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SEMI-SYMMETRIC FOUR DIMENSIONAL NEUTRAL LIE GROUPS

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Abstract. The present paper is concerned with obtaining a classification regarding to four-dimensional semi-symmetric neutral Lie groups. Moreover, we discuss some geometric properties of these spaces. We exhibit a rich class of non-Einstein Ricci soliton examples.

Keywords: semi-symmetric; Lie group, Ricci soliton

MSC 2010: 53C50, 53C30, 53C25

1. Introduction

A symmetric space refers to a connected (pseudo-) Riemannian manifold M which is identified by the existence of global symmetries around any point $p \in M$. A global symmetry at $p \in M$ is an isometry $\zeta_p\colon M \to M$ with differential map $-\mathrm{id}$ on T_pM. Since global isometries reverse the geodesics through points, they are the unique extension of local geodesic isometries and so, symmetric spaces are locally symmetric ($\nabla R = 0$), see [17]. Symmetric spaces are interesting for their simple and friendly structure. Many scientists have studied manifolds which are in some aspects generalizations of the symmetric manifolds, e.g., k-symmetric ($\nabla^k R = 0$, $\nabla^{k-1} R \neq 0$, see [19]) and semi-symmetric manifolds ($R(X,Y) \cdot R = 0$ for all $X, Y \in \mathfrak{X}(M)$, see [20]).

Although locally symmetric spaces are semi-symmetric, there are several kinds of Riemannian spaces which are semi-symmetric but not locally symmetric, see [5], [21]. However, semi-symmetry and locally symmetry are equivalent in several classes of Riemannian manifolds, see [3], [4], [11]. In the pseudo-Riemannian setting, semi-symmetric spaces have been the subject of several researches. Homogeneous Lorentzian semi-symmetric manifolds of dimension three have been studied in [6], showing the existence of semi-symmetric nonsymmetric examples, contrary to the

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Riemannian homogeneous spaces, where symmetry and semi-symmetry are equivalent. A weaker condition, i.e., curvature homogeneous of order one semi-symmetric Lorentzian three-manifolds, has been studied in [7]. A research on semi-symmetric Lorentzian manifolds of dimension three which admitted a degenerate parallel line field has been reported in [8], obtaining semi-symmetric curvature homogeneous examples in the possible different cases. Therefore most of investigations on the pseudo-Riemannian semi-symmetric manifolds have been focused on dimension three.

In this research, we consider pseudo-Riemannian Lie groups with the metric tensor of signature $(2,2)$ and study the semi-symmetry condition

$$R(X,Y)\cdot R=0, \tag{1.1}$$

on such manifolds. In the above formula (1.1), X, Y are arbitrary smooth vector fields on M, and $R(X,Y)$ acts as a derivation on R. This study results in a full classification of Lie groups of neutral signature (i.e., signature $(2,2)$), and we exhibit a rich class of Ricci parallel, Einstein and Ricci soliton examples. Although Einstein homogeneous four dimensional Riemannian manifolds are symmetric, see [15], in the study of the geometry of pseudo-Riemannian semi-symmetric (not locally symmetric) Lie groups (which are obviously homogeneous spaces), several Einstein examples arise, see [22].

This paper is organized in the following way. In Section 2, we recall some general facts for the study of pseudo-Riemannian Lie groups. We fully classify semi-symmetric examples of neutral Lie groups in Section 3, and Section 4 is devoted to the study of the geometry of examples classified in the previous section. Finally, Ricci soliton examples of the neutral Lie groups are studied in the last section.

2. Neutral Lie groups

Four-dimensional homogeneous Riemannian manifolds are classified by Bérard-Bérgery in [2]. He has proved that a simply connected four-dimensional homogeneous Riemannian manifold is either symmetric or isometric to a Lie group equipped with a left-invariant Riemannian metric. This study shows the important role of Lie groups in the study of homogeneous Riemannian spaces of dimension four. During the study of D'Atri spaces (i.e., a space with volume-preserving symmetries), the authors exhibit a classification of four-dimensional simply connected Riemannian Lie groups as follows.

Proposition 2.1 ([1]). *A simply connected four-dimensional Riemannian Lie group is*

(1) *either one of the unsolvable direct products* $\mathbb{R} \times SU(2)$ *and* $\mathbb{R} \times \widetilde{SL}(2,\mathbb{R})$*; or*

(2) *one of the following solvable Lie groups:*

(i) *the nontrivial semi-direct products* $\mathbb{R} \ltimes E(2)$ *and* $\mathbb{R} \ltimes E(1,1)$*;*

(ii) *the nonnilpotent semi-direct products* $\mathbb{R} \ltimes H$*, where* H *denotes the Heisenberg group;*

(iii) *the semi-direct products* $\mathbb{R} \ltimes \mathbb{R}^3$*.*

Following the work [12], if G is a four-dimensional simply connected Lie group, equipped with a left invariant metric of neutral signature, then G is one of the Lie groups listed in the above Proposition 2.1.

Let $(G = \mathbb{R} \ltimes G_3, g)$ be a pseudo-Riemannian Lie group of dimension four, where G_3 is one of the Lie groups $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$, $E(2)$, $E(1,1)$, H or $\mathbb{R}^3$, and denote the Lie algebra of G_3 by $\mathfrak{g}_3$. The study of neutral Lie groups is different from the Riemannian ones; in fact, the restriction of a Riemannian metric g on $\mathfrak{g}_3$ is always nondegenerate and so, $(g|_{G_3}, G_3)$ is again a Riemannian Lie group by its own. But for neutral Lie groups, the restriction of g on $\mathfrak{g}_3$ can be either of signature $(2,1)$ or degenerate, see [12]. We refer to the following lemma for different possibilities in the study of four-dimensional neutral Lie groups.

Lemma 2.2 ([12]). *Let* $\mathfrak{g}$ *denote any four-dimensional Lie algebra and let* g *be an inner product of signature* $(2,2)$ *on* $\mathfrak{g}$*. Then there exists a basis* $\{e_1, e_2, e_3, e_4\}$ *of* $\mathfrak{g}$*, such that*

- ▷ $\mathfrak{g}_3 = \mathrm{Span}\{e_1, e_2, e_3\}$ *is a three-dimensional Lie algebra and* e_4 *acts as a derivation on* $\mathfrak{g}_3$ *(that is,* $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$*, where* $\mathfrak{r} = \mathrm{Span}\{e_4\}$*), and*
- ▷ *with respect to* $\{e_1, e_2, e_3, e_4\}$*, the neutral inner product* g *takes one of the following forms:*

$$\text{(a)}\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Based on the study [12], the authors in [14] have categorized four dimensional neutral Lie groups into the following two classes:

(a) $g|_{\mathfrak{g}_3}$ is Lorentzian and the time-like vector e_4 acts as a derivation on $\mathfrak{g}_3$,

(b) $g|_{\mathfrak{g}_3}$ is degenerate and the light-like vector e_4 acts as a derivation on $\mathfrak{g}_3$,

and we could completely classify these spaces, under the hypothesis of semi-symmetry.

