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ON DUAL RAMSEY THEOREMS FOR RELATIONAL STRUCTURES

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Abstract. We discuss dual Ramsey statements for several classes of finite relational structures (such as finite linearly ordered graphs, finite linearly ordered metric spaces and finite posets with a linear extension) and conclude the paper with another rendering of the Nešetřil-Rödl Theorem for relational structures. Instead of embeddings which are crucial for "direct" Ramsey results, for each class of structures under consideration we propose a special class of quotient maps and prove a dual Ramsey theorem in such a setting. Although our methods are based on reinterpreting the (dual) Ramsey property in the language of category theory, all our results are about classes of finite structures.

Keywords: dual Ramsey property; finite relational structure; category theory

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1. Introduction

Generalizing the classical results of Ramsey from the late 1920's, the structural Ramsey theory originated at the beginning of the 1970's in a series of papers, see [8]. We say that a class K of finite structures has the *Ramsey property* if the following holds: for any number $k \geqslant 2$ of colors and all $\mathcal{A}, \mathcal{B} \in K$ such that $\mathcal{A}$ embeds into $\mathcal{B}$ there is a $\mathcal{C} \in K$ such that no matter how we color the copies of $\mathcal{A}$ in $\mathcal{C}$ with k colors, there is a *monochromatic* copy $\mathcal{B}'$ of $\mathcal{B}$ in $\mathcal{C}$ (that is, all the copies of $\mathcal{A}$ that fall within $\mathcal{B}'$ are colored by the same color). In this parlance the Finite Ramsey Theorem takes the following form:

Theorem 1.1 (Finite Ramsey Theorem [17]). *The class of all finite chains has the Ramsey property.*

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Many natural classes of structures expanded with linear orders have the Ramsey property. For example, the class of all finite linearly ordered graphs $(V, E, \sqsubset)$, where (V, E) is a finite graph and $\sqsubset$ is a linear order on the set V of vertices of the graph has the Ramsey property, see [1], [10]. The same is true for metric spaces, see [9]. In case of finite posets we consider the class of all finite linearly ordered posets $(P, \preceq, \sqsubset)$, where $(P, \preceq)$ is a finite poset and $\sqsubset$ is a linear order on P which extends $\preceq$, see [10]. One of the cornerstones of the structural Ramsey theory is the Nešetřil-Rödl Theorem.

Theorem 1.2 (Nešetřil-Rödl Theorem [1], [10], [12]). *The class of all finite linearly ordered relational structures all having the same, fixed, relational type has the Ramsey property.*[1]

The fact that this result has been proved independently by several research teams, and then reproved in various ways and in various contexts, see [1], [12], [13], [14], clearly demonstrates the importance and justifies the distinguished status this result has in discrete mathematics. The search for a dual version of the Nešetřil-Rödl Theorem was and still is an important research direction and several versions of the dual of the Nešetřil-Rödl Theorem have been published, most notably by Spencer in [20], Prömel in [14], Prömel and Voigt in [16], Frankl, Graham, Rödl in [3] and recently by Solecki in [19].

In this paper we are interested in dual Ramsey statements for classes of finite relational structures and our primary source of motivation is the Finite Dual Ramsey Theorem of Graham and Rothschild, see [4].

Theorem 1.3 (Finite Dual Ramsey Theorem [4], [11]). *For all positive integers k, a, m there is a positive integer n such that for every n-element set C and every k-coloring of the set $\begin{bmatrix} C \\ a \end{bmatrix}$ of all partitions of C with exactly a blocks there is a partition β of C with exactly m blocks such that the set of all partitions from $\begin{bmatrix} C \\ a \end{bmatrix}$ which are coarser than β is monochromatic.*

Since each partition of a finite linearly ordered set can be uniquely represented by the rigid surjection which takes each element of the underlying set to the minimum of the block it belongs to (see Subsection 2.1 for the definition of a rigid surjection), Finite Dual Ramsey Theorem is a structural Ramsey result about finite chains and special surjections between them. The major insight here is that instead of embeddings which are crucial for "direct" Ramsey results, special classes of surjective maps play the key role in proving dual Ramsey results.

[1] Note that this is a restricted version of the Nešetřil-Rödl Theorem which does not account for subclasses defined by forbidden substructures.

The basic setup of this paper relates strongly to [14], where the Nešetřil-Rödl Theorem is interpreted in the language of category theory using the concept of indexed categories. The main result of [14] is the partition theorem for combinatorial cubes. In this sense it can be considered as a dual of the Nešetřil-Rödl Theorem (without forbidden substructures): objects are combinatorial cubes with selected combinatorial subspaces and morphisms preserve the types of the selected subspaces. In this paper, however, we consider a dual of the Nešetřil-Rödl Theorem spelled out in the language of relational structures and base our approach on [7] which can be thought of as a simplified version of the approach taken in [14].

In Section 2 we give a brief overview of standard notions referring to linear orders, total quasiorders and first order structures, and prove several technical results.

In Section 3 we provide basics of category theory and give a categorical reinterpretation of the Ramsey property. We define the Ramsey property and the dual Ramsey property for a category and illustrate these notions using some well-known examples. As our concluding example we prove a dual Ramsey theorem for the category of finite linearly ordered metric spaces and nonexpansive rigid surjections.

In Section 4 we prove dual Ramsey theorems for the following categories: the category $\mathrm{EDig}_{\mathrm{srq}}$ whose objects are finite reflexive digraphs with linear extensions and morphisms are special rigid quotient maps, the category $\mathrm{EPos}_{\mathrm{srq}}$ whose objects are finite posets with linear extensions and morphisms are special rigid quotient maps, the category $\mathrm{OHgr}_{\mathrm{srq}}(r)$, $r \geqslant 2$, whose objects are finite linearly ordered reflexive r-uniform hypergraphs and morphisms are special rigid quotient maps, the category $\mathrm{OGra}_{\mathrm{srq}}$ whose objects are finite linearly ordered reflexive graphs and morphisms are special rigid quotient maps, and a few more subcategories of $\mathrm{OGra}_{\mathrm{srq}}$.

In section 5 we prove that the class of all finite linearly ordered relational structures all having the same, fixed, relational type has the dual Ramsey property with respect to a special class of rigid quotient maps. Note again that this is a restricted formulation of the Nešetřil-Rödl Theorem which does not account for subclasses defined by forbidden "quotients".

