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STRONGLY $(\mathcal{T}, n)$-COHERENT RINGS, $(\mathcal{T}, n)$-SEMIHEREDITARY RINGS AND $(\mathcal{T}, n)$-REGULAR RINGS

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Abstract. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. A left R-module M is called $(\mathcal{T}, n)$-injective if $\mathrm{Ext}_R^n(C, M) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module C; a right R-module M is called $(\mathcal{T}, n)$-flat if $\mathrm{Tor}_n^R(M, C) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module C; a left R-module M is called $(\mathcal{T}, n)$-projective if $\mathrm{Ext}_R^n(M, N) = 0$ for each $(\mathcal{T}, n)$-injective left R-module N; the ring R is called strongly $(\mathcal{T}, n)$-coherent if whenever $0 \to K \to P \to C \to 0$ is exact, where C is $(\mathcal{T}, n+1)$-presented and P is finitely generated projective, then K is $(\mathcal{T}, n)$-projective; the ring R is called $(\mathcal{T}, n)$-semihereditary if whenever $0 \to K \to P \to C \to 0$ is exact, where C is $(\mathcal{T}, n+1)$-presented and P is finitely generated projective, then $\mathrm{pd}(K) \leqslant n-1$. Using the concepts of $(\mathcal{T}, n)$-injectivity and $(\mathcal{T}, n)$-flatness of modules, we present some characterizations of strongly $(\mathcal{T}, n)$-coherent rings, $(\mathcal{T}, n)$-semihereditary rings and $(\mathcal{T}, n)$-regular rings.

Keywords: $(\mathcal{T}, n)$-injective module; $(\mathcal{T}, n)$-flat module; strongly $(\mathcal{T}, n)$-coherent ring; $(\mathcal{T}, n)$-semihereditary ring; $(\mathcal{T}, n)$-regular ring

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules considered are unitary, n is a positive integer. The symbol R-Mod denotes the class of all left R-modules. For any R-module M, $M^+ = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M. Given a class $\mathcal{L}$ of R-modules, we will denote by $\mathcal{L}^\perp = \{M\colon \mathrm{Ext}_R^1(L, M) = 0,\ L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^\perp\mathcal{L} = \{M\colon \mathrm{Ext}_R^1(M, L) = 0,\ L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$.

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Recall that a left *R-module M is FP-injective* (see [7], [11]) or *absolutely pure* (see [10]) if $\mathrm{Ext}_R^1(A, M) = 0$ for every finitely presented left R-module A; a right *R-module M is flat* if $\mathrm{Tor}_1^R(M, A) = 0$ for every finitely presented left R-module A; a *ring R is left coherent* (see [1]) if every finitely generated left ideal of R is finitely presented, or equivalently, if every finitely generated submodule of a projective left R-module is finitely presented, if every finitely presented left R-module is 2-presented; a ring R is *left semihereditary* if every finitely generated left ideal of R is projective, or equivalently, if every finitely generated submodule of a projective left R-module is projective. FP-injective modules, flat modules, coherent rings, semihereditary rings and their generalizations have been studied extensively by many authors. For example, in 1994, Costa introduced the concept of *left n-coherent* rings in [4]. Following [4], a *ring R* is called *left n-coherent* if every n-presented left R-module is $(n+1)$-presented, where a *left R-module A* is called *n-presented* if there exists an exact sequence of left R-modules $F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ in which every F_i is finitely generated free.

In 1996, Chen and Ding introduced the concepts of n-FP-injective modules and n-flat modules in [3]. Following [3], a *left R-module M* is called *n-FP-injective* if $\mathrm{Ext}_R^n(A, M) = 0$ for every n-presented left R-module A, a *right R-module M* is called *n-flat* if $\mathrm{Tor}_n^R(M, A) = 0$ for every n-presented left R-module A. Using the two concepts, they characterized *n-coherent rings.* In 2015, we introduced the concepts of *weakly n-FP-injective* modules and *weakly n-flat* modules in [15]. Following [15], a left R-module M is called weakly n-FP-injective if $\mathrm{Ext}_R^n(A, M) = 0$ for every $(n+1)$-presented left R-module A, a right R-module M is called weakly n-flat if $\mathrm{Tor}_n^R(M, A) = 0$ for every $(n+1)$-presented left R-module A. Using the two concepts, we characterized n-coherent rings in [15], Theorem 2.19. We shall denote by $(\mathcal{FP})_n\mathcal{I}$ (or $\mathcal{W}(\mathcal{FP})_n\mathcal{I}$) the class of all n-FP-injective (or weakly n-FP-injective) left R-modules, and denote by $\mathcal{F}_n$ (or $\mathcal{WF}_n$) the class of all n-flat (or weakly n-flat) right R-modules.

We recall: A subclass $\mathcal{T}$ of left R-modules is called a *weak torsion class* (see [16]) if it is closed under homomorphic images and extensions. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then a left R-module M is called *$\mathcal{T}$-finitely generated* if there exists a finitely generated submodule N such that $M/N \in \mathcal{T}$; a left R-module A is called *$(\mathcal{T}, n)$-presented* if there exists an exact sequence of left R-modules $0 \to K_{n-1} \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ such that $F_0, \ldots, F_{n-1}$ are finitely generated free and K_{n-1} is $\mathcal{T}$-finitely generated. In [16], we extended the concepts of n-FP-injective modules and weakly n-FP-injective modules to $(\mathcal{T}, n)$-injective modules. According to [16] a left R-module M is called *$(\mathcal{T}, n)$-injective* if $\mathrm{Ext}_R^n(C, M) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module C and we extended the concepts of n-flat modules and weakly n-flat modules to

$(\mathcal{T}, n)$-flat modules. According to [16], a right R-module M is called $(\mathcal{T}, n)$-*flat* if $\operatorname{Tor}_n^R(M, C) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module C; and we extended the concepts of n-coherent rings to $(\mathcal{T}, n)$-coherent rings. According to [16], a ring R is called $(\mathcal{T}, n)$-*coherent* if every $(\mathcal{T}, n+1)$-presented module is $(n+1)$-presented. By using the concepts of $(\mathcal{T}, n)$-injective modules and $(\mathcal{T}, n)$-flat modules, we characterized $(\mathcal{T}, n)$-coherent rings.

In this paper, we shall introduce the concepts of strongly $(\mathcal{T}, n)$-coherent rings, $(\mathcal{T}, n)$-semihereditary rings and $(\mathcal{T}, n)$-regular rings. Using the concepts of $(\mathcal{T}, n)$-injectivity and $(\mathcal{T}, n)$-flatness of modules, we shall give a series of characterizations and properties of strongly $(\mathcal{T}, n)$-coherent rings, $(\mathcal{T}, n)$-semihereditary rings and $(\mathcal{T}, n)$-regular rings.

2. Strongly $(\mathcal{T}, n)$-coherent rings

Definition 2.1. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. A left R-module M is called $(\mathcal{T}, n)$-*projective* if $\operatorname{Ext}_R^n(M, N) = 0$ for each $(\mathcal{T}, n)$-injective left R-module N.

We shall denote by $\mathcal{T}_n\mathcal{I}$ (or $\mathcal{T}_n\mathcal{P}$) the class of all $(\mathcal{T}, n)$-injective (or $(\mathcal{T}, n)$-projective) left R-modules, and by $\mathcal{T}_n\mathcal{F}$ the class of all $(\mathcal{T}, n)$-flat right R-modules.

Definition 2.2. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then ring R is called *strongly* $(\mathcal{T}, n)$-*coherent* if whenever $0 \to K \to P \to C \to 0$ is exact, where C is $(\mathcal{T}, n+1)$-presented and P is finitely generated projective, then K is $(\mathcal{T}, n)$-projective.

