$

where $R=S\left[x_{n+1}\right]$. One has

$$
R /\left(I: x_{n+1}\right) \cong K\left[x_{2}, \ldots, x_{n}\right]\left[x_{n+1}\right] /\left(x_{2} x_{3}, x_{3} x_{4}, \ldots, x_{n-1} x_{n}\right)
$$

and

$$
R /\left(I+\left(x_{n+1}\right)\right) \cong S / J
$$

We consider the following three cases:
Case 1: $n \equiv 0(\bmod 3)$. Thus $n-1 \equiv 2(\bmod 3)$. By Proposition 2.1

$$
\operatorname{depth} K\left[x_{2}, \ldots, x_{n}\right] /\left(x_{2} x_{3}, x_{3} x_{4}, \ldots, x_{n-1} x_{n}\right)=((n-1)+1) / 3=n / 3
$$

Hence depth $R /\left(I: x_{n+1}\right)=n / 3+1$. Fact 2.2 provides depth $S / J=n / 3$. Thus by using (2.1) we have

$$
\operatorname{depth} R / I \geqslant \min \{n / 3+1, n / 3\}=n / 3
$$

For computing mdepth $R / I$ in this case, a similar argument as in the proof of Proposition 2.3 shows that the cardinality of a minimal vertex cover of $C_{n} \cup W\left(x_{1}\right)$ can be at most $(2 n+3) / 3$. One observes that the set

$$
C=\{n+1,2,3,5,6, \ldots, n-4, n-3, n-1, n\}
$$

is a maximum minimal vertex cover of $C_{n} \cup W\left(x_{1}\right)$. Thus

$$
\mathfrak{p}_{C}=\left(x_{n+1}, x_{2}, x_{3}, x_{5}, x_{6}, \ldots, x_{n-4}, x_{n-3}, x_{n-1}, x_{n}\right)
$$

is a minimal prime ideal of $I$ with bight $\mathfrak{p}_{C}=(2 n+3) / 3$. Hence

$$
\operatorname{mdepth} R / I=(n+1)-\operatorname{bight} \mathfrak{p}_{C}=(n+1)-(2 n+3) / 3=n / 3
$$

Consequently,

$$
n / 3 \leqslant \operatorname{depth} R / I \leqslant \operatorname{mdepth} R / I=n / 3 .
$$

Thus, we get the desired results in this case.
Case 2: $n \equiv 1(\bmod 3)$. Thus, $n-1 \equiv 0(\bmod 3)$. By Proposition 2.1

$$
\operatorname{depth} K\left[x_{2}, \ldots, x_{n}\right] /\left(x_{2} x_{3}, x_{3} x_{4}, \ldots, x_{n-1} x_{n}\right)=(n-1) / 3
$$

Hence depth $R /\left(I: x_{n+1}\right)=(n+2) / 3$. Fact 2.2 explains depth $S / J=(n-1) / 3$. Thus, by using (2.1) we have

$$
\operatorname{depth} R / I \geqslant \min \{(n+2) / 3,(n-1) / 3\}=(n-1) / 3 .
$$

One observes that the cardinality of a minimal vertex cover of $C_{n} \cup W\left(x_{1}\right)$ in this case can be at most $2(n+2) / 3$ and the set

$$
C=\{n+1,2,3,5,6, \ldots, n-5, n-4, n-2, n\}
$$

is a maximum minimal vertex cover of $C_{n} \cup W\left(x_{1}\right)$. Thus

$$
\mathfrak{p}_{C}=\left(x_{n+1}, x_{2}, x_{3}, x_{5}, x_{6}, \ldots, x_{n-5}, x_{n-4}, x_{n-2}, x_{n}\right)
$$

is a minimal prime ideal of $I$ with bight $\mathfrak{p}_{C}=2(n+2) / 3$. Hence

$$
\text { mdepth } R / I=(n+1)-(2 n+4) / 3=(n-1) / 3 .
$$

We conclude that

$$
(n-1) / 3 \leqslant \operatorname{depth} R / I \leqslant \operatorname{mdepth} R / I=(n-1) / 3 .
$$

Therefore we get the desired conclusions in this case.
Case 3: $n \equiv 2(\bmod 3)$. Thus $n-1 \equiv 1(\bmod 3)$. By Proposition 2.1

$$
\operatorname{depth} K\left[x_{2}, \ldots, x_{n}\right] /\left(x_{2} x_{3}, x_{3} x_{4}, \ldots, x_{n-1} x_{n}\right)=((n-1)+2) / 3=(n+1) / 3 .
$$

