# $q$-ANALOGUES OF TWO SUPERCONGRUENCES OF Z.-W. SUN 

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Abstract. We give several different $q$-analogues of the following two congruences of Z.-W. Sun:

$$
\sum_{k=0}^{\left(p^{r}-1\right) / 2} \frac{1}{8^{k}}\binom{2 k}{k} \equiv\left(\frac{2}{p^{r}}\right)\left(\bmod p^{2}\right) \quad \text { and } \quad \sum_{k=0}^{\left(p^{r}-1\right) / 2} \frac{1}{16^{k}}\binom{2 k}{k} \equiv\left(\frac{3}{p^{r}}\right)\left(\bmod p^{2}\right)
$$

where $p$ is an odd prime, $r$ is a positive integer, and $\left(\frac{m}{n}\right)$ is the Jacobi symbol. The proofs of them require the use of some curious $q$-series identities, two of which are related to Franklin's involution on partitions into distinct parts. We also confirm a conjecture of the latter author and Zeng in 2012.

Keywords: congruences; $q$-binomial coefficient; cyclotomic polynomial; Franklin's involution

MSC 2020: 11B65, 05A10, 05A30, 11A07

## 1. INTRODUCTION

Among other things, Sun in [14], (1.7) and (1.8) proved the congruences

$$
\begin{align*}
\sum_{k=0}^{\left(p^{r}-1\right) / 2} \frac{1}{8^{k}}\binom{2 k}{k} & \equiv\left(\frac{2}{p^{r}}\right)\left(\bmod p^{2}\right)  \tag{1.1}\\
\sum_{k=0}^{\left(p^{r}-1\right) / 2} \frac{1}{16^{k}}\binom{2 k}{k} & \equiv\left(\frac{3}{p^{r}}\right)\left(\bmod p^{2}\right) \tag{1.2}
\end{align*}
$$

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where $p$ is an odd prime, $r$ is a positive integer, and $\left(\frac{m}{n}\right)$ is the Jacobi symbol. Recently, the latter author and Liu in [6], Theorem 1.2 gave the following $q$-analogue of (1.1): for odd $n$,

$$
\begin{equation*}
\sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k} q^{k^{2}}}{\left(q^{4} ; q^{4}\right)_{k}} \equiv(-q)^{\left(1-n^{2}\right) / 8}\left(\bmod \Phi_{n}(q)^{2}\right) \tag{1.3}
\end{equation*}
$$

Here and in what follows, $(a ; q)_{n}=(1-a)(1-a q) \ldots\left(1-a q^{n-1}\right)$ and $\Phi_{n}(q)$ is the $n$th cyclotomic polynomial in $q$.

The first aim of this paper is to give $q$-analogues of (1.1) and (1.2) as follows.

Theorem 1.1. Let $n$ be a positive odd integer. Then

$$
\begin{gather*}
\sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k} q^{2 k}}{\left(q^{2} ; q^{2}\right)_{k}\left(-q ; q^{2}\right)_{k}} \equiv\left(\frac{2}{n}\right) q^{2\lfloor(n+1) / 4\rfloor^{2}}\left(\bmod \Phi_{n}(q)^{2}\right),  \tag{1.4}\\
\sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k} q^{2 k}}{\left(q^{4} ; q^{4}\right)_{k}\left(-q ; q^{2}\right)_{k}} \equiv\left(\frac{3}{n}\right) q^{\left(n^{2}-1\right) / 12}\left(\bmod \Phi_{n}(q)^{2}\right),
\end{gather*}
$$

where $\lfloor x\rfloor$ denotes the largest integer not exceeding $x$.
It is easy to see that the congruences (1.4) and (1.5) reduce to (1.1) and (1.2), respectively, when $q \rightarrow 1$ and $n=p^{r}$.

Recall that the $q$-binomial coefficients $\left[\begin{array}{l}n \\ k\end{array}\right]$ are defined by

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]=\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q}= \begin{cases}\frac{(q ; q)_{n}}{(q ; q)_{k}(q ; q)_{n-k}} & \text { if } 0 \leqslant k \leqslant n \\
0 & \text { otherwise }\end{cases}
$$

Moreover, the $q$-integer is defined as $[n]=[n]_{q}=\left(1-q^{n}\right) /(1-q)$. The second aim of this paper is to give the following result, which in the case $n=p^{r}$ confirms a conjecture of the latter author and Zeng, see [8], Conjecture 5.13.