The paper concludes with Section 6, where we prove that the category of finite linearly ordered reflexive tournaments and rigid surjective homomorphisms does not have the dual Ramsey property.

2. Preliminaries

In order to fix notation and terminology, in this section we give a brief overview of standard notions referring to linear orders, total quasiorders, first-order structures and category theory.

2.1. Linear orders. A *chain* is a pair $\mathcal{A} = (A, \sqsubset)$, where $\sqsubset$ is a linear order on A. In case A is finite we simply write $\mathcal{A} = \{a_1 \sqsubset a_2 \sqsubset \ldots \sqsubset a_n\}$. Following [15] we say that a surjection $f\colon \{a_1 \sqsubset a_2 \sqsubset \ldots \sqsubset a_n\} \to \{b_1 \sqsubset b_2 \sqsubset \ldots \sqsubset b_k\}$ between two finite chains is *rigid* if

$$\min f^{-1}(x) \sqsubset \min f^{-1}(y) \quad \text{whenever } x \sqsubset y.$$

Equivalently, a rigid surjection maps each initial segment of $\{a_1 \sqsubset a_2 \sqsubset \ldots \sqsubset a_n\}$ onto an initial segment of $\{b_1 \sqsubset b_2 \sqsubset \ldots \sqsubset b_k\}$; other than that, a rigid surjection is not required to respect the linear orders in question.

Let $(A_i, \sqsubset_i)$ be finite chains, $1 \leqslant i \leqslant k$. The linear orders $\sqsubset_i$, $1 \leqslant i \leqslant k$, induce the *anti-lexicographic* order $\sqsubset_{\text{alex}}$ on $A_1 \times \ldots \times A_k$ by

$$(a_1, \ldots, a_k) \sqsubset_{\text{alex}} (b_1, \ldots, b_k) \quad \text{if and only if} \quad \text{there is an } s \in \{1, \ldots, k\}$$
$$\text{such that } a_s \sqsubset_s b_s, \text{ and } a_j = b_j \text{ for all } j > s.$$

In particular, every finite chain $(A, \sqsubset)$ induces the anti-lexicographic order $\sqsubset_{\text{alex}}$ on A^n, $n \geqslant 2$. Moreover, every linear order $\sqsubset$ on A induces the *anti-lexicographic* order $\sqsubset_{\text{alex}}$ on $\mathcal{P}(A)$ as follows. For $X \in \mathcal{P}(A)$ let $\vec{X} \in \{0,1\}^{|A|}$ denote the characteristic vector of X. (As A is linearly ordered, we can assign a string of 0's and 1's to each subset of A.) Then for $X, Y \in \mathcal{P}(A)$ we let

$$X \sqsubset_{\text{alex}} Y \quad \text{if and only if} \quad \vec{X} <_{\text{alex}} \vec{Y},$$

where $<$ is the usual ordering $0 < 1$. It is easy to see that for $X, Y \in \mathcal{P}(A)$:

$$X \sqsubset_{\text{alex}} Y \quad \text{if and only if} \quad X \subset Y, \text{ or } \max_{\mathcal{A}}(X \setminus Y) \sqsubset \max_{\mathcal{A}}(Y \setminus X)$$
$$\text{in case } X \text{ and } Y \text{ are incomparable.}$$

2.2. Total quasiorders. A *total quasiorder* is a reflexive and transitive binary relation such that each pair of elements of the underlying set is comparable. Each total quasiorder σ on a set I induces an equivalence relation $\equiv_\sigma$ on I and a linear order $\prec_\sigma$ on $I/\equiv_\sigma$ in a natural way: $i \equiv_\sigma j$ if $(i,j) \in \sigma$ and $(j,i) \in \sigma$, while $(i/\equiv_\sigma) \prec_\sigma (j/\equiv_\sigma)$ if $(i,j) \in \sigma$ and $(j,i) \notin \sigma$.

For the considerations that follow we need to linearly order all the total quasiorders on the same set and have to discuss functions that in a way preserve certain properties of quasiorders. The rest of this subsection is technical.

Definition 2.1. Let σ and τ be two distinct total quasiorders on $I = \{1, 2, \ldots, r\}$. Let

$$\begin{aligned} I/\equiv_\sigma &= \{S_1 <_{\text{alex}} S_2 <_{\text{alex}} \ldots <_{\text{alex}} S_k\}, \\ I/\equiv_\tau &= \{T_1 <_{\text{alex}} T_2 <_{\text{alex}} \ldots <_{\text{alex}} T_l\}. \end{aligned}$$

(Here, $<_{\text{alex}}$ stands for the anti-lexicographic ordering of $\mathcal{P}(\{1, 2, \ldots, r\})$ induced by the usual ordering of the integers.)

We put $\sigma \lhd \tau$ if $k < l$, or $k = l$ and $(S_1, S_2, \ldots, S_k)$ $(<_{\text{alex}})_{\text{alex}}$ $(T_1, T_2, \ldots, T_k)$. (Here, $(<_{\text{alex}})_{\text{alex}}$ denotes the anti-lexicographic ordering on $\mathcal{P}(\{1, 2, \ldots, r\})^k$ induced by $<_{\text{alex}}$ on $\mathcal{P}(\{1, 2, \ldots, r\})$.)

Let $(A, \sqsubset)$ be a linearly ordered set, let r be a positive integer, let $I = \{1, \ldots, r\}$ and let $\bar{a} = (a_1, \ldots, a_r) \in A^r$. Then

$$\operatorname{tp}(\bar{a}) = \{(i, j) \colon\ a_i \sqsubseteq a_j\}$$

is a total quasiorder on I which we refer to as the *type* of $\bar{a}$. Assume that $\sigma = \operatorname{tp}(\bar{a})$. Let $s = |I/\equiv_\sigma|$ and let $i_1, \ldots, i_s$ be the representatives of the classes of $\equiv_\sigma$ enumerated so that $(i_1/\equiv_\sigma) \prec_\sigma \ldots \prec_\sigma (i_s/\equiv_\sigma)$. Then

$$\operatorname{mat}(\bar{a}) = (a_{i_1}, \ldots, a_{i_s})$$

is the *matrix of* $\bar{a}$. Note that $a_{i_1} \sqsubset \ldots \sqsubset a_{i_s}$.