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [5], we say that a homomorphism $\varphi\colon M \to F$, where $F \in \mathcal{F}$, is an $\mathcal{F}$-*preenvelope of* M if for any morphism $f\colon M \to F'$ with $F' \in \mathcal{F}$ there is a $g\colon F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi\colon M \to F$ is said to be an $\mathcal{F}$-*envelope* if every endomorphism $g\colon F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of an $\mathcal{F}$-*precover* and an $\mathcal{F}$-*cover*. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is called a *cotorsion theory* (see [5]) if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called *perfect* (see [6]) if every R-module has a $\mathcal{B}$-envelope and an $\mathcal{A}$-cover. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called *complete* (see [5], Definition 7.1.6 and [12], Lemma 1.13) if for any R-module M there are exact sequences $0 \to M \to B \to A \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $0 \to B' \to A' \to M \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called *hereditary* (see [6], Definition 1.1) if whenever $0 \to A' \to A \to A'' \to 0$ is exact with

$A, A'' \in \mathcal{A}$, then A' is also in $\mathcal{A}$. By [6], Proposition 1.2, a cotorsion theory $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if whenever $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathcal{B}$, then B'' is also in $\mathcal{B}$.

Theorem 2.3. *The following statements are equivalent for the ring R:*

(1) *R is strongly $(\mathcal{T}, n)$-coherent.*

(2) *$({}^{\perp}(\mathcal{T}_n\mathcal{I}), \mathcal{T}_n\mathcal{I})$ is a hereditary cotorsion theory.*

(3) *R is $(\mathcal{T}, n)$-coherent and $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^{\perp})$ is a hereditary cotorsion theory.*

(4) *$\operatorname{Ext}^i_R(C, M) = 0$ for any $i \geqslant n$, any $(\mathcal{T}, n+1)$-presented module C and any $(\mathcal{T}, n)$-injective left R-module M.*

(5) *$\operatorname{Ext}^{n+1}_R(C, M) = 0$ for any $(\mathcal{T}, n+1)$-presented module C and any $(\mathcal{T}, n)$-injective left R-module M.*

(6) *R is $(\mathcal{T}, n)$-coherent and $\operatorname{Tor}^R_i(N, C) = 0$ for any $i \geqslant n$, any $(\mathcal{T}, n+1)$-presented module C and any $(\mathcal{T}, n)$-flat right R-module N.*

(7) *R is $(\mathcal{T}, n)$-coherent and $\operatorname{Tor}^R_{n+1}(N, C) = 0$ for any $(\mathcal{T}, n+1)$-presented module C and any $(\mathcal{T}, n)$-flat right R-module N.*

(8) *If N is a $(\mathcal{T}, n)$-injective left R-module and N_1 is a $(\mathcal{T}, n)$-injective submodule of N, then N/N_1 is $(\mathcal{T}, n)$-injective.*

(9) *For any $(\mathcal{T}, n)$-injective left R-module N, $E(N)/N$ is $(\mathcal{T}, n)$-injective.*

Proof. (2) $\Rightarrow$ (3). If M is a $(\mathcal{T}, n)$-injective left R-module, M_1 is an FP-injective submodule of M, then M_1 is $(\mathcal{T}, n)$-injective, and so M/M_1 is $(\mathcal{T}, n)$-injective by [6], Proposition 1.2 since $({}^{\perp}(\mathcal{T}_n\mathcal{I}), \mathcal{T}_n\mathcal{I})$ is a hereditary cotorsion theory. Thus, R is $(\mathcal{T}, n)$-coherent by [16], Theorem 5.6. Moreover, by [16], Theorem 4.11, statement (2), $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^{\perp})$ is a cotorsion theory. Now let $0 \to A' \to A \to A'' \to 0$ be an exact sequence of right R-modules with $A, A'' \in \mathcal{T}_n\mathcal{F}$. Then we get an exact sequence of left R-modules $0 \to (A'')^+ \to A^+ \to (A')^+ \to 0$. Since A^+ and $(A'')^+$ are $(\mathcal{T}, n)$-injective by [16], Theorem 4.8, $(A')^+$ is also $(\mathcal{T}, n)$-injective by (2), and hence A' is $(\mathcal{T}, n)$-flat. Therefore $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^{\perp})$ is a hereditary cotorsion theory.

(3) $\Rightarrow$ (2). Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence of left R-modules with A, A' $(\mathcal{T}, n)$-injective. Then we get an exact sequence of right R-modules $0 \to (A'')^+ \to A^+ \to (A')^+ \to 0$. Since R is $(\mathcal{T}, n)$-coherent, A^+ and $(A')^+$ are $(\mathcal{T}, n)$-flat by [16], Theorem 5.3, statement (8), and hence $(A'')^+$ is also $(\mathcal{T}, n)$-flat as $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^{\perp})$ is hereditary. And so, A'' is $(\mathcal{T}, n)$-injective by [16], Theorem 5.3, statement (8) again, and (2) follows.

(2) $\Rightarrow$ (4). Let C be a $(\mathcal{T}, n+1)$-presented left R-module with a finite n-presentation $F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} C \longrightarrow 0$. Write $K_{n-2} = \operatorname{Ker}(d_{n-2})$. Then $K_{n-2} \in^{\perp} (\mathcal{T}_n\mathcal{I})$, and so, for any $i \geqslant n$ and any

$(\mathcal{T},n)$-injective left R-module M, we have $\mathrm{Ext}_R^i(C,M)\cong \mathrm{Ext}_R^{i-n+1}(K_{n-2},M)=0$ by [6], Proposition 1.2.

(4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) are obvious.

(5) $\Rightarrow$ (2). Let $0\to A'\to A\to A''\to 0$ be an exact sequence of left R-modules with A,A' $(\mathcal{T},n)$-injective. For any $(\mathcal{T},n+1)$-presented left R-module C we have an exact sequence

$$0=\mathrm{Ext}_R^n(C,A)\to \mathrm{Ext}_R^n(C,A'')\to \mathrm{Ext}_R^{n+1}(C,A')=0.$$

So $\mathrm{Ext}_R^n(C,A'')=0$, and thus A'' is $(\mathcal{T},n)$-injective.

(3), (4) $\Rightarrow$ (6). By (3), R is $(\mathcal{T},n)$-coherent. Let N be a $(\mathcal{T},n)$-flat right R-module. Then N^+ is $(\mathcal{T},n)$-injective. By (4), $\mathrm{Ext}_R^i(C,N^+)=0$ for any $i\geqslant n$ and any $(\mathcal{T},n+1)$-presented left R-module C, and so, by the isomorphism $\mathrm{Tor}_i^R(N,C)^+\cong \mathrm{Ext}_R^i(C,N^+)$ we have that $\mathrm{Tor}_i^R(N,C)=0$ for any $i\geqslant n$ and any $(\mathcal{T},n+1)$-presented left R-module C.

(7) $\Rightarrow$ (3). Assume (7). Then it is clear that R is $(\mathcal{T},n)$-coherent. Now let $0\to A'\to A\to A''\to 0$ be an exact sequence of right R-modules with $A,A''\in \mathcal{T}_n\mathcal{F}$. Then for any $(\mathcal{T},n+1)$-presented left R-module C we get an exact sequence $0=\mathrm{Tor}_{n+1}^R(A'',C)\to \mathrm{Tor}_n^R(A',C)\to \mathrm{Tor}_n^R(A,C)=0$, which shows that $\mathrm{Tor}_n^R(A',C)=0$. So, A' is also $(\mathcal{T},n)$-flat, and therefore $(\mathcal{T}_n\mathcal{F},(\mathcal{T}_n\mathcal{F})^{\perp})$ is a hereditary cotorsion theory.