3. Semi-symmetric examples

This section is devoted to the main theorem of this study, i.e., the classification of left-invariant neutral metrics on four-dimensional Lie groups whose curvature tensors satisfy (1.1). We report strict (nonsymmetric) examples of this kind.

Theorem 3.1. *Let G be a four-dimensional simply connected Lie group. If g is a left invariant neutral metric on G such that its curvature tensor satisfies the strict semi-symmetric (nonsymmetric) condition, then the Lie algebra $\mathfrak{g}$ of G is isometric to $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{g}_3 = \mathrm{Span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \mathrm{Span}\{e_4\}$; then, one of the following cases occurs.*

(a) *$\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis, with e_3,e_4 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:*

(a1) *$\mathbb{R} \ltimes \mathbb{R}^3$, and $\mathfrak{g}$ is described by one of the following sets of conditions:*

(i) $[e_2, e_4] = -Ae_2 + (B + 2\delta A)e_3$, $[e_3, e_4] = Be_2 + Ae_3$, $A(A + \delta B) \neq 0$,

(ii) $[e_1, e_4] = -\delta Ae_2$, $[e_2, e_4] = \delta Be_1 + Be_3$, $[e_3, e_4] = Ae_2$, $A(B - A) \neq 0$,

(iii) $[e_1, e_4] = \frac{1}{2}\delta Ae_1 + Ae_3$, $[e_3, e_4] = -\frac{1}{2}\delta Ae_3$, $A \neq 0$,

(iv) $[e_1, e_4] = \frac{1}{4}\varepsilon\sqrt{2}Ae_1 + \frac{1}{4}\varepsilon\sqrt{2}Ae_2 + Ae_3$,
$[e_2, e_4] = -\frac{1}{4}\varepsilon\delta\sqrt{2}Ae_1 + \frac{1}{4}\varepsilon\sqrt{2}Ae_2 - \delta Ae_3$, $[e_3, e_4] = -\frac{1}{2}\varepsilon\sqrt{2}Ae_3$, $A \neq 0$.

(a2) *$\mathbb{R} \ltimes H$ and $\mathfrak{g}$ can be described by one of the following sets of conditions:*

(i) $[e_1, e_2] = \delta Ae_3$, $[e_1, e_4] = Ae_3$, $[e_2, e_3] = Be_3$, $[e_3, e_4] = -\delta Be_3$, $A \neq 0$,

(ii) $[e_1, e_2] = Ae_1 + \delta Be_3$, $[e_1, e_4] = \delta Ae_1 + Be_3$, $B(A \pm B) \neq 0$,

(iii) $[e_1, e_2] = \dfrac{A\varepsilon\sqrt{B^2 + C^2}}{B}e_3$,
$[e_1, e_3] = Ce_3$, $[e_1, e_4] = Ae_3$, $[e_2, e_3] = Be_3$,
$[e_2, e_4] = -\dfrac{AC}{B}e_3$, $[e_3, e_4] = -\varepsilon\sqrt{B^2 + C^2}e_3$, $AB \neq 0$.

(a3) *Either $\mathbb{R} \ltimes E_2$ or $\mathbb{R} \ltimes E(1, 1)$, with $\mathfrak{g}$ described by one of the following sets of conditions:*

(i) $[e_1, e_2] = Ae_1 + \delta Be_3$, $[e_1, e_4] = \delta Ae_1 + Be_3$,
$[e_2, e_3] = Ce_3$, $[e_3, e_4] = -\delta Ce_3$, $ABC \neq 0$,

(ii) $[e_1, e_2] = Ae_1 + 2\delta Ae_3$, $[e_1, e_4] = \frac{1}{2}\delta Be_1 + Be_3$,
$[e_2, e_3] = Ae_3$, $[e_3, e_4] = -\frac{1}{2}\delta Be_3$, $A \neq 0$,

(iii) $[e_1, e_2] = Ae_1 - Be_2 + 2\varepsilon\sqrt{A^2 + B^2}e_3$,
$[e_1, e_3] = Be_3$, $[e_2, e_3] = Ae_3$, $A^2 + B^2 \neq 0$,

(iv) $[e_1, e_2] = Ae_1 + \delta Ae_2 + 2\varepsilon\sqrt{2}Ae_3$, $[e_1, e_3] = -\delta Ae_3$,
$[e_1, e_4] = \frac{1}{4}\varepsilon\sqrt{2}Be_1 + \frac{1}{4}\delta\varepsilon\sqrt{2}Be_2 + Be_3$, $[e_2, e_3] = Ae_3$,
$[e_2, e_4] = \frac{1}{4}\delta\varepsilon\sqrt{2}Be_1 + \frac{1}{4}\varepsilon\sqrt{2}Be_2 + \delta Be_3$, $[e_3, e_4] = -\frac{1}{2}\varepsilon\sqrt{2}Be_3$, $A \neq 0$,

(v) $[e_1,e_2]=\dfrac{A^2}{4B}e_1-\delta\dfrac{A^2}{4B}e_2+\varepsilon\sqrt{2}Ae_3,\ [e_1,e_3]=\delta Be_3,$

$[e_1,e_4]=\varepsilon\dfrac{\sqrt{2}A^2}{8B}e_1-\delta\varepsilon\dfrac{\sqrt{2}A^2}{8B}e_2+Ae_3,\ [e_2,e_3]=Be_3,$

$[e_2,e_4]=-\delta\varepsilon\dfrac{\sqrt{2}A^2}{8B}e_1+\varepsilon\dfrac{\sqrt{2}A^2}{8B}e_2-\delta Ae_3,\ [e_3,e_4]=-\varepsilon\sqrt{2}Be_3,\ AB\neq 0,$