Hence depth $R /\left(I: x_{n+1}\right)=(n+4) / 3$. Fact 2.2 provides depth $S / J=(n+1) / 3$. Thus by using (2.1) we have

$$
\operatorname{depth} R / I \geqslant \min \{(n+4) / 3,(n+1) / 3\}=(n+1) / 3 .
$$

One observes that the cardinality of a minimal vertex cover of $C_{n} \cup W\left(x_{1}\right)$ can be at most $2(n+1) / 3$ and the set

$$
C=\{n+1,2,3,5,6, \ldots, n-3, n-2, n\}
$$

is a maximum minimal vertex cover of $C_{n} \cup W\left(x_{1}\right)$. Thus

$$
\mathfrak{p}_{C}=\left(x_{n+1}, x_{2}, x_{3}, x_{5}, x_{6}, \ldots, x_{n-3}, x_{n-2}, x_{n}\right)
$$

is a minimal prime ideal of $I$ with bight $\mathfrak{p}_{C}=2(n+1) / 3$. Hence

$$
\operatorname{mdepth} R / I=(n+1)-\operatorname{bight} \mathfrak{p}_{C}=(n+1)-(2 n+2) / 3=(n+1) / 3
$$

Consequently,

$$
(n+1) / 3 \leqslant \operatorname{depth} R / I \leqslant \operatorname{mdepth} R / I=(n+1) / 3
$$

Therefore, we get the desired conclusions in this case too.
We remark that the second part of Proposition 2.4 also follows from [4], Corollary 3.4 in a different way.

## 3. Transversal polymatroids and ideals of Veronese type

In this section, we classify all transversal polymatroidal ideals and all ideals of Veronese type which have maximal depth. Let $F$ be a nonempty subset of $[n]$. We denote by $\mathfrak{p}_{F}$ the monomial prime ideal $\left(\left\{x_{i}: i \in F\right\}\right)$. A transversal polymatroidal ideal is an ideal $I$ of the form $I=\mathfrak{p}_{F_{1}} \ldots \mathfrak{p}_{F_{r}}$, where $F_{1}, \ldots, F_{r}$ is a collection of nonempty subsets of $[n]$ with $r \geqslant 1$. Let $G_{I}$ be the graph with vertex set $\{1, \ldots, r\}$ and for which $\{i, j\}$ is an edge of $G_{I}$ if and only if $F_{i} \cap F_{j} \neq \emptyset$. We recall the following fact from [7], Theorem 4.12.

Fact 3.1. Let $I=\mathfrak{p}_{F_{1}} \ldots \mathfrak{p}_{F_{r}} \subset S$ be a transversal polymatroidal ideal. Then

$$
\operatorname{depth} S / I=c\left(G_{I}\right)-1+n-\left|\bigcup_{i=1}^{r} F_{i}\right|
$$

where by $c\left(G_{I}\right)$ we denote the number of connected components of the graph $G_{I}$.
Let $\mathcal{H}$ be a subgraph of $G_{I}$. We associate the prime ideal $\mathfrak{p}_{\mathcal{H}}=\sum_{i \in \mathcal{V}(\mathcal{H})} \mathfrak{p}_{F_{i}}$. We denote by $\operatorname{Ass}(I)$ the set of associated prime ideals of $R / I$. The set of associated primes of $R / I$ is explicitly described in [7], Theorem 4.7 as follows.

Fact 3.2. Let $I \subset S$ be a transversal polymatroidal ideal. Then

$$
\operatorname{Ass}(I)=\left\{\mathfrak{p}_{\mathcal{T}}: \mathcal{T} \text { is a tree in } G_{I}\right\}
$$

In the following we characterize all transversal polymatroidal ideals which have maximal depth.

Proposition 3.3. Let $I=\mathfrak{p}_{F_{1}} \ldots \mathfrak{p}_{F_{r}} \subset S$ be a transversal polymatroidal ideal. The following conditions are equivalent:
(a) I has maximal depth,
(b) $I$ is a product of monomial prime ideals such that at most one of the factors is not principal.