(1) $\Rightarrow$ (5). Let C be a $(\mathcal{T},n+1)$-presented left R-module and M be a $(\mathcal{T},n)$-injective left R-module. Then there exists an exact sequence $0\to K\to P\to C\to 0$ with P finitely generated projective. By (1), $\mathrm{Ext}_R^n(K,M)=0$. And then from the exact sequence of

$$0=\mathrm{Ext}_R^n(K,M)\to \mathrm{Ext}_R^{n+1}(C,M)\to \mathrm{Ext}_R^{n+1}(P,M)=0$$

we have $\mathrm{Ext}_R^{n+1}(C,M)=0$.

(5) $\Rightarrow$ (8). For any $(\mathcal{T},n+1)$-presented left R-module C, the exact sequence $0\to N_1\to N\to N/N_1\to 0$ induces the exactness of the sequence

$$0=\mathrm{Ext}_R^n(C,N)\to \mathrm{Ext}_R^n(C,N/N_1)\to \mathrm{Ext}_R^{n+1}(C,N_1)=0.$$

This yields that $\mathrm{Ext}_R^n(C,N/N_1)=0$, as desired.

(8) $\Rightarrow$ (9) is obvious.

(9) $\Rightarrow$ (1). Let C be a $(\mathcal{T},n+1)$-presented left R-module. If $0\to K\to P\to C\to 0$ is an exact sequence of left R-modules, where P is finitely generated projective, then for any $(\mathcal{T},n)$-injective module N, $E(N)/N$ is $(\mathcal{T},n)$-injective by (9). From the

exactness of the two sequences

$$0 = \mathrm{Ext}^n_R(P, N) \to \mathrm{Ext}^n_R(K, N) \to \mathrm{Ext}^{n+1}_R(C, N) \to \mathrm{Ext}^{n+1}_R(P, N) = 0$$
$$0 = \mathrm{Ext}^n_R(C, E(N)) \to \mathrm{Ext}^n_R(C, E(N)/N) \to \mathrm{Ext}^{n+1}_R(C, N) \to \mathrm{Ext}^{n+1}_R(C, E(N)) = 0$$

we have $\mathrm{Ext}^n_R(K, N) \cong \mathrm{Ext}^{n+1}_R(C, N) \cong \mathrm{Ext}^n_R(C, E(N)/N) = 0$. Thus, K is $(\mathcal{T}, n)$-projective, as required. □

Corollary 2.4. *Let $\mathcal{T} = R$-Mod. Then the following statements are equivalent for the ring R:*
(1) *R is strongly $(\mathcal{T}, n)$-coherent.*
(2) *R is $(\mathcal{T}, n)$-coherent.*
(3) *R is left n-coherent.*

Proof. (1) ⇒ (2). It follows from Theorem 2.3, statement (3).
(2) ⇒ (3). It follows from [16], Example 5.2, statement (1).
(3) ⇒ (1). Let $0 \to K \to P \to C \to 0$ be exact, where C is $(\mathcal{T}, n+1)$-presented and P is finitely generated projective. Then by (3), K is n-presented, so $\mathrm{Ext}^n_R(K, N) = 0$ for any n-FP-injective left R-modules. This yields that R is strongly $(\mathcal{T}, n)$-coherent. □

Corollary 2.5. *The following statements are equivalent for the ring R:*
(1) *R is left n-coherent.*
(2) *$({}^{\perp}((\mathcal{FP})_n\mathcal{I}), (\mathcal{FP})_n\mathcal{I})$ is a hereditary cotorsion theory.*
(3) *$\mathrm{Ext}^i_R(C, M) = 0$ for any $i \geqslant n$, any n-presented module C and any n-FP-injective left R-module M.*
(4) *$\mathrm{Ext}^{n+1}_R(C, M) = 0$ for any n-presented module C and any n-FP-injective left R-module M.*
(5) *If N is an n-FP-injective left R-module and N_1 is an n-FP-injective submodule of N, then N/N_1 is n-FP-injective.*
(6) *For any n-FP-injective left R-module N, $E(N)/N$ is n-FP-injective.*

Corollary 2.6. *Let $\mathcal{T} = \{0\}$. Then R is strongly $(\mathcal{T}, n)$-coherent if and only if every weakly n-FP-injective left R-module is $(n+1)$-FP-injective.*

Proof. It follows from Theorem 2.3 (5) and [16], Example 4.2, (2). □

Corollary 2.7. *The following statements are equivalent for the ring R:*
(1) *$({}^{\perp}(\mathcal{W}(\mathcal{FP})_n\mathcal{I}), \mathcal{W}(\mathcal{FP})_n\mathcal{I})$ is a hereditary cotorsion theory.*
(2) *$(\mathcal{WF}_n, (\mathcal{WF}_n)^{\perp})$ is a hereditary cotorsion theory.*

(3) $\mathrm{Ext}_R^i(C,M)=0$ *for any* $i \geqslant n$, *any* $(n+1)$*-presented module* C *and any weakly* n*-FP-injective left* R*-module* M.
(4) $\mathrm{Ext}_R^{n+1}(C,M) = 0$ *for any* $(n+1)$*-presented module* C *and any weakly* n*-FP-injective left* R*-module* M.
(5) $\mathrm{Tor}_i^R(N,C)=0$ *for any* $i \geqslant n$, *any* $(n+1)$*-presented module* C *and any weakly* n*-flat right* R*-module* N.
(6) $\mathrm{Tor}_{n+1}^R(N,C) = 0$ *for any* $(n+1)$*-presented module* C *and any weakly* n*-flat right* R*-module* N.
(7) *If* N *is a weakly* n*-FP-injective left* R*-module and* N_1 *is a weakly* n*-FP-injective submodule of* N, *then* N/N_1 *is weakly* n*-FP-injective.*
(8) *For any weakly* n*-FP-injective left* R*-module* N *and* $E(N)/N$ *is weakly* n*-FP-injective.*

Let $\mathcal{F}$ be a class of left R-modules. As usual, we write ${}^{\perp_n}\mathcal{F}=\{M\colon \mathrm{Ext}_R^n(M,F)=0, F\in\mathcal{F}\}$, and $\mathcal{F}^{\perp_n}=\{M\colon \mathrm{Ext}_R^n(F,M)=0,\ F\in\mathcal{F}\}$.

Definition 2.8. Let n be a positive integer. A pair $(\mathcal{L},\mathcal{C})$ of classes of R-modules is called an *n-cotorsion theory* if $\mathcal{L}^{\perp_n}=\mathcal{C}$ and ${}^{\perp_n}\mathcal{C}=\mathcal{L}$. An n-cotorsion theory $(\mathcal{L},\mathcal{C})$ is called *hereditary* if whenever $0\to L'\to L\to L''\to 0$ is exact with $L,L''\in\mathcal{L}$, then L' is also in $\mathcal{L}$.

It is easy to see that the pair $(\mathcal{T}_n\mathcal{P},\mathcal{T}_n\mathcal{I})$ is an n-cotorsion theory.

Theorem 2.9. *Let* $(\mathcal{L},\mathcal{C})$ *be an* n*-cotorsion theory. Then the following statements are equivalent:*
(1) $(\mathcal{L},\mathcal{C})$ *is hereditary.*
(2) *If* $0\to L'\to P\to L''\to 0$ *is exact with* P *projective and* $L''\in\mathcal{L}$, *then* L' *is also in* $\mathcal{L}$.
(3) $\mathrm{Ext}_R^{n+i}(L,C)=0$ *for any non-negative integer* i *and any* $L\in\mathcal{L}$ *and* $C\in\mathcal{C}$.
(4) $\mathrm{Ext}_R^{n+1}(L,C)=0$ *for any* $L\in\mathcal{L}$ *and* $C\in\mathcal{C}$.
(5) *If* $0\to C'\to C\to C''\to 0$ *is exact with* $C',C\in\mathcal{C}$, *then* C'' *is also in* $\mathcal{C}$.
(6) *If* $0\to C'\to E\to C''\to 0$ *is exact with* $C'\in\mathcal{C}$ *and* E *injective, then* C'' *is also in* $\mathcal{C}$.
(7) *If* $C\in\mathcal{C}$, *then* $E(C)/C\in\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(2) $\Rightarrow$ (3). We only need to prove the case, where $i\geqslant 1$. Let $L_0=L$. Then by (2) we have exact sequences $0\to L_k\to P_k\to L_{k-1}\to 0$, $k=1,2,\ldots,i$, where each $L_k\in\mathcal{L}$ and P_k is projective. So we have that $\mathrm{Ext}_R^{n+i}(L,C)\cong \mathrm{Ext}_R^{n+i-1}(L_1,C)\cong\ldots\cong\mathrm{Ext}_R^n(L_i,C)=0$.