(vi) $[e_1,e_2]=\dfrac{A^2}{4B}e_1-\dfrac{A^2C}{4B^2}e_2+\dfrac{\varepsilon\sqrt{C^2+B^2}A}{B}e_3,\ [e_1,e_3]=Ce_3,$

$[e_1,e_4]=\dfrac{A^2}{4\varepsilon\sqrt{C^2+B^2}}e_1-\dfrac{A^2C}{4B\varepsilon\sqrt{C^2+B^2}}e_2+Ae_3,\ [e_2,e_3]=Be_3,$

$[e_2,e_4]=-\dfrac{A^2C}{4B\varepsilon\sqrt{C^2+B^2}}e_1+\dfrac{A^2C^2}{4B^2\varepsilon\sqrt{C^2+B^2}}e_2-\dfrac{CA}{B}e_3,$

$[e_3,e_4]=-\varepsilon\sqrt{C^2+B^2}e_3,\ AB\neq 0,$

(vii) $[e_1,e_2]=\dfrac{A\varepsilon\sqrt{B^2+C^2}}{B}e_1-\dfrac{AC\varepsilon\sqrt{B^2+C^2}}{B^2}e_2+\dfrac{D\varepsilon\sqrt{B^2+C^2}}{B}e_3,$

$[e_1,e_3]=Ce_3,\ [e_1,e_4]=Ae_1-\dfrac{AC}{B}e_2+De_3,\ [e_2,e_3]=Be_3,$

$[e_2,e_4]=-\dfrac{AC}{B}e_1+\dfrac{AC^2}{B^2}e_2-\dfrac{CD}{B}e_3,\ [e_3,e_4]=-\varepsilon\sqrt{C^2+B^2}e_3,\ ABD\neq 0.$

(b) $\{e_1,e_2,e_3,e_4\}$ *is a basis, with the inner product* g *on* $\mathfrak{g}$*, completely determined by* $g(e_1,e_1)=-g(e_2,e_2)=g(e_3,e_4)=g(e_4,e_3)=1$ *and* $g(e_i,e_j)=0$ *otherwise. In this case,* G *is isometric to one of the following semi-direct products* $\mathbb{R}\ltimes G_3$*:*

(b1) $\mathbb{R}\ltimes\mathbb{R}^3$ *and* $\mathfrak{g}$ *is described by one of the following sets of conditions:*

(i) $[e_1,e_4]=Ae_1+Be_2+Ce_3,\ [e_2,e_4]=De_1+Ee_2+Fe_3,$
$[e_3,e_4]=Ge_3,\ G^2+(B+D)^2\neq 0,\ G^2+(A-E)^2+(B-D)^2\neq 0,$
$G^2+(A+\varepsilon B)^2+(D+\varepsilon E)^2\neq 0,\ (A+\varepsilon D)^2+(E+\varepsilon B)^2+(2G+\varepsilon(D+B))^2\neq 0,$
$(B-D)^2+(A-E)^2+(A-G)^2\neq 0,$
$(A+\varepsilon D+G)^2+(B-2\varepsilon G+D)^2+(E+\varepsilon D-G)^2\neq 0.$

(ii) $[e_1,e_4]=\delta Ae_1+Ae_2+Be_3,\ [e_2,e_4]=\delta Ce_1+Ce_2+De_3,$
$[e_3,e_4]=Ee_1+\delta Ee_2+Fe_3,\ (A-\delta F)^2+(C-F)^2+E^2\neq 0,$
$(BE-\delta AF)^2+(DE-\delta CF)^2\neq 0.$

(b2) $\mathbb{R}\ltimes H$ *and* $\mathfrak{g}$ *is described by one of the following sets of conditions:*

(i) $[e_1,e_2]=Ae_3,\ [e_1,e_3]=\delta Be_3,$
$[e_1,e_4]=Ce_1-\delta Ce_2+\dfrac{CA-DA+\delta BE-\delta AF}{B}e_3,$
$[e_2,e_3]=Be_3,\ [e_2,e_4]=Fe_1-\delta Fe_2+Ee_3,\ [e_3,e_4]=De_3,\ B\neq 0,$
$C^2+F^2\neq 0,\ C^2+(D+\delta A)^2\neq 0,\ A+\delta D\neq 0.$

(ii) $[e_1,e_2]=Ae_3,\ [e_1,e_4]=(-B+C)e_1+De_2+Ee_3,$
$[e_2,e_4]=Fe_1+Be_2+Ge_3,\ [e_3,e_4]=Ce_3,\ A\neq 0,$

$C^2+(F-A+D)^2\neq 0$, $(AD+B^2-CB-D^2)^2+(FD+(B-C)B)^2\neq 0$, $B^2+C^2+(D-F)^2\neq 0$, $(A+2\varepsilon B)^2+(C+\varepsilon A)^2+(D-F)^2\neq 0$, $\varepsilon^2=1$.

(iii) $[e_1,e_2]=-Ae_2-\dfrac{A^2}{B}e_3$, $[e_1,e_3]=Be_2+Ae_3$,
$[e_1,e_4]=\dfrac{A^2+C^2}{2B}e_2+\dfrac{A(A^2+C^2)}{2B^2}e_3$,
$[e_2,e_4]=-\dfrac{CA}{B}e_2-\dfrac{CA^2}{B^2}e_3$, $[e_3,e_4]=Ce_2+\dfrac{AC}{B}e_3$, $B\neq 0$,

(iv) $[e_1,e_3]=Ae_1+Be_2, B(A\pm B)\neq 0$.

(v) $[e_1,e_2]=Ae_3$, $[e_1,e_3]=\delta Be_3$,
$[e_1,e_4]=Ce_1-\delta Ce_2+De_3$, $[e_2,e_3]=Be_3$,
$[e_2,e_4]=Ee_1-\delta Ee_2+Fe_3$,
$[e_3,e_4]=-\dfrac{BD-AC-\delta BF+\delta AE}{A}e_3$, $A\neq 0$,
$C^2+E^2\neq 0$, $(A+2E)^2+B^2+(C+\delta E)^2\neq 0$, $(A-E+\delta C)^2+B^2\neq 0$,
$AE+\delta(BD-AC)-BF+A^2\neq 0$.