For a total quasiorder σ on I such that $|I/\equiv_\sigma| = s$ and an arbitrary s-tuple $\bar{b} = (b_1, \ldots, b_s) \in A^s$ define an r-tuple

$$\operatorname{tup}(\sigma, \bar{b}) = (a_1, \ldots, a_r) \in A^r$$

as follows. Let $i_1, \ldots, i_s$ be the representatives of the classes of $\equiv_\sigma$ enumerated so that $(i_1/\equiv_\sigma) \prec_\sigma \ldots \prec_\sigma (i_s/\equiv_\sigma)$. Then put

$$a_\eta = b_\xi \quad \text{if and only if} \quad \eta \equiv_\sigma i_\xi.$$

(In other words, we put b_1 on all the entries in $i_1/\equiv_\sigma$, we put b_2 on all the entries in $i_2/\equiv_\sigma$, and so on.)

It is a matter of routine to check that for every tuple $\bar{a}$ and every tuple $\bar{b} = (b_1, b_2, \ldots, b_s)$ such that $b_1 \sqsubset b_2 \sqsubset \ldots \sqsubset b_s$ we have

$$\text{(2.1)} \qquad \operatorname{tp}(\operatorname{tup}(\sigma, \bar{b})) = \sigma, \quad \operatorname{mat}(\operatorname{tup}(\sigma, \bar{b})) = \bar{b}, \quad \text{and} \quad \operatorname{tup}(\operatorname{tp}(\bar{a}), \operatorname{mat}(\bar{a})) = \bar{a}.$$

Definition 2.2. Let $(A, \sqsubset)$ be a finite chain and let $n \geqslant 2$. Define the linear order $\sqsubset_{\text{sal}}$ on A^n as follows ("sal" in the subscript stands for "special anti-lexicographic"). Take any $\bar{a}, \bar{b} \in A^n$, where $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$.

▷ If $\mathrm{tp}(\bar{a}) \lhd \mathrm{tp}(\bar{b})$ then $\bar{a} \sqsubset_{\mathrm{sal}} \bar{b}$.

▷ If $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ and $\{a_1, a_2, \ldots, a_n\} \neq \{b_1, b_2, \ldots, b_n\}$ then $\bar{a} \sqsubset_{\mathrm{sal}} \bar{b}$ if and only if $\{a_1, a_2, \ldots, a_n\} \sqsubset_{\mathrm{alex}} \{b_1, b_2, \ldots, b_n\}$.

(Note that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ and $\{a_1, a_2, \ldots, a_n\} = \{b_1, b_2, \ldots, b_n\}$ imply $\bar{a} = \bar{b}$.)

Lemma 2.3. *Let $(A, \sqsubset)$ be a finite chain and let $n \geqslant 2$ be an integer.*

(a) *For all $\bar{a}, \bar{b} \in A^n$ we have that $\bar{a} = \bar{b}$ if and only if $\mathrm{mat}(\bar{a}) = \mathrm{mat}(\bar{b})$ and $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$.*

(b) *Assume that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ for some $\bar{a}, \bar{b} \in A^n$. Then $\bar{a} \sqsubset_{\mathrm{sal}} \bar{b}$ if and only if $\mathrm{mat}(\bar{a}) \sqsubset_{\mathrm{sal}} \mathrm{mat}(\bar{b})$.*

(c) *Assume that $\mathrm{tp}(\bar{a}^1) = \mathrm{tp}(\bar{a}^2) = \ldots = \mathrm{tp}(\bar{a}^k)$ for some $\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^k \in A^n$. Then $\min\{\mathrm{mat}(\bar{a}^1), \mathrm{mat}(\bar{a}^2), \ldots, \mathrm{mat}(\bar{a}^k)\} = \mathrm{mat}(\min\{\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^k\})$, where both minima are taken with respect to $\sqsubset_{\mathrm{sal}}$.*

P r o o f. Case (a) is obvious, and case (c) follows directly from case (b). So, let us show case (b). Let $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$. If $\{a_1, a_2, \ldots, a_n\} = \{b_1, b_2, \ldots, b_n\}$ then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ implies $\bar{a} = \bar{b}$. Assume that $\{a_1, a_2, \ldots, a_n\} \neq \{b_1, b_2, \ldots, b_n\}$. Let $\mathrm{mat}(\bar{a}) = (a_{i_1}, a_{i_2}, \ldots, a_{i_r})$ for some indices $i_1, i_2, \ldots, i_r$. Then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ implies that $\mathrm{mat}(\bar{b}) = (b_{i_1}, b_{i_2}, \ldots, b_{i_r})$. Note, also, that $\{a_1, a_2, \ldots, a_n\} = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ and that $\{b_1, b_2, \ldots, b_n\} = \{b_{i_1}, b_{i_2}, \ldots, b_{i_r}\}$. Therefore, $\bar{a} \sqsubset_{\mathrm{sal}} \bar{b}$ if and only if $\{a_1, a_2, \ldots, a_n\} \sqsubset_{\mathrm{alex}} \{b_1, b_2, \ldots, b_n\}$ if and only if $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \sqsubset_{\mathrm{alex}} \{b_{i_1}, b_{i_2}, \ldots, b_{i_r}\}$ if and only if $\mathrm{mat}(\bar{a}) \sqsubset_{\mathrm{sal}} \mathrm{mat}(\bar{b})$. □

Lemma 2.4. *Let $(A, \sqsubset)$ and $(B, \sqsubset')$ be finite chains. For every $n \geqslant 2$ and every mapping $f\colon A \to B$ define $\hat{f}\colon A^n \to B^n$ by*

$$\hat{f}(a_1, a_2, \ldots, a_n) = (f(a_1), f(a_2), \ldots, f(a_n)).$$

(a) *For every total quasiorder σ such that $|A/\equiv_\sigma| = n$ and every $\bar{a} \in A^n$ we have that $\mathrm{tup}(\sigma, \hat{f}(\bar{a})) = \hat{f}(\mathrm{tup}(\sigma, \bar{a}))$.*