(4) $\Rightarrow$ (1). Let $0 \to L' \to L \to L'' \to 0$ be exact with $L, L'' \in \mathcal{L}$. Then for any $C \in \mathcal{C}$, by (4) we have an exact sequence $0 = \operatorname{Ext}_R^n(L, C) \to \operatorname{Ext}_R^n(L', C) \to \operatorname{Ext}_R^{n+1}(L'', C) = 0$, so $\operatorname{Ext}_R^n(L', C) = 0$, and thus $L' \in \mathcal{L}$.

(4) $\Rightarrow$ (5). Let $L \in \mathcal{L}$. Then by (4) we have an exact sequence $0 = \operatorname{Ext}_R^n(L, C) \to \operatorname{Ext}_R^n(L, C'') \to \operatorname{Ext}_R^{n+1}(L, C') = 0$, so $\operatorname{Ext}_R^n(L, C'') = 0$, and hence $C'' \in \mathcal{C}$.

(7) $\Rightarrow$ (4). Let $L \in \mathcal{L}$ and $C \in \mathcal{C}$. Then by (7), $E(C)/C \in \mathcal{C}$, and so

$$\operatorname{Ext}_R^n(L, E(C)/C) = 0.$$

Thus, by the exactness of

$$0 = \operatorname{Ext}_R^n(L, E(C)/C) \to \operatorname{Ext}_R^{n+1}(L, C) \to \operatorname{Ext}_R^{n+1}(L, E(C) = 0,$$

we get that $\operatorname{Ext}_R^{n+1}(L, C) = 0$. $\square$

By Theorems 2.3 and 2.9, we have the following result.

Corollary 2.10. *Let R be a strongly $(\mathcal{T}, n)$-coherent if and only if $(\mathcal{T}_n\mathcal{P}, \mathcal{T}_n\mathcal{I})$ is a hereditary n-cotorsion theory.*

Definition 2.11.

(1) The *$(\mathcal{T}, n)$-injective dimension of a module* $_RM$ is defined by

$$\mathcal{T}_n\mathcal{I}{-}\dim(_RM) = \inf\{k\colon \operatorname{Ext}_R^{n+k}(C, M) = 0 \text{ for every } (\mathcal{T}, n{+}1)\text{-presented module } C\}.$$

(2) The *$(\mathcal{T}, n)$-injective global dimension of a ring* R is defined by

$$\mathcal{T}_n\mathcal{I} - \mathrm{GLD}(R) = \sup\{\mathcal{T}_n\mathcal{I} - \dim(M)\colon M \text{ is a left } R\text{-module}\}.$$

Theorem 2.12. *Let R be a strongly $(\mathcal{T}, n)$-coherent ring, M a left R-module and k a non-negative integer. Then the following statements are equivalent:*

(1) $\mathcal{T}_n\mathcal{I} - \dim(_RM) \leqslant k$.

(2) $\operatorname{Ext}_R^{n+k+l}(C, M) = 0$ *for any $(\mathcal{T}, n+1)$-presented module C and any non-negative integer l.*

(3) $\operatorname{Ext}_R^{n+k}(C, M) = 0$ *for any $(\mathcal{T}, n+1)$-presented module C.*

(4) *If the sequence* $0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} \ldots \longrightarrow E_{k-1} \xrightarrow{d_{k-1}} E_k \longrightarrow 0$ *is exact with $E_0, \ldots, E_{k-1}$ $(\mathcal{T}, n)$-injective, then E_k is also $(\mathcal{T}, n)$-injective.*

(5) *There exists an exact sequence of left R-modules* $0 \to M \to E_0 \to \ldots \to E_{k-1} \to E_k \to 0$ *such that $E_0, \ldots, E_{k-1}, E_k$ are $(\mathcal{T}, n)$-injective.*

P r o o f. (1) $\Rightarrow$ (2). Use induction on k. If $k = 0$, then (2) holds by Theorem 2.3, statement (4). So let $k > 0$. Assume that $\operatorname{Ext}_R^{n+k-1+l}(C,N) = 0$ for any $(\mathcal{T}, n+1)$-presented module C, any non-negative integer l and any left R-module N with $\mathcal{T}_n\mathcal{I} - \dim(N) \leqslant k-1$. Then there exists a positive integer $r \leqslant k$ such that $\operatorname{Ext}_R^{n+r}(C,M) = 0$ for any $(\mathcal{T}, n+1)$-presented module C, which implies that $\operatorname{Ext}_R^{n+r-1}(C, E(M)/M) = 0$ for any $(\mathcal{T}, n+1)$-presented module C. So $\mathcal{T}_n\mathcal{I} - \dim(E(M)/M) \leqslant r-1$, and hence $\mathcal{T}_n\mathcal{I} - \dim(E(M)/M) \leqslant k-1$. By hypothesis, we have $\operatorname{Ext}_R^{n+k-1+l}(C, E(M)/M) = 0$ for any $(\mathcal{T}, n+1)$-presented module C and any non-negative integer l, it yields that $\operatorname{Ext}_R^{n+k+l}(C,M) = 0$. Therefore statement (2) holds by induction axioms.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since R is strongly $(\mathcal{T}, n)$-coherent and $E_0, \ldots, E_{k-1}$ is $(\mathcal{T}, n)$-injective, by Theorem 2.3, statement (4) we have $\operatorname{Ext}_R^{n+k}(C,M) \cong \operatorname{Ext}_R^{n+k-1}(C, \operatorname{im}(d_0)) \cong \operatorname{Ext}_R^{n+k-2}(C, \operatorname{im}(d_1)) \cong \ldots \cong \operatorname{Ext}_R^{n}(C, \operatorname{im}(d_{k-1})) = \operatorname{Ext}_R^{n}(C, E_k)$ for any $(\mathcal{T}, n+1)$-presented module C. So statement (4) follows from statement (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\operatorname{Ext}_R^{n+k}(C,M) \cong \operatorname{Ext}_R^{n}(C, E_k)$. $\square$

Definition 2.13.

(1) The $(\mathcal{T}, n)$-flat dimension of a module M_R is defined by

$$\mathcal{T}_n\mathcal{F} - \dim(M_R) = \inf\{k\colon \operatorname{Tor}_{n+k}^R(M,C) = 0 \text{ for every } (\mathcal{T}, n+1)\text{-presented module } C\}.$$

(2) The $(\mathcal{T}, n)$-weak global dimension of a ring R is defined by

$$\mathcal{T}_n - \mathrm{WD}(R) = \sup\{\mathcal{T}_n\mathcal{F} - \dim(M)\colon M \text{ is a right } R\text{-module}\}.$$

Theorem 2.14. *Let M be a right R-module. Then*

$$\mathcal{T}_n\mathcal{F} - \dim(M) = \mathcal{T}_n\mathcal{I} - \dim(M^+).$$

P r o o f. By the isomorphism $\operatorname{Tor}_{n+k}^R(M,C)^+ \cong \operatorname{Ext}_R^{n+k}(C, M^+)$. $\square$

Theorem 2.15. *Let R be a strongly $(\mathcal{T}, n)$-coherent ring, M a right R-module and k a non-negative integer. Then the following statements are equivalent:*

(1) $\mathcal{T}_n\mathcal{F} - \dim(M_R) \leqslant k$.