Theorem 1.2. Let $n$ be a positive integer. Then

$$
\sum_{k=0}^{n-1} q^{(n-k)^{2}}\left[\begin{array}{c}
n+k  \tag{1.6}\\
k
\end{array}\right]^{2}\left[\begin{array}{c}
n-1 \\
k
\end{array}\right]^{2} \equiv q[n]\left(\bmod \Phi_{n}(q)^{2}\right)
$$

Note that, exactly similarly to the proof of Theorem 5.3 in [8], we can show that

$$
\sum_{k=0}^{n-1} q^{(n-k)^{2}}\left[\begin{array}{c}
n+k  \tag{1.7}\\
k
\end{array}\right]^{2}\left[\begin{array}{c}
n-1 \\
k
\end{array}\right]^{2} \equiv 0(\bmod [n])
$$

Therefore, combining the congruences (1.6) and (1.7), we see that the congruence (1.6) also holds modulo $[n] \Phi_{n}(q)$. We refer the reader to [7] and references therein for other congruences on sums involving $q$-binomial coefficients.

Suggested by the referee, we would like to make the following conjecture.
Conjecture 1.3. The congruence (1.6) still holds modulo $[n] \Phi_{n}(q)^{2}$.
The paper is organized as follows. In the next section, we give a new proof of a curious $q$-series identity of Liu, see [12] and also provide two similar identities. We prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. Finally in Section 5, motivated by the recent work of the latter author and Zudilin, see [9], we give parameter generalizations of (1.3)-(1.5) and show more complicated $q$-analogues of (1.1) and (1.2).

## 2. A curious $q$-Series identity of J.-C. Liu

Liu in [12], (2.3) presented the following $q$-series identity:

$$
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
2 n-k  \tag{2.1}\\
k
\end{array}\right](-q ; q)_{n-k}=(-1)^{\binom{n}{2}} q^{\binom{n}{2}+\lfloor(n+1) / 2\rfloor^{2}},
$$

which will be used in our proof of (1.4). We give a new proof (2.1) here for two reasons. Firstly, Liu's proof of (1.4) is a little complicated and cannot be generalized to prove similar identities. Secondly, we want the paper to be more self-contained.

Pro of of (2.1). By the $q$-binomial theorem (see, for example, [1], page 36, Theorem 3.3), we have

$$
(x ; q)_{N}=\sum_{k=0}^{N}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
N \\
k
\end{array}\right] x^{k}, \quad \frac{1}{(x ; q)_{N}}=\sum_{k=0}^{\infty}\left[\begin{array}{c}
N+k-1 \\
k
\end{array}\right] x^{k},
$$

and so

$$
\left(\sum_{k=0}^{n+a}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
n+a  \tag{2.2}\\
k
\end{array}\right] x^{k}\right)\left(\sum_{k=0}^{\infty}\left[\begin{array}{c}
n+a+k \\
k
\end{array}\right] x^{k}\right)=\frac{(x ; q)_{n+a}}{(x ; q)_{n+a+1}}=\frac{1}{1-x q^{n+a}}
$$

Equating the coefficients of $x^{n-a}$ on both sides of (2.2), we obtain

$$
\sum_{k=0}^{n-a}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
n+a \\
k
\end{array}\right]\left[\begin{array}{c}
2 n-k \\
n-a-k
\end{array}\right]=q^{n^{2}-a^{2}}
$$

which can be written as

$$
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
2 n-k  \tag{2.3}\\
k
\end{array}\right]\left[\begin{array}{c}
2 n-2 k \\
n-k+a
\end{array}\right]=q^{n^{2}-a^{2}}
$$

With the help of (2.3), we are now able to prove (2.1). By Slater's Bailey pair C(1) in [13], we have

$$
(-q ; q)_{n}=\sum_{k=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{k} q^{3 k^{2}+k}\left[\begin{array}{c}
2 n \\
n+2 k
\end{array}\right] .
$$

It follows that

$$
\left.\begin{array}{rl}
\left.\sum_{k=0}^{n}(-1)^{k} q^{k} \begin{array}{c}
k \\
2
\end{array}\right)
\end{array} \begin{array}{c}
2 n-k  \tag{2.4}\\
k
\end{array}\right](-q ; q)_{n-k} .
$$