(b) *Take any $\bar{a} = (a_1, a_2, \ldots, a_n) \in A^n$ and assume that $a_i \sqsubset a_j \Rightarrow f(a_i) \sqsubset' f(a_j)$ for all i and j. Then $\mathrm{tp}(\hat{f}(\bar{a})) = \mathrm{tp}(\bar{a})$ and $\mathrm{mat}(\hat{f}(\bar{a})) = \hat{f}(\mathrm{mat}(\bar{a}))$.*

P r o o f. Both (a) and (b) are straightforward. □

Lemma 2.5. *Let $(A, \sqsubset)$ and $(B, \sqsubset')$ be finite chains. Let $n \geqslant 2$ be an integer and let $\theta \subseteq A^n$ and $\theta' \subseteq B^n$ be relations. Let σ be a total quasiorder on $\{1, 2, \ldots, n\}$,* *let* $r = |\{1, 2, \ldots, n\}/\equiv_\sigma| \geqslant 2$ *and let*

$$\varrho = \{\mathrm{mat}(\bar{x})\colon\ \bar{x} \in \theta \quad \text{and} \quad \mathrm{tp}(\bar{x}) = \sigma\} \subseteq A^r,$$
$$\varrho' = \{\mathrm{mat}(\bar{x})\colon\ \bar{x} \in \theta' \quad \text{and} \quad \mathrm{tp}(\bar{x}) = \sigma\} \subseteq B^r.$$

Furthermore, let $f\colon A \to B$ *be a mapping such that*

- ▷ *for every* $(x_1, x_2, \ldots, x_n) \in \theta$, *if* $f{\restriction}_{\{x_1,x_2,\ldots,x_n\}}$ *is not a constant map then* $x_i \sqsubset x_j \Rightarrow f(x_i) \sqsubset' f(x_j)$ *for all* i *and* j, *and*
- ▷ $\hat{f}{\restriction}_\theta\colon \theta \to \theta'$ *is well defined (that is,* $\hat{f}(\bar{x}) \in \theta'$ *for all* $\bar{x} \in \theta$*) and surjective.*

Then $\hat{f}{\restriction}_\varrho\colon \varrho \to \varrho'$ *is well defined, surjective, and for all* $\bar{p} \in \theta'$ *such that* $\mathrm{tp}(\bar{p}) = \sigma$ *we have that* $\min(\hat{f}{\restriction}_\varrho^{-1}(\mathrm{mat}(\bar{p}))) = \mathrm{mat}(\min \hat{f}{\restriction}_\theta^{-1}(\bar{p}))$, *where both the minima are taken with respect to* $\sqsubset_{\mathrm{sal}}$.

P r o o f. For notational convenience let $\hat{f}_\theta = \hat{f}{\restriction}_\theta$ and $\hat{f}_\varrho = \hat{f}{\restriction}_\varrho$. Lemmas 2.3 and 2.4 ensure that $\hat{f}_\varrho\colon \varrho \to \varrho'$ is well defined and surjective. Take any $\bar{p} \in \theta'$ such that $\mathrm{tp}(\bar{p}) = \sigma$ and let $\hat{f}_\theta^{-1}(\bar{p}) = \{\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^k\}$. Let us first show that $\hat{f}_\varrho^{-1}(\mathrm{mat}(\bar{p})) = \{\mathrm{mat}(\bar{a}^1), \mathrm{mat}(\bar{a}^2), \ldots, \mathrm{mat}(\bar{a}^k)\}$.

$\supseteq$: Take any $i \in \{1, 2, \ldots, k\}$. Let us first show that $\mathrm{tp}(\bar{a}^i) = \sigma$ for all i. Since $\hat{f}(\bar{a}^i) = \bar{p}$ and $r \geqslant 2$ it follows that $f{\restriction}_{\{a_1^i, a_2^i, \ldots, a_n^i\}}$ is not a constant map. Then by the assumption (the first item above) we have that $a_s^i \sqsubset a_t^i \Rightarrow f(a_s^i) \sqsubset' f(a_t^i)$ for all s and t. This now yields that $\mathrm{tp}(\bar{a}^i) = \mathrm{tp}(\hat{f}(\bar{a}^i))$. Since $\hat{f}(\bar{a}^i) = \hat{f}_\theta(\bar{a}^i) = \bar{p}$, it follows that $\mathrm{tp}(\bar{a}^i) = \mathrm{tp}(\bar{p}) = \sigma$. Applying Lemma 2.4 (b) once again gives $\hat{f}_\varrho(\mathrm{mat}(\bar{a}^i)) = \mathrm{mat}(\hat{f}_\theta(\bar{a}^i)) = \mathrm{mat}(\bar{p})$.

$\subseteq$: Take any $\bar{u} \in \hat{f}_\varrho^{-1}(\mathrm{mat}(\bar{p}))$. Then $\bar{u} = \mathrm{mat}(\bar{x})$ for some $\bar{x} \in \theta$ such that $\mathrm{tp}(\bar{x}) = \sigma$, so $\mathrm{mat}(\bar{p}) = \hat{f}_\varrho(\bar{u}) = \hat{f}_\varrho(\mathrm{mat}(\bar{x})) = \mathrm{mat}(\hat{f}_\theta(\bar{x}))$ (see Lemma 2.4). On the other hand, $\mathrm{tp}(\bar{p}) = \sigma = \mathrm{tp}(\bar{x}) = \mathrm{tp}(\hat{f}_\theta(\bar{x}))$ using Lemma 2.4 once more. So, $\mathrm{mat}(\bar{p}) = \mathrm{mat}(\hat{f}_\theta(\bar{x}))$ and $\mathrm{tp}(\bar{p}) = \mathrm{tp}(\hat{f}_\theta(\bar{x}))$ whence, by Lemma 2.3, $\bar{p} = \hat{f}_\theta(\bar{x})$. In other words, $\bar{x} \in \hat{f}_\theta^{-1}(\bar{p})$ whence $\bar{x} = \bar{a}^i$ for some i. Then $\bar{u} = \mathrm{mat}(\bar{x}) = \mathrm{mat}(\bar{a}^i)$ for some i.