(2) $\operatorname{Tor}_{n+k+l}^R(M,C) = 0$ *for any $(\mathcal{T}, n+1)$-presented module C and any non-negative integer l.*

(3) $\operatorname{Tor}_{n+k}^R(M,C) = 0$ *for any $(\mathcal{T}, n+1)$-presented module C.*

(4) *If the sequence $0 \longrightarrow F_k \xrightarrow{\varepsilon} F_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ is exact with $F_0, \ldots, F_{k-1}$ $(\mathcal{T}, n)$-flat, then F_k is also $(\mathcal{T}, n)$-flat.*

(5) *There exists an exact sequence of right R-modules* $0 \longrightarrow F_k \xrightarrow{\varepsilon} F_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ *such that* $F_0, \dots, F_{k-1}, F_k$ *are* $(\mathcal{T}, n)$*-flat.*

Proof. (1) $\Rightarrow$ (2). Let C be a $(\mathcal{T}, n+1)$-presented module and l be any non-negative integer. By (1), there exists a non-negative integer $r \leqslant k$ such that $\mathrm{Tor}^R_{n+r}(M, C) = 0$. And so, by the isomorphism $\mathrm{Tor}^R_{n+r}(M, C)^+ \cong \mathrm{Ext}^{n+r}_R(C, M^+)$, we have $\mathrm{Ext}^{n+r}_R(C, M^+) = 0$. Since R is strongly $(\mathcal{T}, n)$-coherent, by Theorem 2.12 we have $\mathrm{Ext}^{n+k+l}_R(C, M^+) = 0$, and then $\mathrm{Tor}^R_{n+k+l}(M, C) = 0$ by the isomorphism $\mathrm{Tor}^R_{n+k+l}(M, C)^+ \cong \mathrm{Ext}^{n+k+l}_R(C, M^+)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Since R is strongly $(\mathcal{T}, n)$-coherent and $F_0, \dots, F_{k-1}$ is $(\mathcal{T}, n)$-flat, by Theorem 2.3, statement (6) we have $\mathrm{Tor}^R_{n+k}(M, C) \cong \mathrm{Tor}^R_{n+k-1}(\mathrm{Ker}(d_0), C) \cong \mathrm{Tor}^R_{n+k-2}(\mathrm{Ker}(d_1), C) \cong \dots \cong \mathrm{Tor}^R_n(\mathrm{Ker}(d_{k-1}), C) = \mathrm{Tor}^R_n(F_k, C)$. So statement (4) follows from statement (3).

(5) $\Rightarrow$ (3). It follows from the above isomorphism $\mathrm{Tor}^R_{n+k}(M, C) \cong \mathrm{Tor}^R_n(F_k, C)$. $\square$

Lemma 2.16. *Let R be a strongly $(\mathcal{T}, n)$-coherent ring. Then every $(\mathcal{T}, n+1)$-presented module C is m-presented for any positive integer m.*

Proof. If $m < n$, then it is clear that the result holds. Assume that every $(\mathcal{T}, n+1)$-presented module is m-presented for some $m \geqslant n$. Then for any $(\mathcal{T}, n+1)$-presented module C and any FP-injective module N we have $\mathrm{Ext}^{m+1}_R(C, N) = 0$ by Theorem 2.3, statement (4) because R is strongly $(\mathcal{T}, n)$-coherent. Let $0 \to K_{m-n-1} \to F_{m-n-1} \to \dots \to F_1 \to F_0 \to C \to 0$ be an exact sequence of left R-modules with $F_0, \dots, F_{m-n-1}$ finitely generated free left R-modules and K_{m-n-1} n-presented. Then $\mathrm{Ext}^{n+1}_R(K_{m-n-1}, N) \cong \mathrm{Ext}^{m+1}_R(C, N) = 0$, so K_{m-n-1} is $(n+1)$-presented by [16], Lemma 5.5, and hence C is $(m+1)$-presented. Therefore this lemma holds by induction axioms. $\square$

Theorem 2.17. *Let R be a left strongly $(\mathcal{T}, n)$-coherent ring and M a left R-module. Then*

$$\mathcal{T}_n\mathcal{I} - \dim(M) = \mathcal{T}_n\mathcal{F} - \dim(M^+).$$

Proof. Let k be a positive integer and C be a $(\mathcal{T}, n+1)$-presented module. Since R is left strongly $(\mathcal{T}, n)$-coherent, by Lemma 2.16, C is $(n+k+2)$-presented. So, by [3], Lemma 2.7, statement (2), we have $\mathrm{Tor}^R_{n+k+1}(M^+, C) \cong \mathrm{Ext}^{n+k+1}_R(C, M)^+$. Consequently, $\mathcal{T}_n\mathcal{I} - \dim(M) = \mathcal{T}_n\mathcal{F} - \dim(M^+)$ by Theorems 2.12 and 2.15. $\square$

Corollary 2.18. *Let R be a strongly $(\mathcal{T}, n)$-coherent ring. Then*

$$\mathcal{T}_n - \mathrm{WD}(R) = \mathcal{T}_n\mathcal{I} - \mathrm{GLD}(R).$$

Proof. It follows from Theorems 2.14 and 2.17. □

3. $(\mathcal{T}, n)$-semihereditary rings

Recall that a ring R is called *left semihereditary* if every finitely generated left ideal of R is projective, or equivalently, if every finitely generated submodule of a projective right R-module is projective. It is easy to see that a ring R is left semihereditary if and only if the projective dimension of every finitely presented left R-module is less than or equal to 1. The concept of semihereditary rings has been generalized by many authors. For example, a commutative ring R is called a (n, d)*-ring* (see [4]) if every n-presented R-module has the projective dimension at most d; a ring R is called a *left* (n, d)*-ring* (see [13]) if every n-presented left R-module has the projective dimension at most d; a ring R is called a *left n-hereditary ring* (see [14]) if it is a left $(n, 1)$-ring; a ring R is called a *left n-regular ring* (see [14]) if it is a left $(n, 0)$-ring.

Definition 3.1. A *ring* R is called *left weakly n-hereditary* if it is a left (n, n)-ring.

Clearly, left n-hereditary ring is left weakly n-hereditary. A ring R is left semihereditary if and only if R is left 1-hereditary if and only if R is left weakly 1-hereditary.

Example 3.2. Let R be a non-coherent commutative ring of weak dimension one. Then $R[x]$ is a (2,2)-ring but not a (2,1)-ring by [4], Example 6.5, and so $R[x]$ is a weakly 2-hereditary ring which is not 2-hereditary.

Next, we generalize the concept of left n-regular rings.

Definition 3.3. A *ring* R is called *left weakly n-regular* if it is a left $(n, n-1)$-ring.

Clearly, R is regular if and only if it is left weakly 1-regular. Left n-regular ring is left weakly n-regular. If $n \geqslant 2$, then left n-hereditary ring is left weakly n-regular. Since left $(2, 2)$-rings need not be left $(2, 1)$-rings by Example 3.2, left weakly 2-hereditary rings need not be left weakly 2-regular.

Example 3.4. Let A be an arbitrary Prüfer domain (i.e. (1,1)-domain) and let R be the trivial ring extension of A by its quotient field. Then by [8], Example 3.4, R is a commutative (2,1)-ring which is not a (2,0)-ring. So, in general, left weakly 2-regular rings need not be left 2-regular.

Definition 3.5. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then the ring R is called $(\mathcal{T}, n)$-semihereditary if $\operatorname{pd}(C) \leqslant n$ for each $(\mathcal{T}, n+1)$-presented module C.