(vi) $[e_1,e_2]=Ae_1-\delta Ae_2+Be_3$
$[e_1,e_4]=Ce_1-\delta Ce_2+De_3$
$[e_2,e_4]=Ee_1-\delta Ee_2+Fe_3$,
$[e_3,e_4]=-\dfrac{-BC-\delta AF+AD+\delta BE}{B}e_3$, $B\neq 0$,
$(4AD-2BC+\delta(E^2-B^2-C^2))^2+(4AF+B^2-2BE+E^2-C^2)^2\neq 0$,
$(AD-BC)^2+(EB-AF)^2\neq 0$,
$(B+E-\delta C)^2+(AEF+(\delta C-E)(C^2-E^2)-ACD)^2\neq 0$.

(vii) $[e_1,e_3]=\delta Ae_3$,
$[e_1,e_4]=Be_1-\delta Be_2+\delta Ce_3$, $[e_2,e_3]=Ae_3$,
$[e_2,e_4]=De_1-\delta De_2+Ce_3$, $[e_3,e_4]=Ee_3$, $AE(B^2+D^2)\neq 0$,

(viii) $[e_1,e_2]=Ae_1-\delta Ae_2$
$[e_1,e_4]=Be_1-\delta Be_2+\delta Ce_3$
$[e_2,e_4]=De_1-\delta De_2+Ce_3$, $[e_3,e_4]=Ee_3$, $A\neq 0$,
$C^2+E^2\neq 0$, $(B-\delta D-2E)^2+(AC-E(\delta D+E))^2\neq 0$,
$(AC-BE)^2+(D-\delta B)^2\neq 0$.

(ix) $[e_1,e_2]=Ae_1+\delta Ae_2+Be_3$,
$[e_1,e_4]=Ce_1+\delta Ce_2+\dfrac{CB-DB-\delta AE+\delta BF}{A}e_3$,
$[e_2,e_4]=Fe_1+\delta Fe_2+Ee_3$, $[e_3,e_4]=De_3$, $A\neq 0$,
$D^2+(AE-BF)^2\neq 0$, $(B+F+\delta(C-2D))^2+(AE-DC+\delta F(C-2D)+F^2+D^2)^2\neq 0$,
$(B+F+\delta C)^2+(AE+\delta CF-CD+F^2)^2\neq 0$.

(b3) *Either* $\mathbb{R} \ltimes E_2$ *or* $\mathbb{R} \ltimes E(1,1)$, *with* $\mathfrak{g}$ *being described by one of the following sets of conditions:*

(i) $[e_1,e_2] = -\frac{1}{2}Ae_1 + \frac{1}{2}\delta Ae_2 + Be_3$, $[e_1,e_3] = \delta Ae_3$,
$[e_1,e_4] = Ce_1 - \delta Ce_2 + \dfrac{2CB - 2DB + \delta AE - 2\delta BF}{A}e_3$,
$[e_2,e_3] = Ae_3$, $[e_2,e_4] = Fe_1 - \delta Fe_2 + Ee_3$, $[e_3,e_4] = De_3$, $A \neq 0$,
$(B + \delta C - F)^2 + (AE - 2C(C - D - \delta F))^2 \neq 0$.

(ii) $[e_1,e_2] = Ae_1 - \delta Ae_2 + Be_3$, $[e_1,e_3] = \delta Ce_3$,
$[e_1,e_4] = De_1 - \delta De_2 + Ee_3$, $[e_2,e_3] = Ce_3$,
$[e_2,e_4] = Fe_1 - \delta Fe_2 + Ge_3$,
$[e_3,e_4] = -\dfrac{EC - DB - \delta AG + AE - \delta CG + \delta BF}{B}e_3$, $ABC \neq 0$,
$(C + 2A)^2 + (ED - G(B + \delta D))^2 + (F - B - \delta D)^2 \neq 0$.

(iii) $[e_1,e_2] = Ae_1 - \delta Ae_2$, $[e_1,e_3] = \delta Be_3$,
$[e_1,e_4] = Ce_1 - \delta Ce_2 + \delta De_3$, $[e_2,e_3] = Be_3$,
$[e_2,e_4] = Ee_1 - \delta Ee_2 + De_3$, $[e_3,e_4] = Fe_3$, $AB \neq 0$,
$(B + 2A)^2 + (AD - CF)^2 + (E - \delta C)^2 \neq 0$.

(iv) $[e_1,e_2] = Ae_1 - \delta Ae_2$, $[e_1,e_3] = -\delta Ae_3$,
$[e_1,e_4] = -\delta Be_1 + Be_2 + Ce_3$, $[e_2,e_3] = -Ae_3$,
$[e_2,e_4] = Be_1 - \delta Be_2 + De_3$, $[e_3,e_4] = -\delta Ae_1 + Ae_2 + Ee_3$, $AE \neq 0$,
$A(C - \delta D) - EB - \delta B^2 \neq 0$.

(v) $[e_1,e_2] = -Ae_1 - \delta Ae_2$, $[e_1,e_3] = -\delta Ae_3$, $[e_1,e_4] = Be_3$,
$[e_2,e_3] = Ae_3$, $[e_2,e_4] = Ce_3$, $[e_3,e_4] = \delta Ae_1 + Ae_2 + De_3$, $A(B + C) \neq 0$,

(vi) $[e_1,e_2] = -Ae_1 - \delta Ae_2$, $[e_1,e_3] = -\delta Ae_3$,
$[e_1,e_4] = Be_1 + \delta Be_2 + Ce_3$, $[e_2,e_3] = Ae_3$,
$[e_2,e_4] = De_1 + \delta De_2 + Ee_3$, $[e_3,e_4] = Fe_1 + \delta Fe_2 + Ge_3$, $A \neq 0$,
$(A+\delta F)^2+G^2 \neq 0$, $(B-\delta D)^2+(AC+\delta(AE-D^2)+GD)^2+(F+\delta A)^2 \neq 0$,
$(A - \delta F)^2 + (BG - F(C - \delta E))^2 + (DG + F(\delta C - E))^2 \neq 0$.

In the cases listed above, $\varepsilon = \pm 1$ *and* $\delta = \pm 1$.