Let us now show that $\min(\hat{f}_\varrho^{-1}(\mathrm{mat}(\bar{p}))) = \mathrm{mat}(\min \hat{f}_\theta^{-1}(\bar{p}))$. Lemma 2.4 yields that $\mathrm{tp}(\bar{a}^1) = \mathrm{tp}(\bar{a}^2) = \ldots = \mathrm{tp}(\bar{a}^k)$ so by Lemma 2.3 we have that

$$\begin{aligned}\min(\hat{f}_\varrho^{-1}(\mathrm{mat}(\bar{p}))) &= \min\{\mathrm{mat}(\bar{a}^1), \mathrm{mat}(\bar{a}^2), \ldots, \mathrm{mat}(\bar{a}^k)\}\\ &= \mathrm{mat}(\min\{\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^k\}) = \mathrm{mat}(\min \hat{f}_\theta^{-1}(\bar{p})).\end{aligned}$$

This concludes the proof. □

2.3. Structures. Let Θ be a set of relational symbols. A Θ-*structure* $\mathcal{A} = (A, \Theta^{\mathcal{A}})$ is a set A together with a set $\Theta^{\mathcal{A}}$ of relations on A which are interpretations of the corresponding symbols in Θ. The interpretation of a symbol $\theta \in \Theta$ in the Θ-structure $\mathcal{A}$

is denoted by $\theta^{\mathcal{A}}$. The underlying sets of structures $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ are always denoted by the corresponding *roman* letter $A, B, C, \ldots$, respectively. A structure $\mathcal{A} = (A, \Theta^{\mathcal{A}})$ is *finite* if A is a finite set. A structure $\mathcal{A} = (A, \Theta^{\mathcal{A}})$ is *reflexive* if the following holds for every $\theta \in \Theta$, where $r = \operatorname{ar}(\theta)$:

$$\Delta_{A,r} = \{(\underbrace{a, a, \ldots, a}_{r})\colon\ a \in A\} \subseteq \theta^{\mathcal{A}}.$$

In case of reflexive structures unary relations play no role because every reflexive unary relation is trivial.

Let $\mathcal{A}$ and $\mathcal{B}$ be Θ-structures. A mapping $f\colon\ A \to B$ is a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$, and we write $f\colon\ \mathcal{A} \to \mathcal{B}$, if

$$(a_1, \ldots, a_r) \in \theta^{\mathcal{A}} \Rightarrow (f(a_1), \ldots f(a_r)) \in \theta^{\mathcal{B}}$$

for each $\theta \in \Theta$ and $a_1, \ldots, a_r \in A$. A homomorphism $f\colon\ \mathcal{A} \to \mathcal{B}$ is an *embedding* if f is injective and

$$(f(a_1), \ldots f(a_r)) \in \theta^{\mathcal{B}} \Leftrightarrow (a_1, \ldots, a_r) \in \theta^{\mathcal{A}}$$

for each $\theta \in \Theta$ and $a_1, \ldots, a_r \in A$. A homomorphism $f\colon\ \mathcal{A} \to \mathcal{B}$ is a *quotient map* if f is surjective and for every $\theta \in \Theta$ and $(b_1, \ldots, b_r) \in \theta^{\mathcal{B}}$ there exists an $(a_1, \ldots, a_r) \in \theta^{\mathcal{A}}$ such that $f(a_i) = b_i$, $1 \leqslant i \leqslant r$.

Let Θ be a relational language and let $\sqsubset \notin \Theta$ be a binary relational symbol. A *linearly ordered* Θ*-structure* is a $(\Theta \cup \{\sqsubset\})$-structure $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$, where $(A, \Theta^{\mathcal{A}})$ is a Θ-structure and $\sqsubset^{\mathcal{A}}$ is a linear order on A. A linearly ordered Θ-structure $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ is *reflexive* if $(A, \Theta^{\mathcal{A}})$ is a reflexive Θ-structure.

3. Category theory and the Ramsey property

In order to specify a *category* $\mathbf{C}$ one has to specify a class of objects $\operatorname{Ob}(\mathbf{C})$, a set of morphisms $\hom_{\mathbf{C}}(A, B)$ for all $A, B \in \operatorname{Ob}(\mathbf{C})$, the identity morphism id_A for all $A \in \operatorname{Ob}(\mathbf{C})$, and the composition of morphisms $\cdot$ so that $\mathrm{id}_B \cdot f = f \cdot \mathrm{id}_A = f$ for all $f \in \hom_{\mathbf{C}}(A, B)$, and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ whenever the compositions are defined. A morphism $f \in \hom_{\mathbf{C}}(B, C)$ is *monic* or *left cancellable* if $f \cdot g = f \cdot h$ implies $g = h$ for all $g, h \in \hom_{\mathbf{C}}(A, B)$, where $A \in \operatorname{Ob}(\mathbf{C})$ is arbitrary. A morphism $f \in \hom_{\mathbf{C}}(B, C)$ is *epimorphism* or *right cancellable* if $g \cdot f = h \cdot f$ implies $g = h$ for all $g, h \in \hom_{\mathbf{C}}(C, D)$, where $D \in \operatorname{Ob}(\mathbf{C})$ is arbitrary. A morphism $f \in \hom_{\mathbf{C}}(A, B)$ is *invertible* if there exists a morphism $g \in \hom_{\mathbf{C}}(B, A)$ such that $g \cdot f = \mathrm{id}_A$ and $f \cdot g = \mathrm{id}_B$. Let $\operatorname{Aut}_{\mathbf{C}}(A)$ denote the set of all the invertible morphisms in $\hom_{\mathbf{C}}(A, A)$. An object $A \in \operatorname{Ob}(\mathbf{C})$ is *rigid* if $\operatorname{Aut}_{\mathbf{C}}(A) = \{\mathrm{id}_A\}$.

Example 3.1. Let $\mathbf{Ch}_{\text{emb}}$ denote the category whose objects are finite chains and whose morphisms are embeddings.

Example 3.2. By $\text{Rel}(\Theta, \sqsubset)$ we denote the category whose objects are finite linearly ordered Θ-structures and whose morphisms are embeddings.

Example 3.3. The composition of two rigid surjections is again a rigid surjection, so finite chains and rigid surjections constitute a category which we denote by $\mathbf{Ch}_{\text{rs}}$.