Interchanging the summation order on the right-hand side of (2.4) and using (2.3), we get

$$
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
2 n-k  \tag{2.5}\\
k
\end{array}\right](-q ; q)_{n-k}=\sum_{j=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{j} q^{n^{2}-j^{2}+j} .
$$

Since the $j$ th and $(1-j)$ th terms on the right-hand side of (2.5) cancel each other for $j=1, \ldots,\lfloor n / 2\rfloor$, only the term corresponding to $j=-\lfloor n / 2\rfloor$ on the right-hand side of (2.5) survives. This proves (2.1).

Similarly, we can show that

$$
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k+1}{2}}\left[\begin{array}{c}
2 n-k  \tag{2.6}\\
k
\end{array}\right](-q ; q)_{n-k}=\sum_{k=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{k} q^{3 k^{2}+k} .
$$

There are many more identities similar to (2.1) and (2.6). For example, using the identities (see [10], Proposition 2)

$$
\begin{aligned}
\left(-q ; q^{2}\right)_{n} & =\sum_{k=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{k} q^{2 k^{2}}\left[\begin{array}{c}
2 n \\
n+2 k
\end{array}\right] \\
\left(1+q^{n}\right)\left(-q^{2} ; q^{2}\right)_{n-1} & =\sum_{k=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{k} q^{2 k^{2}+k}\left[\begin{array}{c}
2 n \\
n+2 k
\end{array}\right]
\end{aligned}
$$

we can prove that

$$
\begin{align*}
\sum_{k=0}^{n}(-1)^{k}\left[\begin{array}{c}
2 n-k \\
k
\end{array}\right]\left(-q ; q^{2}\right)_{n-k} q^{\binom{k}{2}} & =\sum_{k=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{k} q^{n^{2}-2 k^{2}},  \tag{2.7}\\
\sum_{k=0}^{n}(-1)^{k}\left[\begin{array}{c}
2 n-k \\
k
\end{array}\right]\left(1+q^{n-k}\right)\left(-q^{2} ; q^{2}\right)_{n-k-1} q^{\binom{k}{2}} & =\sum_{k=-\lfloor n / 2\rfloor}^{\lfloor n / 2\rfloor}(-1)^{k} q^{n^{2}-2 k^{2}+k},
\end{align*}
$$

where $\left(-q^{2} ; q^{2}\right)_{-1}=1 / 2$.

## 3. Proof of Theorem 1.1

It is easy to see that

$$
\left(1-q^{n-2 j+1}\right)\left(1-q^{n+2 j-1}\right)+\left(1-q^{2 j-1}\right)^{2} q^{n-2 j+1}=\left(1-q^{n}\right)^{2}
$$

$1-q^{n} \equiv 0\left(\bmod \Phi_{n}(q)\right)$, and so

$$
\left(1-q^{n-2 j+1}\right)\left(1-q^{n+2 j-1}\right) \equiv-\left(1-q^{2 j-1}\right)^{2} q^{n-2 j+1}\left(\bmod \Phi_{n}(q)^{2}\right)
$$

Therefore,

$$
\begin{align*}
&\left(q^{1-n} ; q^{2}\right)_{k}\left(q^{n+1} ; q^{2}\right)_{k}=(-1)^{k} \prod_{j=1}^{k}\left(1-q^{n-2 j+1}\right)\left(1-q^{n+2 j-1}\right)  \tag{3.1}\\
& \equiv q^{k^{2}-n k} \prod_{j=1}^{k}\left(1-q^{2 j-1}\right)^{2} q^{n-2 j+1}=\left(q ; q^{2}\right)_{k}^{2}\left(\bmod \Phi_{n}(q)^{2}\right)
\end{align*}
$$