Proof. We start the proof by considering G to be a four-dimensional Lie group, and let g be a left-invariant neutral metric on that. By Lemma 2.2, the Lie algebra $\mathfrak{g}$ of G is a semi-direct product $\mathfrak{r} \ltimes \mathfrak{g}_3$, where $\mathfrak{r} = \mathrm{Span}\{e_4\}$ acts on $\mathfrak{g}_3 = \mathrm{Span}\{e_1,e_2,e_3\}$. So, the general form of the Lie algebra $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$ can be given by

$$\begin{aligned}
&[e_1,e_2] = a_1e_1 + a_2e_2 + a_3e_3, \quad &[e_1,e_3] = b_1e_1 + b_2e_2 + b_3e_3,\\
&[e_1,e_4] = c_1e_1 + c_2e_2 + c_3e_3, \quad &[e_2,e_3] = d_1e_1 + d_2e_2 + d_3e_3,\\
&[e_2,e_4] = p_1e_1 + p_2e_2 + p_3e_3, \quad &[e_3,e_4] = q_1e_1 + q_2e_2 + q_3e_3,
\end{aligned} \tag{3.1}$$

where the coefficients $a_i, \ldots, q_i$ are real constants. Validity of the Jacobi identity will be shown by

$$[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0, \quad i, j, k = 1, \ldots, 4, \tag{3.2}$$

for the Lie algebra $\mathfrak{g}$ of (3.1). Two cases (a) and (b) may occur for the inner product g on $\mathfrak{g}$. We will investigate them separately.

Case (a): We first choose a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with e_3 and e_4 time-like, then we apply the *Koszul formula* on the above Lie algebra $\mathfrak{g}$. By setting $\Lambda_i = \nabla_{e_i}$ for all indices $i = 1, \ldots, 4$; the components of the Levi-Civita connection are

$$\Lambda_1 = \begin{pmatrix} 0 & a_1 & b_1 & c_1 \\ -a_1 & 0 & \frac{1}{2}(b_2 + d_1 + a_3) & \frac{1}{2}(c_2 + p_1) \\ b_1 & \frac{1}{2}(b_2 + d_1 + a_3) & 0 & \frac{1}{2}(c_3 - q_1) \\ c_1 & \frac{1}{2}(c_2 + p_1) & \frac{1}{2}(q_1 - c_3) & 0 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} 0 & a_2 & \frac{1}{2}(d_1 + b_2 - a_3) & \frac{1}{2}(c_2 + p_1) \\ -a_2 & 0 & d_2 & p_2 \\ \frac{1}{2}(d_1 + b_2 - a_3) & d_2 & 0 & \frac{1}{2}(p_3 - q_2) \\ \frac{1}{2}(c_2 + p_1) & p_2 & \frac{1}{2}(q_2 - p_3) & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & \frac{1}{2}(b_2 - d_1 - a_3) & -b_3 & \frac{1}{2}(q_1 - c_3) \\ \frac{1}{2}(a_3 - b_2 + d_1) & 0 & -d_3 & \frac{1}{2}(q_2 - p_3) \\ -b_3 & -d_3 & 0 & q_3 \\ \frac{1}{2}(q_1 - c_3) & \frac{1}{2}(-p_3 + q_2) & -q_3 & 0 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0 & \frac{1}{2}(c_2 - p_1) & -\frac{1}{2}(q_1 + c_3) & 0 \\ \frac{1}{2}(p_1 - c_2) & 0 & -\frac{1}{2}(q_2 + p_3) & 0 \\ -\frac{1}{2}(q_1 + c_3) & -\frac{1}{2}(q_2 + p_3) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then for all indices i, j applying the Riemann curvature formula

$$R(e_i, e_j) = \Lambda_i \Lambda_j - \Lambda_j \Lambda_i - \Lambda_{[e_i, e_j]},$$

gives the curvature components $R_{ijkl} := g(R(e_i, e_j)e_k, e_h)$ (for all components of the curvature tensor, see [12]). Next, Equation (1.1) yields that

$$\begin{aligned} \mathbf{R}(R(e_i, e_j)e_k, e_l, e_m, e_n) &+ \mathbf{R}(e_k, R(e_i, e_j)e_l, e_m, e_n) \\ &+ \mathbf{R}(e_k, e_l, R(e_i, e_j)e_m, e_n) \\ &+ \mathbf{R}(e_k, e_l, e_m, R(e_i, e_j)e_n) = 0, \quad 1 \leqslant i, j, k, l, m, n \leqslant 4, \end{aligned} \tag{3.3}$$

where $\mathbf{R}$ is the $(0, 4)$-curvature tensor.

In order to classify semi-symmetric examples listed in the case (a) of the statement, it suffices to solve the algebraic systems (3.2) and (3.3) simultaneously; generally nonisometric solutions are not symmetric (since symmetric examples are trivially semi-symmetric), they are exactly those 14 sets of Lie algebras listed in the case (a) of the statement.

For cases (a1) (i)–(a1) (iv), we have $[\mathfrak{g}, \mathfrak{g}] = 0$; thus they correspond to the semi-direct products of the type $\mathbb{R} \ltimes \mathbb{R}^3$. On the other hand, in cases (a2) (i)–(a2) (iii), $[\mathfrak{g}, \mathfrak{g}]$ is one-dimensional and by comparison with the classification of three-dimensional Lorentzian Lie algebras given in [18], they correspond to $\mathbb{R} \ltimes H$. Finally, in cases (a3) (i)–(a3) (vii), $[\mathfrak{g}, \mathfrak{g}]$ is two-dimensional and they correspond to $\mathbb{R} \ltimes E_2$ or $\mathbb{R} \ltimes E(1,1)$.

Case (b): To state our results in this case, we use a method similar to that applied in the previous case, however, obviously, the nonzero components of the metric are $g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_4) = g(e_4, e_3) = 1$. Here we avoid bringing the Levi-Civita connection and curvature component, so for more information we can refer to [12], [16].

Combining (3.2) and (3.3) (again, we exclude symmetric examples), we obtain the nonisometric classification listed for the case (b) of the statement. By the study of the dimension of the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, we determine the correct class of the Lie algebras of case (b) and this ends the proof. □

4. Geometry of Semi-Symmetric four dimensional neutral Lie groups

Now, we can study the curvature properties of our classification on semi-symmetric four dimensional Lie groups. For this purpose, we consider in each of the classes listed in Theorem 3.1 the tensor components with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$, with e_3 and e_4 time-like being used to describe the Lie algebra $\mathfrak{g}$. The Ricci parallel condition means that the covariant derivative of the Ricci tensor vanishes identically. Clearly, each Ricci flat manifold (i.e., a manifold on which the Ricci tensor vanishes) is Einstein (i.e., $\varrho = \mu g$), and every Einstein manifold is Ricci parallel.