For a category $\mathbf{C}$, the *opposite category*, denoted by $\mathbf{C}^{\text{op}}$, is the category whose objects are the objects of $\mathbf{C}$, morphisms are formally reversed so that $\hom_{\mathbf{C}^{\text{op}}}(A, B) = \hom_{\mathbf{C}}(B, A)$, and so is the composition $f \cdot_{\mathbf{C}^{\text{op}}} g = g \cdot_{\mathbf{C}} f$.

A category $\mathbf{D}$ is a *subcategory* of a category $\mathbf{C}$ if $\text{Ob}(\mathbf{D}) \subseteq \text{Ob}(\mathbf{C})$ and $\hom_{\mathbf{D}}(A, B) \subseteq \hom_{\mathbf{C}}(A, B)$ for all $A, B \in \text{Ob}(\mathbf{D})$. A category $\mathbf{D}$ is a *full subcategory* of a category $\mathbf{C}$ if $\text{Ob}(\mathbf{D}) \subseteq \text{Ob}(\mathbf{C})$ and $\hom_{\mathbf{D}}(A, B) = \hom_{\mathbf{C}}(A, B)$ for all $A, B \in \text{Ob}(\mathbf{D})$.

A *functor* $F \colon \mathbf{C} \to \mathbf{D}$ from a category $\mathbf{C}$ to a category $\mathbf{D}$ maps $\text{Ob}(\mathbf{C})$ to $\text{Ob}(\mathbf{D})$ and maps morphisms of $\mathbf{C}$ to morphisms of $\mathbf{D}$ so that $F(f) \in \hom_{\mathbf{D}}(F(A), F(B))$ whenever $f \in \hom_{\mathbf{C}}(A, B)$, $F(f \cdot g) = F(f) \cdot F(g)$ whenever $f \cdot g$ is defined, and $F(\text{id}_A) = \text{id}_{F(A)}$.

Categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* if there exist functors $F \colon \mathbf{C} \to \mathbf{D}$ and $G \colon \mathbf{D} \to \mathbf{C}$ which are inverses of one another both on objects and on morphisms.

The *product* of categories $\mathbf{C}_1$ and $\mathbf{C}_2$ is the category $\mathbf{C}_1 \times \mathbf{C}_2$ whose objects are pairs (A_1, A_2), where $A_1 \in \text{Ob}(\mathbf{C}_1)$ and $A_2 \in \text{Ob}(\mathbf{C}_2)$, morphisms are pairs $(f_1, f_2) \colon (A_1, A_2) \to (B_1, B_2)$, where $f_1 \colon A_1 \to B_1$ is a morphism in $\mathbf{C}_1$ and $f_2 \colon A_2 \to B_2$ is a morphism in $\mathbf{C}_2$. The composition of morphisms is carried out componentwise: $(f_1, f_2) \cdot (g_1, g_2) = (f_1 \cdot g_1, f_2 \cdot g_2)$.

Let $\mathbf{C}$ be a category and $\mathcal{S}$ a set. We say that $\mathcal{S} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ is a *k-coloring* of $\mathcal{S}$ if $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ whenever $i \neq j$. Equivalently, a k-coloring of $\mathcal{S}$ is any map $\chi \colon \mathcal{S} \to \{1, 2, \ldots, k\}$. For an integer $k \geqslant 2$ and $A, B, C \in \text{Ob}(\mathbf{C})$ we write $C \to (B)^A_k$ to indicate that for every k-coloring $\hom_{\mathbf{C}}(A, C) = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ there is an $i \in \{1, \ldots, k\}$ and a morphism $w \in \hom_{\mathbf{C}}(B, C)$ such that $w \cdot \hom_{\mathbf{C}}(A, B) \subseteq \mathcal{X}_i$.

Definition 3.4. A category $\mathbf{C}$ has the *Ramsey property* if for every integer $k \geqslant 2$ and all $A, B \in \text{Ob}(\mathbf{C})$ such that $\hom_{\mathbf{C}}(A, B) \neq \emptyset$ there is a $C \in \text{Ob}(\mathbf{C})$ such that $C \to (B)^A_k$. A category $\mathbf{C}$ has the *dual Ramsey property* if $\mathbf{C}^{\text{op}}$ has the Ramsey property.

Clearly, if $\mathbf{C}$ and $\mathbf{D}$ are isomorphic categories and one of them has the (dual) Ramsey property, then so does the other. Actually, even more is true: if $\mathbf{C}$ and $\mathbf{D}$ are equivalent categories and one of them has the (dual) Ramsey property, then so does the other. We refrain from providing the definition of (the fairly standard notion of) categorical equivalence as we will have no use for it in this paper, and for the proof we refer the reader to [7]. Nevertheless, this fact is important because it demonstrates that the Ramsey property is a genuine categorical property.

Example 3.5. The category $\mathbf{Ch}_{\mathrm{emb}}$ of finite chains and embeddings has the Ramsey property. This is just a reformulation of the Finite Ramsey Theorem (see Theorem 1.1).

Example 3.6. The category $\mathrm{Rel}(\Theta, \sqsubset)$ has the Ramsey property. This is the Nešetřil-Rödl Theorem (see Theorem 1.2).

Example 3.7. The category $\mathbf{Ch}_{\mathrm{rs}}$ of finite chains and rigid surjections has the dual Ramsey property. This is just a reformulation of the Finite Dual Ramsey Theorem (see Theorem 1.3; see also the discussion in the Introduction.)

One of the main benefits of considering the Ramsey property in the setting of category theory is the Duality Principle which is a metatheorem of category theory.

The Duality Principle. If a statement φ is true in a category $\mathbf{C}$, then the opposite statement φ^{op} is true in $\mathbf{C}^{\mathrm{op}}$.

For a detailed technical discussion and the precise definition of φ^{op} we refer the reader to [2]. Here, however, we would like to stress that the Duality Principle saves quite a lot of work, in particular in situations, where we want to reuse the existing Ramsey-type results to infer dual Ramsey-type results. For example, in [7], Proposition 2.3 we proved that if $\mathbf{C}$ is a category, where morphisms are monic and $\mathbf{C}$ has the Ramsey property then all the objects in $\mathbf{C}$ are rigid. As an immediate consequence of the Duality Principle we have the following corollary (bearing in mind that rigidity is a self-dual notion).