Proof. $\quad(\mathrm{a}) \Rightarrow(\mathrm{b})$ : We may assume that $\bigcup_{i=1}^{r} F_{i}=[n]$. Let $k=c\left(G_{I}\right)$ and $G_{1}, \ldots, G_{k}$ be connected components of $G_{I}$. Fact 3.1 provides depth $S / I=k-1$. We denote by $I_{1}, \ldots, I_{k}$ the transversal polymatroidal ideals for which the associated graphs are the connected components of $G_{I}$. Hence $I=I_{1} \ldots I_{k}=I_{1} \cap \ldots \cap I_{k}$, since the ideals $I_{j}$ are generated in pairwise disjoint sets of variables. Thus $\operatorname{Ass}(I)=$ $\bigcup_{i=1}^{k} \operatorname{Ass}\left(I_{i}\right)$. We may assume that $1 \leqslant l_{1} \leqslant \ldots \leqslant l_{k}$, where for all $j$ we have $l_{j}=$ $\left|\bigcup_{i \in \mathcal{V}\left(G_{j}\right)}^{i=1} F_{i}\right|$. Note that $l_{1}+\ldots+l_{k}=n$. In view of Fact 3.2, we have mdepth $S / I=$ $n-l_{k}$. Since $I$ has maximal depth, it follows that $n-l_{k}=k-1$, and hence $l_{1}+\ldots+l_{k-1}=k-1$. Consequently, $I=\left(x_{1}\right) \ldots\left(x_{k-1}\right)\left(x_{k}, \ldots, x_{n}\right)$, as desired.
(b) $\Rightarrow$ (a): If $I$ is a product of monomial prime ideals such that all the factors are principal, then $S / I$ is Cohen-Macaulay and hence $I$ has maximal depth. Thus, we may assume that $I=\left(x_{1}\right) \ldots\left(x_{k-1}\right)\left(x_{k}, \ldots, x_{n}\right)$. As

$$
\operatorname{Ass}(I)=\left\{\left(x_{1}\right), \ldots,\left(x_{k-1}\right),\left(x_{k}, \ldots, x_{n}\right)\right\}
$$

we have mdepth $S / I=n-(n-k+1)=k-1$. The ideal $I$ is a transversal polymatroidal ideal. It follows from Theorem 3.1 that depth $S / I=k-1$. Here $c\left(G_{I}\right)=k$ and $\left|\bigcup_{i=1}^{r} F_{i}\right|=n$. Therefore $I$ has maximal depth.

As a consequence one has

Corollary 3.4. Let $I \subset S$ be the intersection of monomial prime ideals in pairwise disjoint sets of variables. Then $I$ has maximal depth if and only if $I$ is a product of monomial prime ideals such that at most one of the factors is not principal.

One of the most distinguished polymatroidal ideals is the ideal of Veronese type. Let $S=K\left[x_{1}, \ldots, x_{n}\right]$ and fix positive integers $d$ and $a_{1}, \ldots, a_{n}$ with $1 \leqslant a_{1} \leqslant \ldots \leqslant$ $a_{n} \leqslant d$. The ideal of Veronese type of $S$ indexed by $d$ and $\left(a_{1}, \ldots, a_{n}\right)$ is the ideal $I_{d ; a_{1}, \ldots, a_{n}}$ which is generated by those monomials $u=x_{1}^{u_{1}} \ldots x_{n}^{u_{n}}$ of $S$ of degree $d$ with $u_{i} \leqslant a_{i}$ for each $1 \leqslant i \leqslant n$.

The set of associated prime ideals and depth of the ideal of Veronese type are described in [7], Proposition 5.2 and Corollary 5.7 as

$$
\begin{equation*}
\operatorname{Ass}(S / I)=\left\{\mathfrak{p}_{F}: F \subset[n], \sum_{i=1}^{n} a_{i} \geqslant d-1+|F| \text { and } \sum_{i \notin F} a_{i} \leqslant d-1\right\} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{depth} S / I=\max \left\{0, d+n-1-\sum_{i=1}^{n} a_{i}\right\} \tag{3.2}
\end{equation*}
$$

Proposition 3.5. The ideal of Veronese type has maximal depth if and only if there exists a $\mathfrak{p}_{F} \in \operatorname{Ass}(S / I)$, where $|F|=\sum_{i=1}^{n} a_{i}-(d-1)$.

Proof. In view of (3.1), $\mathfrak{p}_{F}$ has the maximum height if $|F|=\sum_{i=1}^{n} a_{i}-(d-1)$. Thus mdepth $S / I=n-|F|=n+d-1-\sum_{i=1}^{n} a_{i}$, which is the same as depth $S / I$ by (3.2). Therefore the conclusion follows.

Here is an example:
Example 3.6. Consider $I=I_{5 ; 1,2,3} \subset S=K\left[x_{1}, x_{2}, x_{3}\right]$. Then

$$
I=\left(x_{1}^{2} x_{2}^{2} x_{3}, x_{1}^{3} x_{2} x_{3}, x_{1}^{3} x_{2}^{2}\right)
$$

Formula (3.1) yields

$$
\operatorname{Ass}(S / I)=\left\{\left(x_{1}\right),\left(x_{2}\right),\left(x_{1}, x_{2}\right),\left(x_{1}, x_{3}\right),\left(x_{2}, x_{3}\right)\right\}
$$

As $\mathfrak{p}_{F} \in \operatorname{Ass}(S / I)$ with $|F|=2, I$ has maximal depth and depth $S / I=$ mdepth $S / I=1$.