The concept of conformal flatness is deduced by the following system of algebraic equations:

$$(4.1)\quad W_{ijkh} = R_{ijkh} + \tfrac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{ih}\varrho_{jk} - g_{jk}\varrho_{ih}) - \tfrac{1}{6}r(g_{ik}g_{jh} - g_{ih}g_{jk}) = 0,$$

for all indices $i, j, k, h = 1, \dots, 4$, where W and r denote the Weyl tensor and the scalar curvature, respectively.

Let us now study the case (a1) (i) in Theorem 3.1. The nonzero components of the Ricci tensor are (assuming $\delta = -1$)

$$\varrho_{22} = -\varrho_{23} = \varrho_{33} = 2(AB - A^2),$$

it is clear that this case will be Ricci flat if and only if

$$A = 0 \quad \text{or} \quad A = B,$$

and also is Ricci parallel, i.e., $\nabla \varrho = 0$, if and only if

$$A = 0 \quad \text{or} \quad A = B.$$

But these solutions satisfy the Einstein equation; therefore, there is no non-Einstein Ricci parallel solution for this case. Based on a straightforward computation using the components of the metric tensor, the curvature tensor, the Ricci tensor, and by the fact that the scalar curvature vanishes in our case, we will have the following nonzero components for the Weyl tensor:

$$\begin{aligned} &W_{1212} = W_{1313} = W_{2424} = W_{3434} = A^2 - AB, \\ &W_{1213} = W_{2434} = -A^2 + AB. \end{aligned}$$

So, the Weyl tensor vanishes if and only if

$$A = 0 \quad \text{or} \quad A = B.$$

Now (again by the assumption $\delta = -1$), we consider another case, for example (b1) (ii), so, the nonzero components of the Ricci tensor are

$$\begin{aligned} &\varrho_{11} = \varrho_{12} = \varrho_{22} = -\tfrac{1}{2}E^2, \\ &\varrho_{14} = \varrho_{24} = \tfrac{1}{2}(EC - AE), \\ &\varrho_{44} = \ - AF - EB + AC + FC + ED - \tfrac{1}{2}(A^2 + C^2). \end{aligned}$$

To have the Ricci flat example, these components must be zero. The resulting system has two sets of solutions:

$$A = C, \ \ E = 0, \quad \text{or} \quad A = C - 2F, \ \ E = 0.$$

Also, the Ricci parallel condition gives a system of algebraic equations with four sets of solutions, e.g.,

$$E = 0, \quad F = 0.$$

But, if $A = C$, the corresponding Lie algebra is Einstein, so we exclude this condition to have a non-Einstein example. For the Weyl tensor, the only nonzero components are

$$W_{1414} = W_{1424} = W_{2424} = -\tfrac{1}{2}(FA + FC + BE + ED).$$

One solution for the above equation is

$$E = 0, \quad F = 0.$$

According to (4.1), by using the information related to metrics, curvature and Ricci tensors for the semi-symmetric examples in Theorem 3.1, an immediate result is:

Theorem 4.1. *Let (G, g) be one of the Lie groups listed in Theorem* 3.1*. Ricci flat, non-Einstein Ricci parallel and conformally flat examples are listed in Table* 1.

Remark 4.2. In Table 1, one can easily show that there is non-Einstein example which is not Ricci flat, for all cases of Theorem 3.1.

5. Ricci soliton examples

A complete classification of semi-symmetric four dimensional neutral Lie groups was given throughout Theorem 3.1. Based on this classification, we will study Ricci solitons in this section. This consideration will complete the study of Ricci solitons in [10], where the authors study the Ricci solitons on the four dimensional homogeneous spaces with nontrivial isotropy. A pseudo-Riemannian manifold (M, g) is called a *Ricci soliton* if it admits a smooth vector field X such that

$$\mathcal{L}_X g + \varrho = \lambda g, \tag{5.1}$$

where $\mathcal{L}_X$ and ϱ denote the Lie derivative in the direction of X and the Ricci tensor, respectively, and λ is an arbitrary real number. Depending on the value of λ, a Ricci soliton is called *shrinking, steady*, or*expanding* according to whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

Ricci solitons play an important role in understanding the singularities of the Ricci flow, since they are self-similar solutions. A survey and further references on the geometry of Ricci solitons may be found in [13]. Ricci solitons have been the subject of several studies on the homogeneous manifolds (see for example [9], [14], [16]).

case	Ricci flat	Ricci parallel non-Einstein	Conformally flat
(a1) (ii)	$A+B=0$	$A+B\neq 0$	$\times$
(a2) (i)	$A\pm\sqrt{2}B=0$	$A\pm\sqrt{2}B\neq 0$	$\times$
(a2) (ii)	$B\pm\sqrt{2}A=0$	$B\pm\sqrt{2}A\neq 0$	$\times$
(a2) (iii)	$A\pm\sqrt{2}B=0$	$A\pm\sqrt{2}B\neq 0$	$\times$
(a3) (i)	$2C^2-B^2+2A^2=0$	$2C^2-B^2+2A^2\neq 0$	$\times$
(a3) (ii)	$B\pm 2A=0$	$B\pm 2A\neq 0$	$\times$
(a3) (iv)	$B\pm 2A=0$	$B\pm 2A\neq 0$	$\times$
(a3) (v),(vi)	$A\pm 2B=0$	$A\pm 2B\neq 0$	$\times$
(a3) (vii)	$2A^2(B^2+C^2)$ $-B^2(D^2-2B^2)=0$	$2A^2(B^2+C^2)$ $-B^2(D^2-2B^2)\neq 0$	$\times$
(b1) (i)	$2G(A+E)+(B-D)^2$ $-2(A^2+E^2)=0$	$G=0,$ $2G(A+E)+(B-D)^2$ $-2(A^2+E^2)\neq 0$	$E-A+\varepsilon(B-D)$ $=G+2(\varepsilon B-A)=0,$ $D+B=G-E-A=0,$ $D-B=E-A=0,\ \varepsilon=\pm 1$
(b1) (ii)	$A+\delta(C-2F)=E=0,$ $(A+\delta C)=E=0$	$AF-ED-\delta(EB-CF)=0,$ $E\neq 0$	$AF+ED-\delta(EB+CF)=0$
(b2) (i)	$\times$	$C-\delta F=0$	$C+\delta F=0$
(b2) (ii)	$A^2+4B(B-C)-(F-D)^2=0$	$C=0,$ $A^2-(F-D)^2+4B^2\neq 0$	$C-2B=F-D=0,$ $A-F-D=0$
(b2) (iii)	$\times$	$\checkmark$	$\times$
(b2) (iv)	$\sqrt{2}A\pm B=0$	$\sqrt{2}A\pm B\neq 0$	$\times$
(b2) (v)	$A-\delta C+E=B=0$	$C-\delta E=0$	$C+\delta E=0$
(b2) (vi)	$B-\delta C+E=0,$ $BC+2A(\delta F-D)$ $+\delta B(B-E)=0$	$BC-AD+\delta(AF-BE)=0,$ $AD-\delta(AF+B^2)\neq 0$	$AD(B+C+E)-B^2C-ACF$ $+\delta(AF-BE)(B-E)$ $-\delta C^2B=0$
(b2) (vii)	$\times$	$B-\delta D=0$	$B+\delta D=0$
(b2) (viii)	$B-\delta D=0,$ $B-\delta D-2E=0$	$E=0,$ $B-\delta D\neq 0$	$2AC-E(B+\delta D)=0$
(b2) (ix)	$B+\delta C+F=0,$ $B+\delta(C-2D)+F=0$	$D=0,$ $B+\delta C+F\neq 0$	$2AE+\delta D(B+F)$ $-CD-2BF=0$
(b3) (i)	$B+\delta C-F=0$	$\times$	$AE+CD$ $+\delta(2BC+DF-DB)=0$
(b3) (ii)	$C+2A=F-B-\delta D=0$	$\times$	$D+\delta F=G+\delta E=0$
(b3) (iii)	$B+2A=C-\delta E=0$	$\times$	$2AD-F(C+\delta E)=0$
(b3) (iv)	$\times$	$\times$	$\checkmark$
(b3) (v)	$\checkmark$	$\times$	$\times$
(b3) (vi)	$A-\delta F=B+\delta D=0$	$\times$	$BG+AE$ $-\delta(CA+GD-EF)-CF=0$