It follows that

$$
\begin{aligned}
\sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k} q^{2 k}}{\left(q^{4} ; q^{4}\right)_{k}\left(-q ; q^{2}\right)_{k}} & \equiv \sum_{k=0}^{(n-1) / 2} \frac{\left(q^{1-n} ; q^{2}\right)_{k}\left(q^{n+1} ; q^{2}\right)_{k}}{\left(q^{2} ; q^{4}\right)_{k}\left(q^{4} ; q^{4}\right)_{k}} q^{2 k} \\
& =\sum_{k=0}^{(n-1) / 2}(-1)^{k} q^{k(k-n+2)}\left[\begin{array}{c}
(n-1) / 2+k \\
(n-1) / 2-k
\end{array}\right]_{q^{2}} \\
& =\sum_{k=0}^{(n-1) / 2}(-1)^{(n-1) / 2-k} q^{n-\left(n^{2}+3\right) / 4+2\binom{k}{2}}\left[\begin{array}{c}
n-1-k \\
k
\end{array}\right]_{q^{2}} \\
& =(-1)^{(n-1) / 2}\left(\frac{n}{3}\right) q^{\left(n^{2}-1\right) / 12}\left(\bmod \Phi_{n}(q)^{2}\right)
\end{aligned}
$$

The last equality holds because of the identity (see [2], [3], [4])

$$
\sum_{k=0}^{\lfloor n / 2\rfloor}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{c}
n-k  \tag{3.2}\\
k
\end{array}\right]= \begin{cases}(-1)^{\lfloor n / 3\rfloor} q^{n(n-1) / 6}, & \text { if } n \not \equiv 2(\bmod 3) \\
0, & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

The proof of (1.5) then follows from the quadratic reciprocity law

$$
\left(\frac{3}{n}\right)=(-1)^{(n-1) / 2}\left(\frac{n}{3}\right)
$$

Similarly, we have

$$
\begin{aligned}
& \sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k} q^{2 k}}{\left(q^{2} ; q^{2}\right)_{k}\left(-q ; q^{2}\right)_{k}} \\
& \equiv \sum_{k=0}^{(n-1) / 2}(-1)^{(n-1) / 2-k} q^{n-\left(n^{2}+3\right) / 4+2\binom{k}{2}}\left[\begin{array}{c}
n-1-k \\
k
\end{array}\right]_{q^{2}}\left(-q^{2} ; q^{2}\right)_{(n-1) / 2-k} \\
& \quad=\left(\frac{2}{n}\right) q^{2\lfloor(n+1) / 4\rfloor^{2}}\left(\bmod \Phi_{n}(q)^{2}\right) .
\end{aligned}
$$

The last equality follows from (2.1) by replacing $n$ with $(n-1) / 2$ and $q$ with $q \rightarrow q^{2}$.

## 4. Proof of Theorem 1.2

We can easily prove the congruence

$$
\left(q^{1-n} ; q\right)_{k}\left(q^{n+1} ; q\right)_{k}=(q ; q)_{k}^{2}\left(\bmod \Phi_{n}(q)^{2}\right)
$$

similar to (3.1). It follows that

$$
\left[\begin{array}{c}
n+k \\
k
\end{array}\right]^{2}\left[\begin{array}{c}
n-1 \\
k
\end{array}\right]^{2}=\frac{\left(q^{n+1} ; q\right)_{k}^{2}\left(q^{1-n} ; q\right)_{k}^{2}}{(q ; q)_{k}^{4}} q^{2 n k-k^{2}-k} \equiv q^{2 n k-k^{2}-k}\left(\bmod \Phi_{n}(q)^{2}\right)
$$

and so

$$
\sum_{k=0}^{n-1} q^{(n-k)^{2}}\left[\begin{array}{c}
n+k \\
k
\end{array}\right]^{2}\left[\begin{array}{c}
n-1 \\
k
\end{array}\right]^{2} \equiv \sum_{k=0}^{n-1} q^{n^{2}-k}=q^{n^{2}-n+1}[n] \equiv q[n]\left(\bmod \Phi_{n}(q)^{2}\right)
$$

as desired.

## 5. Concluding remarks

Very recently, the latter author and Zudilin in [9] developed a creative microscoping method to prove $q$-supercongruences by adding a parameter $a$ (see also [5]). Along the same lines, we can generalize (1.3)-(1.5) as follows: for any positive odd integer $n$ modulo $\left(1-a q^{n}\right)\left(a-q^{n}\right)$ :