Example 3.6. Let $\mathcal{T} = R - \mathrm{Mod}$. Then R is $(\mathcal{T}, n)$-semihereditary if and only if it is left weakly n-hereditary.

Example 3.7. Let $\mathcal{T} = \{0\}$. Then R is $(\mathcal{T}, n)$-semihereditary if and only if it is left weakly $(n+1)$-regular.

Theorem 3.8. *Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then the following statements are equivalent for the ring R:*

(1) *R is a left $(\mathcal{T}, n)$-semihereditary ring.*

(2) *If $0 \to K \to P \to C \to 0$ is exact, where C is $(\mathcal{T}, n+1)$-presented, P is finitely generated projective, then $\operatorname{pd}(K) \leqslant n-1$.*

(3) *R is $(\mathcal{T}, n)$-coherent and every submodule of a $(\mathcal{T}, n)$-flat right R-module is $(\mathcal{T}, n)$-flat.*

(4) *R is $(\mathcal{T}, n)$-coherent and every right ideal is $(\mathcal{T}, n)$-flat.*

(5) *R is $(\mathcal{T}, n)$-coherent and every finitely generated right ideal is $(\mathcal{T}, n)$-flat.*

(6) *Every quotient module of a $(\mathcal{T}, n)$-injective left R-module is $(\mathcal{T}, n)$-injective.*

(7) *Every quotient module of an injective left R-module is $(\mathcal{T}, n)$-injective.*

(8) *Every left R-module has a monic $(\mathcal{T}, n)$-injective cover.*

(9) *Every right R-module has an epic $(\mathcal{T}, n)$-flat envelope.*

(10) *For every left R-module A, the sum of an arbitrary family of $(\mathcal{T}, n)$-injective submodules of A is $(\mathcal{T}, n)$-injective.*

(11) *Every torsionless right R-module is $(\mathcal{T}, n)$-flat.*

(12) *R is strongly $(\mathcal{T}, n)$-coherent and $\mathcal{T}_n\mathcal{I} - \mathrm{GLD}(R) \leqslant 1$.*

(13) *R is strongly $(\mathcal{T}, n)$-coherent and $\mathcal{T}_n - \mathrm{WD}(R) \leqslant 1$.*

Proof. (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) are trivial.

(2) $\Rightarrow$ (3). Assume (2). Then R is clearly $(\mathcal{T}, n)$-coherent by [16], Lemma 5.5. Let A be a submodule of a $(\mathcal{T}, n)$-flat right R-module B and let C be a $(\mathcal{T}, n+1)$-presented left R-module. Then there exists an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective. By (1), $\operatorname{pd}(K) \leqslant n-1$ and so $fd(K) \leqslant n-1$. Then the exactness of $0 = \operatorname{Tor}^R_{n+1}(B/A, P) \to \operatorname{Tor}^R_{n+1}(B/A, C) \to \operatorname{Tor}^R_n(B/A, K) = 0$ implies that $\operatorname{Tor}^R_{n+1}(B/A, C) = 0$. Thus, from the exactness of the sequence $0 = \operatorname{Tor}^R_{n+1}(B/A, C) \to \operatorname{Tor}^R_n(A, C) \to \operatorname{Tor}^R_n(B, C) = 0$ we have $\operatorname{Tor}^R_n(A, C) = 0$, that is, A is $(\mathcal{T}, n)$-flat.

(5) $\Rightarrow$ (2). Let C be a $(\mathcal{T}, n+1)$-presented left R-module. If $0 \to K \to P \to C \to 0$ is an exact sequence of left R-modules, where P is finitely generated projective. Since R is $(\mathcal{T}, n)$-coherent, K is n-presented. For any finitely generated right ideal I

of R we have an exact sequence $0 \to \mathrm{Tor}_{n+1}^R(R/I, C) \to \mathrm{Tor}_n^R(I, C) = 0$ since I is $(\mathcal{T}, n)$-flat. So $\mathrm{Tor}_{n+1}^R(R/I, C) = 0$, and hence we obtain an exact sequence $0 = \mathrm{Tor}_{n+1}^R(R/I, C) \to \mathrm{Tor}_n^R(R/I, K) \to 0$. Thus, $\mathrm{Tor}_n^R(R/I, K) = 0$. Let K have a finite n-presentation $F_n \xrightarrow{d_n} \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} K \longrightarrow 0$. Then $\mathrm{Ker}(d_{n-2})$ is finitely presented and $\mathrm{Tor}_1^R(R/I, \mathrm{Ker}(d_{n-2}) = 0$, so $\mathrm{Ker}(d_{n-2})$ is projective. Therefore $\mathrm{pd}(K) \leqslant n-1$.

(2) $\Rightarrow$ (6). Let M be a $(\mathcal{T}, n)$-injective left R-module and N be a submodule of M. Then for any $(\mathcal{T}, n+1)$-presented left R-module C, there exists an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective and $\mathrm{pd}(K) \leqslant n-1$ by (2). And so the exact sequence $0 = \mathrm{Ext}_R^n(K, N) \to \mathrm{Ext}_R^{n+1}(C, N) \to \mathrm{Ext}_R^{n+1}(P, N) = 0$ implies that $\mathrm{Ext}_R^{n+1}(C, N) = 0$. Thus, the exact sequence $0 = \mathrm{Ext}_R^n(C, M) \to \mathrm{Ext}_R^n(C, M/N) \to \mathrm{Ext}_R^{n+1}(C, N) = 0$ implies that $\mathrm{Ext}_R^n(C, M/N) = 0$. Consequently, M/N is $(\mathcal{T}, n)$-injective.

(7) $\Rightarrow$ (2). Let C be a $(\mathcal{T}, n+1)$-presented left R-module and there is an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective. Then for any left R-module M, by hypothesis, $E(M)/M$ is $(\mathcal{T}, n)$-injective, and so $\mathrm{Ext}_R^n(C, E(M)/M) = 0$. Thus, the exactness of the sequence $0 = \mathrm{Ext}_R^n(C, E(M)/M) \to \mathrm{Ext}_R^{n+1}(C, M) \to \mathrm{Ext}_R^{n+1}(C, E(M)) = 0$ implies that $\mathrm{Ext}_R^{n+1}(C, M) = 0$. Hence, the exactness of the sequence $0 = \mathrm{Ext}_R^n(P, M) \to \mathrm{Ext}_R^n(K, M) \to \mathrm{Ext}_R^{n+1}(C, M) = 0$ implies that $\mathrm{Ext}_R^n(K, M) = 0$, as required.

(3) $\Leftrightarrow$ (9). It follows from [2], Theorem 2 and [16], Theorem 5.3, statement (5).

(3), (6) $\Rightarrow$ (8). Since R is $(\mathcal{T}, n)$-coherent by (3) for any left R-module M there is a $(\mathcal{T}, n)$-injective cover $f\colon E \to M$ by [16], Corollary 5.8. Note that $\mathrm{im}(f)$ is $(\mathcal{T}, n)$-injective by (6), and $f\colon E \to M$ is a $(\mathcal{T}, n)$-injective precover, so for the inclusion map $i\colon \mathrm{im}(f) \to M$ there is a homomorphism $g\colon \mathrm{im}(f) \to E$ such that $i = fg$. Hence $f = f(gf)$. Observing that $f\colon E \to M$ is a $(\mathcal{T}, n)$-injective cover and gf is an endomorphism of E, gf is an automorphisms of E, and thus $f\colon E \to M$ is a monic $(\mathcal{T}, n)$-injective cover.