Table 1. Ricci flat, non-Einstein Ricci parallel, conformally flat examples of Theorem 3.1. Note: ✓ in the column "Ricci parallel non-Einstein" of table, means the corresponding space is always Ricci parallel.

Clearly, Einstein manifolds are trivially Ricci solitons, so here we just report non-Einstein Ricci soliton examples of Theorems 3.1. Therefore, we treat all cases in Theorem 3.1 for possible Ricci soliton examples. For instance, we consider the Lie algebra in the case (a1) (ii) of Theorem 3.1; the Levi-Civita connection with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with e_3 and e_4 time-like is

$$\Lambda_1 = \Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(A-B) \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(A-B) & 0 & 0 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}(A-B) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(B-A) \\ \frac{1}{2}(A-B) & 0 & \frac{1}{2}(A-B) & 0 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0 & \frac{1}{2}(A+B) & 0 & 0 \\ -\frac{1}{2}(A+B) & 0 & -\frac{1}{2}(A+B) & 0 \\ 0 & -\frac{1}{2}(A+B) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, we have the nonzero components of the Ricci tensor with respect to the basis $\{e_1, e_2, e_3, e_4\}$ as

$$\varrho_{11} = \varrho_{13} = \varrho_{33} = \tfrac{1}{2}(A^2 - B^2).$$

Let $X = X_1e_1 + X_2e_2 + X_3e_3 + X_4e_4$, for arbitrary real coefficients $X_1, \dots, X_4$, be an arbitrary vector filed on G. By direct calculation, one can easily obtain the Lie derivative of g as

$$\mathcal{L}_X g = \begin{pmatrix} 0 & X_4(A-B) & 0 & X_2B \\ X_4(A-B) & 0 & X_4(A-B) & -X_1A - X_3A \\ 0 & X_4(A-B) & 0 & X_2B \\ X_2B & -X_1A - X_3A & X_2B & 0 \end{pmatrix}.$$

Now, (5.1) gives the system of equations

$$\begin{cases} \lambda = X_2B = 0, \\ X_4(A-B) = 0, \\ A^2 - B^2 = 0, \\ -X_1A - X_3A = 0, \\ A^2 - B^2 - 2\lambda = 0, \\ A^2 - B^2 + 2\lambda = 0. \end{cases}$$

So the possible solution is $A + \varepsilon B = \lambda = 0$, which is Einstein, and there is no non-Einstein Ricci soliton example in this case.

Let us now consider, for example, the Lie algebra in the case (b1) (i) of Theorem 3.1, so, the Levi-Civita connection with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with e_3 and e_4 time-like is

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & A \\ 0 & 0 & 0 & \frac{1}{2}(B-D) \\ -A & \frac{1}{2}(B-D) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}(-B+D) \\ 0 & 0 & 0 & E \\ \frac{1}{2}(B-D) & E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0 & -\frac{1}{2}(D+B) & 0 & C \\ -\frac{1}{2}(D+B) & 0 & 0 & -F \\ -C & -F & -G & 0 \\ 0 & 0 & 0 & G \end{pmatrix},$$

and $\Lambda_3 = 0$. The nonzero components of the Ricci tensor with respect to the basis $\{e_1, e_2, e_3, e_4\}$ are:

$$\varrho_{44} = AG - A^2 - DB + EG - E^2 + \tfrac{1}{2}(D^2 + B^2).$$

For an arbitrary vector filed X, one can easily obtain the Lie derivative of g by direct calculation as

$$\mathcal{L}_X g = \begin{pmatrix} 2X_4A & -X_4(-D+B) & 0 & -X_1A - X_2D + X_4C \\ -X_4(-D+B) & -2X_4E & 0 & X_1B + X_2E + X_4F \\ 0 & 0 & 0 & X_4G \\ -X_1A-X_2D+X_4C & X_1B+X_2E+X_4F & X_4G & -2X_1C-2X_2F-2X_3G \end{pmatrix}.$$

Now, (5.1) gives the system of equations

$$\begin{cases} -X_4(-D+B) = 0, \\ 2X_4A - \lambda = 0, \\ -2\,X_4E + \lambda = 0, \\ X_4G - \lambda = 0, \\ -X_1A - X_2D + X_4C = 0, \\ X_1B + X_2E + X_4F = 0, \\ -2X_1C - 2X_2F - 2X_3G + GA + \frac{1}{2}D^2 + \frac{1}{2}B^2 - A^2 - DB + GE - E^2 = 0. \end{cases}$$

These equations have the set of exactly seven solutions, e.g., one nontrivial solution is

$$A = B = X_2 = X_4 = \lambda = 0,$$
$$X_1 = \frac{-4X_3G + D^2 + 2GE - 2E^2}{4C}.$$

Recall that Einstein manifolds are trivially Ricci solitons. In this case, the Einstein equation is satisfied if and only if

$$G = \frac{D^2 - 2E^2}{-2E},$$

so we exclude this trivial Ricci soliton. By the same method, we can consider Ricci solitons for all cases appearing in Theorems 3.1. The results are summarized in Theorem 5.1.