Corollary 3.8. *Let $\mathbf{C}$ be a category, where morphisms are epimorphisms. If $\mathbf{C}$ has the dual Ramsey property then all the objects in $\mathbf{C}$ are rigid.*

As another example, the following result is a categorical rendering of the Product Ramsey Theorem for Finite Structures of Sokić, see [18].

Theorem 3.9 ([5])**.** *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories such that $\hom_{\mathbf{C}_i}(A, B)$ is finite for all $A, B \in \mathrm{Ob}(\mathbf{C}_i)$, $i \in \{1, 2\}$. If $\mathbf{C}_1$ and $\mathbf{C}_2$ both have the Ramsey property then $\mathbf{C}_1 \times \mathbf{C}_2$ has the Ramsey property.*

The following result now follows for free by The Duality Principle.

Corollary 3.10. *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories such that $\hom_{\mathbf{C}_i}(A, B)$ is finite for all $A, B \in \mathrm{Ob}(\mathbf{C}_i)$, $i \in \{1, 2\}$. If $\mathbf{C}_1$ and $\mathbf{C}_2$ both have the dual Ramsey property then $\mathbf{C}_1 \times \mathbf{C}_2$ has the dual Ramsey property.*

We will need this corollary in the proof of Proposition 5.3. We will also need the following simple technical result (actually its categorical dual).

Lemma 3.11. *Let* $\mathbf{C}$ *be a category, let* $A, B, C, D \in \mathrm{Ob}(\mathbf{C})$ *be arbitrary and let* $k \geqslant 2$ *be an integer. If* $C \to (B)^A_k$ *and* $\hom(C, D) \neq \emptyset$ *then* $D \to (B)^A_k$.

As our concluding example we prove the dual Ramsey theorem for linearly ordered metric spaces. A *linearly ordered metric space* is a triple $\mathcal{M} = (M, d, \sqsubset)$, where $d\colon M^2 \to \mathbb{R}$ is a metric on M and $\sqsubset$ is a linear order on M. For a positive integer n and a positive real number δ let

$$\mathcal{M}_{n,\delta} = (\{1, 2, \ldots, n\}, d^\delta_n, <)$$

be the linearly ordered metric space, where $<$ is the usual ordering of the integers and $d^\delta_n(x, y) = \delta$ whenever $x \neq y$. A mapping $f\colon M \to M'$ is a *nonexpansive rigid surjection* from $(M, d, \sqsubset)$ to $(M', d', \sqsubset')$ if

- $\triangleright$ $f\colon (M, d) \to (M', d')$ is nonexpansive, that is, $d'(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in M$, and
- $\triangleright$ $f\colon (M, \sqsubset) \to (M', \sqsubset')$ is a rigid surjection.

Let $\mathrm{OMet}_{\mathrm{ners}}$ be the category whose objects are finite linearly ordered metric spaces and morphisms are nonexpansive rigid surjections.

For a linearly ordered metric space $\mathcal{M} = (M, d, \sqsubset)$ let $\mathrm{Spec}(\mathcal{M}) = \{d(x, y)\colon x, y \in M \text{ and } x \neq y\}$. For a subcategory $\mathbf{C}$ of $\mathrm{OMet}_{\mathrm{ners}}$ let

$$\mathrm{Spec}(\mathbf{C}) = \bigcup \{\mathrm{Spec}(\mathcal{M})\colon \mathcal{M} \in \mathrm{Ob}(\mathbf{C})\}.$$

Lemma 3.12. *Let* $\mathbf{C}$ *be a full subcategory of* $\mathrm{OMet}_{\mathrm{ners}}$ *such that for every positive integer* m *and every* $\delta \in \mathrm{Spec}(\mathbf{C})$ *there is an integer* $n \geqslant m$ *such that* $\mathcal{M}_{n,\delta} \in \mathrm{Ob}(\mathbf{C})$. *Then* $\mathbf{C}$ *has the dual Ramsey property. In particular, the following categories have the dual Ramsey property:*

- $\triangleright$ *the category* $\mathrm{OMet}_{\mathrm{ners}}$;
- $\triangleright$ *the category* $\mathrm{OMet}_{\mathrm{ners}}(S)$, $S \subseteq \mathbb{R}$, *which stands for the full subcategory of* $\mathrm{OMet}_{\mathrm{ners}}$ *spanned by all those finite linearly ordered metric spaces* $\mathcal{M}$ *such that* $\mathrm{Spec}(\mathcal{M}) \subseteq S$.

Proof. Take any $k \geqslant 2$, any $\mathcal{A} = (A, d^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ and $\mathcal{B} = (B, d^{\mathcal{B}}, \sqsubset^{\mathcal{B}})$ in $\mathbf{C}$ such that there is a nonexpansive rigid surjection $\mathcal{B} \to \mathcal{A}$, and let $(C, \sqsubset^{\mathcal{C}})$ be a finite chain such that $(C, \sqsubset^{\mathcal{C}}) \to (B, \sqsubset^{\mathcal{B}})^{(A, \sqsubset^{\mathcal{A}})}_k$ in $\mathbf{Ch}^{\mathrm{op}}_{\mathrm{rs}}$. Such a chain exists because $\mathbf{Ch}^{\mathrm{op}}_{\mathrm{rs}}$ has the Ramsey property (see Example 3.7). Let $\delta = \max\{d^{\mathcal{B}}(x, y)\colon x, y \in B\}$. By the assumption, there is an integer $n \geqslant |C|$ such that $\mathcal{M}_{n,\delta} \in \mathrm{Ob}(\mathbf{C})$.

Since $n \geqslant |C|$, there is a rigid surjection $(\{1, 2, \ldots, n\}, <) \to (C, \sqsubset^{\mathcal{C}})$, so Lemma 3.11 yields that $(\{1, 2, \ldots, n\}, <) \to (B, \sqsubset^{\mathcal{B}})_k^{(A, \sqsubset^{\mathcal{A}})}$ in $\mathbf{Ch}_{\mathrm{rs}}^{\mathrm{op}}$. Then it is easy to show that $\mathcal{M}_{n,\delta} \to (\mathcal{B})^{\mathcal{A}}_k$ in $\mathbf{C}^{\mathrm{op}}$ because $f\colon \mathcal{M}_{n,\delta} \to \mathcal{A}$ is a nonexpansive rigid surjection if and only if $f\colon (\{1, 2, \ldots, n\}, <) \to (A, \sqsubset^{\mathcal{A}})$ is a rigid surjection.