$$
\begin{align*}
& \sum_{k=0}^{(n-1) / 2} \frac{\left(a q ; q^{2}\right)_{k}\left(q / a ; q^{2}\right)_{k} q^{k^{2}}}{\left(q ; q^{2}\right)_{k}\left(q^{4} ; q^{4}\right)_{k}} \equiv(-q)^{\left(1-n^{2}\right) / 8},  \tag{5.1}\\
& \sum_{k=0}^{(n-1) / 2} \frac{\left(a q ; q^{2}\right)_{k}\left(q / a ; q^{2}\right)_{k} q^{2 k}}{\left(q^{2} ; q^{2}\right)_{k}\left(q^{2} ; q^{4}\right)_{k}} \equiv\left(\frac{2}{n}\right) q^{2\lfloor(n+1) / 4\rfloor^{2}},  \tag{5.2}\\
& \sum_{k=0}^{(n-1) / 2} \frac{\left(a q ; q^{2}\right)_{k}\left(q / a ; q^{2}\right)_{k} q^{2 k}}{\left(q^{4} ; q^{4}\right)_{k}\left(q^{2} ; q^{4}\right)_{k}} \equiv\left(\frac{3}{n}\right) q^{\left(n^{2}-1\right) / 12} . \tag{5.3}
\end{align*}
$$

It is easy to see that, letting $a \rightarrow 1$ in (5.1)-(5.3), we recover (1.3)-(1.5), respectively.
Moreover, there are other different $q$-analogues of (1.1) and (1.2). For example, applying the identities (2.6)-(2.8) (replacing $n$ with $(n-1) / 2$ and $q$ with $q^{2}$ ), we have the following more complicated $q$-analogues of (1.1) modulo $\Phi_{n}(q)^{2}$ :

$$
\begin{gathered}
\sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k}}{\left(q^{2} ; q^{2}\right)_{k}\left(-q ; q^{2}\right)_{k}} \equiv(-1)^{(n-1) / 2} q^{\left(1-n^{2}\right) / 4} \sum_{k=-\lfloor(n-1) / 4\rfloor}^{\lfloor(n-1) / 4\rfloor}(-1)^{k} q^{6 k^{2}+2 k}, \\
\sum_{k=0}^{(n-1) / 2} \frac{\left(-q^{2} ; q^{4}\right)_{k}\left(q ; q^{2}\right)_{k} q^{2 k}}{\left(q^{4} ; q^{4}\right)_{k}\left(-q ; q^{2}\right)_{k}} \equiv(-1)^{(n-1) / 2} q^{\left(n^{2}-1\right) / 4} \sum_{k=-\lfloor(n-1) / 4\rfloor}^{\lfloor(n-1) / 4\rfloor}(-1)^{k} q^{-4 k^{2}}, \\
\sum_{k=0}^{(n-1) / 2} \frac{\left(1+q^{2 k}\right)\left(-q^{4} ; q^{4}\right)_{k-1}\left(q ; q^{2}\right)_{k} q^{2 k}}{\left(q^{4} ; q^{4}\right)_{k}\left(-q ; q^{2}\right)_{k}} \\
\equiv(-1)^{(n-1) / 2} q^{\left(n^{2}-1\right) / 4} \sum_{k=-\lfloor(n-1) / 4\rfloor}^{\lfloor(n-1) / 4\rfloor}(-1)^{k} q^{2 k-4 k^{2}} .
\end{gathered}
$$

Similarly, applying the invariant of (3.2) (see, for example, [11], (1.5))

$$
\sum_{k=0}^{\lfloor n / 2\rfloor}(-1)^{k} q^{\binom{k+1}{2}}\left[\begin{array}{c}
n-k \\
k
\end{array}\right]=\sum_{k=-\lfloor(n+1) / 3\rfloor}^{\lfloor n / 3\rfloor}(-1)^{k} q^{k(3 k+1) / 2},
$$

which follows readily from Franklin's involution on partitions into distinct parts (see the proof of [1], Theorem 1.6), we have the following $q$-analogue of (1.2): modulo $\Phi_{n}(q)^{2}$,

$$
\sum_{k=0}^{(n-1) / 2} \frac{\left(q ; q^{2}\right)_{k}}{\left(q^{4} ; q^{4}\right)_{k}\left(-q ; q^{2}\right)_{k}} \equiv(-1)^{(n-1) / 2} q^{\left(1-n^{2}\right) / 4} \sum_{k=-\lfloor n / 3\rfloor}^{\lfloor(n-1) / 3\rfloor}(-1)^{k} q^{3 k^{2}+k}
$$

There are also parameter generalizations of the above four congruences, which are omitted here.

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