Theorem 5.1. *Let* (G, g) *be a four-dimensional semi-symmetric neutral Lie group of Theorem* 3.1. *Then* (G, g) *is a Ricci soliton if one of the following cases in Table* 2 *occurs.*

Lie algebra	Ricci soliton non-Einstein
(a1) (i)	$\lambda = X_2 = X_3 = X_4 + A + \delta = 0$
(a1) (ii)	$\times$
(a1) (iii)	$A - 2\delta X_4 = \lambda = X_1 = X_3 = 0,\ X_4 \neq 0$
(a1) (iv)	$A - \varepsilon\sqrt{2}X_4 = \lambda = X_1 + X_2 = X_3 = 0,\ X_4 \neq 0,\ \delta = -1$
(a3) (i)	$\times$
(a3) (ii)	$\lambda = X_1 = 4AX_2 + 4A^2 + 2\delta X_4 B - B^2 = X_3 = 0,\ B \neq \pm 2A$
(a3) (iii)	$\lambda = X_3 = 0,\ X_1 = -B,\ X_2 = -A$
(a3) (iv)	$X_1 = -X_2 = \dfrac{\varepsilon\sqrt{2}BX_4 + 4A^2 - B^2}{4A},\ \lambda = X_3 = 0,\ B \pm 2A \neq 0$
(b1) (i)	$A = B = X_2 = X_4 = \lambda = 0,$ $X_1 = \dfrac{-4X_3G + D^2 + 2GE - 2E^2}{4C},\ 2GE + D^2 - 2E^2 \neq 0$
(b1) (ii)	$A = E = X_2 = X_4 = \lambda = 0,$ $X_1 = -\dfrac{4X_3F + C^2 - 2FC}{4B},\ C(C - 2F) \neq 0$
(b2) (i)	$A + \delta D = F - \delta C = \lambda = 0,\ X_2 = \dfrac{X_4D - \delta X_1 B}{B},$ $X_3 = -\dfrac{BD + 4\delta X_1 D + 4X_4E}{4B},\ X_4 = \dfrac{B^2}{4C},\ BD \neq 0$

Table 2. Ricci soliton examples of Theorem 3.1 (continued).

Lie algebra	Ricci soliton non-Einstein
(b2) (ii)	$A-F=B=X_1=X_4=\lambda=0,$ $X_2=\dfrac{4X_3C-D^2+2FD}{-4G},\ D(D-2F)\neq 0$
(b2) (iii), (iv)	$\times$
(b2) (v)	$B=\lambda=X_1=X_2=X_4=0,$ $X_3=-\dfrac{A^2-E^2+2\delta CE-C^2}{4(C-\delta E)},\ A\pm(\delta C-E)\neq 0$
(b2) (vi)	$A=\lambda=X_1=X_2=X_4=0,$ $X_3=-\dfrac{B^2-E^2-C^2-2\delta CE}{4(C-\delta E)},\ B\pm(\delta C-E)\neq 0$
(b2) (vii)	$B-\delta D=E=\lambda=CX_4+AX_3=0,\ X_4=\dfrac{A^2}{4B},\ A\neq 0$
(b2) (viii)	$\lambda=X_1=X_2=X_4=0,$ $X_3=-\dfrac{B^2-2EB+\delta(2ED-2BD)+D^2}{4B},\ (B-\delta D)(B-\delta D-2E)\neq 0$
(b2) (ix)	$\lambda=X_1=X_2=X_4=0,$ $X_3=-\dfrac{B^2+(C+\delta F)^2-2(D-\delta B)(C+\delta F)-\delta 2DB}{4D},$ $(B+\delta C+F)(B+\delta C+F-2\delta D)\neq 0$
(b3) (i)	$\lambda=\dfrac{A^2(B+C-F)^2}{4((C-B+F)D+2CB+AE)},$ $X_1=\varphi(A(B+C-F)(2C-D)),\ X_2=-\varphi(A(B+C-F)(D+2F)),$ $X_3=\varphi(2C(B-C)^2+2(F-B)(F+D)^2+3AE(F-C+\frac{1}{3}(2D+B))$ $+6BC(F+D)+2C(D^2-F^2)+D(F^2-B^2)-C^2(5D+2F)),$ $X_4=-\varphi A^2(B+C-F),$ $\varphi=-\dfrac{(B+C-F)}{4(2C-2F-D)(2BC+CD-BD+AE+DF)},$ $B-F-C\neq 0\quad\text{for } \delta=1$
(b3) (ii)	$G=E=B-F-\delta D=X_3=0,\ X_1=\dfrac{-\delta(C^2+2AC)+2\delta X_4D+2X_4F}{4A},$ $X_2=\dfrac{2AC-2X_4D-2\delta X_4F+C^2}{4A},\ X_4=\dfrac{\lambda}{D-\delta F},\ C(2A+C)\neq 0$
(b3) (iii)	$C-\delta E=\lambda=X_3=X_4=0,\ -\delta X_1=X_2=\dfrac{B(2A+B)}{4A},\ B+2A\neq 0$
(b3) (iv)	$X_4=0,\ \delta X_1=X_2=\dfrac{\lambda}{2A},$ $X_3=\dfrac{-X_2(C+\delta D)-\delta AD+AC-EB-\delta B^2}{\delta E},\ A(C-\delta D)-\delta EB-B^2\neq 0$

Table 2. Ricci soliton examples of Theorem 3.1 (continued).

Lie algebra	Ricci soliton non-Einstein
(b3) (v)	$B-\delta C=X_3=X_4=0,\ \delta X_1=X_2=-\dfrac{\lambda}{2A}\neq 0$
(b3) (vi)	$A-\delta F=B+\delta D=C-\delta E=G=0,\ X_1=-\delta X_2=\dfrac{\lambda}{2F},$ $X_3=-\dfrac{X_2D}{F},\ F\neq 0$

Table 2. Ricci soliton examples of Theorem 3.1.

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