(8) $\Rightarrow$ (6). Let M be a $(\mathcal{T}, n)$-injective left R-module and N be a submodule of M. By (8), M/N has a monic $(\mathcal{T}, n)$-injective cover $f\colon E \to M/N$. Let $\pi\colon M \to M/N$ be the natural epimorphism. Then there exists a homomorphism $g\colon M \to E$ such that $\pi = fg$. Thus, f is an isomorphism, and therefore $M/N \cong E$ is $(\mathcal{T}, n)$-injective.

(6) $\Rightarrow$ (10). Let A be a left R-module and $\{A_\gamma\colon \gamma \in \Gamma\}$ be an arbitrary family of $(\mathcal{T}, n)$-injective submodules of A. Since the direct sum of $(\mathcal{T}, n)$-injective modules is $(\mathcal{T}, n)$-injective and $\sum\limits_{\gamma\in\Gamma} A_\gamma$ is a homomorphic image of $\oplus_{\gamma\in\Gamma} A_\gamma$, by (6), $\sum\limits_{\gamma\in\Gamma} A_\gamma$ is $(\mathcal{T}, n)$-injective.

(10) $\Rightarrow$ (7). Let E be an injective left R-module and $K \leqslant E$. Take $E_1 = E_2 = E$, $N = E_1 \oplus E_2$, $D = \{(x, -x)\colon x \in K\}$. Define $f_1\colon E_1 \to N/D$ by $x_1 \mapsto (x_1, 0) + D$,

$f_2\colon E_2 \to N/D$ by $x_2 \mapsto (0, x_2) + D$ and write $\overline{E}_i = f_i(E_i)$, $i = 1, 2$. Then $\overline{E}_i \cong E_i$ is injective, $i = 1, 2$, and so $N/D = \overline{E}_1 + \overline{E}_2$ is $(\mathcal{T}, n)$-injective. By the injectivity of $\overline{E}_i$, $(N/D)/\overline{E}_i$ is isomorphic to a summand of N/D and thus it is $(\mathcal{T}, n)$-injective. Now, we define $f\colon E \to (N/D)/\overline{E}_1$; $e \mapsto f_2(e) + \overline{E}_1$, then f is an epimorphism with $\mathrm{Ker}(f) = K$, and hence $E/K \cong (N/D)/\overline{E}_1$ is $(\mathcal{T}, n)$-injective.

(3) $\Rightarrow$ (11). Let M be a torsionless right R-module. Then there exists an exact sequence $0 \to M \to \prod R_R$. Since R is $(\mathcal{T}, n)$-coherent, by [16], Theorem 5.3, statement (4), $\prod R_R$ is $(\mathcal{T}, n)$-flat. By hypothesis, every submodule of a $(\mathcal{T}, n)$-flat R-module is $(\mathcal{T}, n)$-flat, so M is $(\mathcal{T}, n)$-flat.

(11) $\Rightarrow$ (3). Assume (11). Then $\prod R_R$ is $(\mathcal{T}, n)$-flat, and thus R is $(\mathcal{T}, n)$-coherent by [16], Theorem 5.3, statement (4). Moreover, every right ideal of R is torsionless and so $(\mathcal{T}, n)$-flat.

(2) $\Rightarrow$ (12). Let $0 \to K \to P \to C \to 0$ be exact with C $(\mathcal{T}, n+1)$-presented and P finitely generated projective. Then by (2), $\mathrm{pd}(K) \leqslant n-1$, and so K is $(\mathcal{T}, n)$-projective, which shows that R is strongly $(\mathcal{T}, n)$-coherent. Now let M be any left R-module. Then for any $(\mathcal{T}, n+1)$-presented module C we have an exact sequence $0 \to K \to P \to C \to 0$ of left R-modules, where P is finitely generated projective. By (2), $\mathrm{pd}(K) \leqslant n-1$. Thus, the exact sequence $0 = \mathrm{Ext}_R^n(K, M) \to \mathrm{Ext}_R^{n+1}(C, M) \to \mathrm{Ext}_R^{n+1}(P, M) = 0$ implies that $\mathrm{Ext}_R^{n+1}(C, M) = 0$. This yields that $\mathcal{T}_n\mathcal{I} - \mathrm{GLD}(R) \leqslant 1$ by Definition 2.11.

(12) $\Rightarrow$ (13). It follows from Theorem 2.12 and the isomorphism

$$\mathrm{Tor}_{n+1}^R(M, C)^+ \cong \mathrm{Ext}_R^{n+1}(C, M^+).$$

(13) $\Rightarrow$ (3). Assume (13). Then R is clearly $(\mathcal{T}, n)$-coherent. Let A be a submodule of a $(\mathcal{T}, n)$-flat right R-module B and let C be a $(\mathcal{T}, n+1)$-presented left R-module. Since R is strongly $(\mathcal{T}, n)$-coherent and $\mathcal{T}_n$-WD(R)$\leqslant 1$, by Theorem 2.15 we have $\mathrm{Tor}_{n+1}^R(B/A, C) = 0$. Then, from the exactness of the sequence $0 = \mathrm{Tor}_{n+1}^R(B/A, C) \to \mathrm{Tor}_n^R(A, C) \to \mathrm{Tor}_n^R(B, C) = 0$ we have $\mathrm{Tor}_n^R(A, C) = 0$, which shows that A is $\mathcal{T}_n$-flat. $\square$

Corollary 3.9. *The following statements are equivalent for the ring R:*

(1) *R is a left weakly n-hereditary ring.*

(2) *If $0 \to K \to P \to C \to 0$ is exact, where C is n-presented, P is finitely generated projective, then $\mathrm{pd}(K) \leqslant n-1$.*

(3) *R is left n-coherent and every submodule of an n-flat right R-module is n-flat.*

(4) *R is left n-coherent and every right ideal is n-flat.*

(5) *R is left n-coherent and every finitely generated right ideal is n-flat.*

(6) *Every quotient module of an n-FP-injective left R-module is n-FP-injective.*

(7) *Every quotient module of an injective left R-module is n-FP-injective.*
(8) *Every left R-module has a monic n-FP-injective cover.*
(9) *Every right R-module has an epic n-flat envelope.*
(10) *For every left R-module A, the sum of an arbitrary family of n-FP-injective submodules of A is n-FP-injective.*
(11) *Every torsionless right R-module is n-flat.*
(12) *R is left n-coherent and $(\mathcal{FP})_n\mathcal{I} - \mathrm{GLD}(R) \leqslant 1$.*
(13) *R is left n-coherent and $n - \mathrm{WD}(R) \leqslant 1$.*

Proof. It follows from Theorem 3.8 and Corollary 2.4. □

Let $n = 1$, then by Corollary 3.9, we can obtain a series of characterizations of left semihereditary rings.

Corollary 3.10. *The following statements are equivalent for the ring R:*
(1) *R is a left semihereditary ring.*
(2) *If $0 \to K \to P \to C \to 0$ is exact, where C is finitely presented, P is finitely generated projective, then K is projective.*
(3) *R is left coherent and every submodule of a flat right R-module is flat.*
(4) *R is left coherent and every right ideal is flat.*
(5) *R is left coherent and every finitely generated right ideal is flat.*
(6) *Every quotient module of an FP-injective left R-module is FP-injective.*
(7) *Every quotient module of an injective left R-module is FP-injective.*
(8) *Every left R-module has a monic FP-injective cover.*
(9) *Every right R-module has an epic flat envelope.*
(10) *For every left R-module A, the sum of an arbitrary family of FP-injective submodules of A is FP-injective.*
(11) *Every torsionless right R-module is flat.*
(12) *R is left coherent and $\mathcal{FPI} - \mathrm{GLD}(R) \leqslant 1$.*
(13) *R is left coherent and $\mathrm{WD}(R) \leqslant 1$.*