## 4. Powers of ideals

A subset $D \subset[n]$ is called an independent set of $G$ if $D$ contains no set $\{i, j\}$ which is an edge of $G$. The graph $G$ is called bipartite if $V(G)$ is the disjoint union of $V_{1}$ and $V_{2}$ such that $V_{1}$ and $V_{2}$ are independent sets. The bipartite graph $G$ is called a complete bipartite graph if $\{i, j\} \in E(G)$ for all $i \in V_{1}$ and $j \in V_{2}$.

Proposition 4.1. Let $G$ be a complete bipartite graph on the vertex set $V$ with bipartition $V=V_{1} \cup V_{2}$, where $V_{1}=\left\{v_{1}, \ldots, v_{n}\right\}$ and $V_{2}=\left\{w_{1}, \ldots, w_{m}\right\}$ with $1 \leqslant n \leqslant m$. Then $G$ has maximal depth if and only if $n=1$, i.e. $G$ is a star graph.

Proof. The edge ideal of $G$ is $\mathfrak{p}_{1} \cap \mathfrak{p}_{2}$, where $\mathfrak{p}_{1}=\left(x_{1}, \ldots, x_{n}\right)$ and $\mathfrak{p}_{2}=$ $\left(y_{1}, \ldots, y_{m}\right)$. We set $S=K\left[x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right]$ and $R=S /\left(\mathfrak{p}_{1} \cap \mathfrak{p}_{2}\right)$. Consider the exact sequence $0 \rightarrow S /\left(\mathfrak{p}_{1} \cap \mathfrak{p}_{2}\right) \rightarrow S / \mathfrak{p}_{1} \oplus S / \mathfrak{p}_{2} \rightarrow S /\left(\mathfrak{p}_{1}+\mathfrak{p}_{2}\right) \rightarrow 0$. Since depth $S /\left(\mathfrak{p}_{1}+\mathfrak{p}_{2}\right)=0$, it follows from Lemma 1.5 (Depth lemma) that depth $R=1$. On the other hand, $\operatorname{Ass}(R)=\left\{\mathfrak{p}_{1}, \mathfrak{p}_{2}\right\}$. It follows that mdepth $R=n$. Consequently, mdepth $R$ - depth $R=n-1$. Therefore the conclusion follows.

Remark 4.2. In Proposition 4.1, we showed mdepth $R-\operatorname{depth} R=n-1$. Thus, the difference between depth and mdepth can be any number.

In the following we classify all connected bipartite graphs such that $I^{k}$ has maximal depth for all $k \gg 0$.

Proposition 4.3. Let $G$ be a connected bipartite graph and $I=I(G)$ its edge ideal. Then $I^{k}$ has maximal depth for $k \gg 0$ if and only if $G$ is a star graph.

Proof. Suppose $I^{k}$ has maximal depth for $k \gg 0$. By [6], Corollary 10.3.18, we have depth $S / I^{k}=1$ for $k \gg 0$. Hence mdepth $S / I^{k}=1$ for $k \gg 0$. As $G$ is bipartite, we have $\operatorname{Ass}(I)=\operatorname{Ass}\left(I^{k}\right)$ for all $k$, see [9]. It follows that mdepth $S / I=1$ and hence depth $S / I=1$. Thus there exists a minimal vertex cover $F$ of $G$ such that $|[n] \backslash F|=1$. Therefore $G$ is a star graph.

Now, suppose $G$ is a star graph and $J$ is its edge ideal. By Proposition 4.1 we have depth $S / J=$ mdepth $S / J=1$. As $G$ is bipartite, we have mdepth $S / J^{k}=1$ for all $k$. It follows that depth $S / J^{k}=1$ for all $k$ and hence $J^{k}$ has maximal depth for $k \gg 0$.

Remark 4.4. Let $I$ be an ideal in a Noetherian ring $R$. Brodmann in [1] showed that $\operatorname{Ass}\left(I^{k}\right)=\operatorname{Ass}\left(I^{k+1}\right)$ for all $k \gg 0$. The ideal $I$ for which $\operatorname{Ass}\left(I^{k}\right) \subset \operatorname{Ass}\left(I^{k+1}\right)$ for all $k \geqslant 1$ is said to satisfy the persistence property. Edge ideals of graphs and polymatroidal ideals have persistence property, see [7], [10]. In this case, we have mdepth $S / I^{k+1} \leqslant \operatorname{mdepth} S / I^{k}$ for all $k$ and say the ideal $I$ has nonincreasing mdepth.

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