Corollary 3.11. *The following statements are equivalent for the ring R:*
(1) *R is a left weakly $(n+1)$-regular ring.*
(2) *If $0 \to K \to P \to C \to 0$ is exact, where C is (n+1)-presented, P is finitely generated projective, then $\mathrm{pd}(K) \leqslant n-1$.*
(3) *Every submodule of a weakly n-flat right R-module is weakly n-flat.*
(4) *Every right ideal is weakly n-flat.*
(5) *Every finitely generated right ideal is weakly n-flat.*
(6) *Every quotient module of a weakly n-FP-injective left R-module is weakly n-FP-injective.*

(7) *Every quotient module of an injective left R-module is weakly n-FP-injective.*
(8) *Every left R-module has a monic weakly n-FP-injective cover.*
(9) *Every right R-module has an epic weakly n-flat envelope.*
(10) *For every left R-module A, the sum of an arbitrary family of weakly n-FP-injective submodules of A is weakly n-FP-injective.*
(11) *Every torsionless right R-module is weakly n-flat.*
(12) *Every weakly n-FP-injective left R-module is $(n+1)$-FP-injective and*

$$\mathcal{W}(\mathcal{FP})_n\mathcal{I} - \mathrm{GLD}(R) \leqslant 1.$$

(13) *Every weakly n-FP-injective left R-module is $(n+1)$-FP-injective and $\mathcal{W}_n - \mathrm{WD}(R) \leqslant 1$.*

Proof. It follows from Theorem 3.8 and Corollary 2.6. $\square$

4. $(\mathcal{T}, n)$-regular rings

Definition 4.1. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then the ring R is called *$(\mathcal{T}, n)$-regular* if $\mathrm{pd}(C) \leqslant n-1$ for each $(\mathcal{T}, n+1)$-presented module C.

Example 4.2. Let $\mathcal{T} = R - \mathrm{Mod}$. Then R is $(\mathcal{T}, n)$-regular if and only if it is left weakly n-regular.

Example 4.3. Let $\mathcal{T} = \{0\}$. Then R is $(\mathcal{T}, n)$-regular if and only if it is a left $(n+1, n-1)$-ring.

Theorem 4.4. *Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then the following conditions are equivalent for R:*
(1) *R is $(\mathcal{T}, n)$-regular.*
(2) *Every left R-module is $(\mathcal{T}, n)$-injective.*
(3) *Every right R-module is $(\mathcal{T}, n)$-flat.*
(4) *Every cotorsion right R-module is $(\mathcal{T}, n)$-flat.*
(5) *Every right R-module in $(\mathcal{T}_n\mathcal{F})^{\perp}$ is injective.*
(6) *Every left R-module in ${}^{\perp}(\mathcal{T}_n\mathcal{I})$ is projective.*
(7) *R is $(\mathcal{T}, n)$-semihereditary and ${}_RR$ is $(\mathcal{T}, n)$-injective.*
(8) *R is strongly $(\mathcal{T}, n)$-coherent and every left R-module in ${}^{\perp}(\mathcal{T}_n\mathcal{I})$ is $(\mathcal{T}, n)$-injective.*
(9) *R is strongly $(\mathcal{T}, n)$-coherent and every right R-module in $(\mathcal{T}_n\mathcal{F})^{\perp}$ is $(\mathcal{T}, n)$-flat.*

P r o o f. (1) $\Leftrightarrow$ (2); (3) $\Rightarrow$ (4), (5); (2) $\Rightarrow$ (6); (1), (2) $\Rightarrow$ (7); and (2), (7) $\Rightarrow$ (8) are clear.

(2) $\Rightarrow$ (3). It follows from the isomorphism $\mathrm{Tor}_n^R(M, C)^+ \cong \mathrm{Ext}_R^n(C, M^+)$.

(4) $\Rightarrow$ (2). Let M be any left R-module. Since M^+ is pure injective by [5], Proposition 5.3.7, M^+ is a cotorsion by [5], Lemma 5.3.23, and so M^+ is $(\mathcal{T}, n)$-flat by (4). Hence, by [16], Theorem 4.8, M^{++} is $(\mathcal{T}, n)$-injective. Note that M is a pure submodule of M^{++}. By [16], Proposition 4.9, statement (1), M is $(\mathcal{T}, n)$-injective.

(5) $\Rightarrow$ (3). It follows from the fact that $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^\perp)$ is a cotorsion theory (see [16], Theorem 4.11, statement (2)).

(6) $\Rightarrow$ (2). It follows from the fact that $({}^\perp(\mathcal{T}_n\mathcal{I}), \mathcal{T}_n\mathcal{I})$ is a cotorsion theory (see [16], Theorem 4.11, statement (1)).

(7) $\Rightarrow$ (2) Let M be any left R-module. Then there exists an exact sequence $F \to M \to 0$ with F free. Since ${}_RR$ is $(\mathcal{T}, n)$-injective, by [16], Proposition 4.6, F is $(\mathcal{T}, n)$-injective. Since R is $(\mathcal{T}, n)$-semihereditary, by Theorem 3.8, statement (6), M is $(\mathcal{T}, n)$-injective.

(8) $\Rightarrow$ (2). Let M be any left R-module. By [16], Theorem 4.11, statement (1), there exists an exact sequence $0 \to K \to F \to M \to 0$ with $F \in^\perp (\mathcal{T}_n\mathcal{I})$ and $K \in \mathcal{T}_n\mathcal{I}$. Then $F \in \mathcal{T}_n\mathcal{I}$ by (8). Note that R is strongly $(\mathcal{T}, n)$-coherent, by Theorem 2.3, statement (8), we have that $M \in \mathcal{T}_n\mathcal{I}$.

(3), (8) $\Rightarrow$ (9). It is obvious.

(9) $\Rightarrow$ (3). Let $E \in (\mathcal{T}_n\mathcal{F})^\perp$. Then for any right R-module M, by [16], Theorem 4.11, statement (2), $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^\perp)$ is a perfect cotorsion theory, so it is a complete cotorsion theory, and hence there exists an exact sequence $0 \to M \to F \to L \to 0$, where $F \in (\mathcal{T}_n\mathcal{F})^\perp$ and $L \in \mathcal{T}_n\mathcal{F}$. By (9), F is $(\mathcal{T}, n)$-flat. Since R is strongly $(\mathcal{T}, n)$-coherent, by Theorem 2.3, statement (3), $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^\perp)$ is a hereditary cotorsion theory, and thus, M is $(\mathcal{T}, n)$-flat. $\square$

Corollary 4.5. *Let n be a positive integer. Then the following conditions are equivalent for R:*

(1) *R is left weakly n-regular.*
(2) *Every left R-module is n-FP-injective.*
(3) *Every right R-module is n-flat.*
(4) *Every cotorsion right R-module is n-flat.*
(5) *Every right R-module in $\mathcal{F}_n^\perp$ is injective.*
(6) *Every left R-module in ${}^\perp((\mathcal{FP})_n\mathcal{I})$ is projective.*
(7) *R is left weakly n-hereditary and ${}_RR$ is n-FP-injective.*
(8) *R is left n-coherent and every left R-module in ${}^\perp((\mathcal{FP})_n\mathcal{I})$ is n-FP-injective.*
(9) *R is left n-coherent and every right R-module in $(\mathcal{F}_n)^\perp$ is n-flat.*

Recall that a left R-module N is said to be *FP-projective* (see [9]) if $\mathrm{Ext}^1_R(N,M)=0$ for any FP-injective left R-module M.

Corollary 4.6. *The following conditions are equivalent for a ring R:*

(1) *R is regular.*
(2) *Every left R-module is FP-injective.*
(3) *Every right R-module is flat.*
(4) *Every cotorsion right R-module is flat.*
(5) *Every cotorsion right R-module is injective.*
(6) *Every FP-projective left R-module is projective.*
(7) *R is left semihereditary and ${}_RR$ is FP-injective.*
(8) *R is left coherent and every FP-projective left R-module is FP-injective.*

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