□

What is an acceptable kind of objects? Having in mind Corollary 3.8, a necessary requirement for a category to have the dual Ramsey property is that all of its objects be rigid. In this paper we consider categories of finite linearly ordered relational structures, as adding linear orders to finite structures turns out to be technically the easiest way of achieving rigidity. The morphisms we will be working with will be surjective so all the structures in the paper will necessarily be reflexive.

What is an acceptable kind of morphisms? Embeddings have established themselves as the only kind of morphisms of interest when considering "direct" Ramsey results in structural Ramsey theory. For dual Ramsey results, though, there is no obvious notion that parallels in full the notion of embedding. For example, fix a relational language Θ and consider a category $\mathbf{C}$ whose objects are some finite linearly ordered Θ-structures $\mathcal{A} = (A, \Theta^{\mathcal{A}}, <^{\mathcal{A}})$ and morphisms are just rigid surjections $f\colon (A, <^{\mathcal{A}}) \to (B, <^{\mathcal{B}})$. Then $\mathbf{C}$ has the dual Ramsey property provided it contains arbitrarily large finite structures (see Example 3.7 and Lemma 3.11). This is clearly far from satisfactory. The same holds if we require each morphism $f\colon (A, \Theta^{\mathcal{A}}, <^{\mathcal{A}}) \to (B, \Theta^{\mathcal{B}}, <^{\mathcal{B}})$ to be a surjective homomorphism $f\colon (A, \Theta^{\mathcal{A}}) \to (B, \Theta^{\mathcal{B}})$ and at the same time a rigid surjection $f\colon (A, <^{\mathcal{A}}) \to (B, <^{\mathcal{B}})$: the dual Ramsey property follows as soon as the category has arbitrarily large empty structures.

Therefore, when dealing with surjective homomorphisms quotient maps are usually seen as the more appropriate structure maps. Our main results shall be, therefore, spelled in the context, where each morphism $f\colon (A, \Theta^{\mathcal{A}}, <^{\mathcal{A}}) \to (B, \Theta^{\mathcal{B}}, <^{\mathcal{B}})$ under consideration is a quotient map $f\colon (A, \Theta^{\mathcal{A}}) \to (B, \Theta^{\mathcal{B}})$ and at the same time a rigid surjection $f\colon (A, <^{\mathcal{A}}) \to (B, <^{\mathcal{B}})$.

4. Dual Ramsey theorems for structures and special quotient maps

In this section we turn to the main goal of the paper, which is to prove dual Ramsey theorems for various categories of structures and special quotient maps. We first prove our main technical result (see Theorem 4.4), and as a consequence derive dual Ramsey theorems for acyclic digraphs with linear extensions, posets with linear extensions, linearly ordered uniform hypergraphs and linearly ordered graphs.

For an integer $r \geqslant 2$, a *reflexive r-structure with a linear extension* (or *r-erst* for short) is a linearly ordered reflexive structure $\mathcal{A} = (A, \varrho, \sqsubset)$ such that $\sqsubset$ is a *linear extension* of ϱ in the sense that

$$\text{if } (a_1, a_2, \ldots, a_r) \in \varrho \setminus \Delta_{A,r} \text{ then } a_1 \sqsubset a_2 \sqsubset \ldots \sqsubset a_r,$$

where $r = \operatorname{ar}(\varrho)$.

Definition 4.1. Let $\mathcal{A} = (A, \varrho, \sqsubset)$ and $\mathcal{A}' = (A', \varrho', \sqsubset')$ be two linearly ordered relational structures, where $\operatorname{ar}(\varrho) = \operatorname{ar}(\varrho') = r$. Then each homomorphism $f\colon (A, \varrho) \to (A', \varrho')$ induces a mapping $\hat{f}\colon \varrho \to \varrho'$ by $\hat{f}(a_1, a_2, \ldots, a_r) = (f(a_1), f(a_2), \ldots, f(a_r))$. A homomorphism $f\colon (A, \varrho) \to (A', \varrho')$ is a *strong rigid quotient map* from $\mathcal{A}$ to $\mathcal{A}'$ if $\hat{f}\colon (\varrho, \sqsubset_{\mathrm{sal}}) \to (\varrho', \sqsubset'_{\mathrm{sal}})$ is a rigid surjection.

The following lemma justifies the name for these morphisms: it shows that a strong rigid quotient map is a rigid surjection and a quotient map. The converse is not true, see Figure 1.

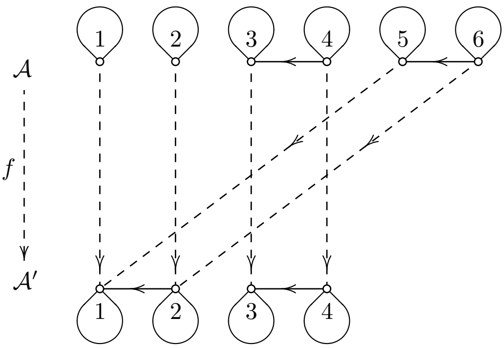


Figure 1. A rigid surjection and a quotient map which is not strong.

Lemma 4.2. *Let $\mathcal{A} = (A, \varrho, \sqsubset)$ and $\mathcal{A}' = (A', \varrho', \sqsubset')$ be two linearly ordered reflexive relational structures, where $\operatorname{ar}(\varrho) = \operatorname{ar}(\varrho') = r \geqslant 2$, and let $f\colon (A, \varrho) \to (A', \varrho')$ be a homomorphism.*

(a) *For any $u \in A'$, if $\hat{f}^{-1}(u, u, \ldots, u) \neq \emptyset$ then $\min \hat{f}^{-1}(u, u, \ldots, u) = (x, x, \ldots, x)$, where $x = \min f^{-1}(u)$.*

(b) *Assume that $\hat{f}\colon (\varrho, \sqsubset_{\mathrm{sal}}) \to (\varrho', \sqsubset'_{\mathrm{sal}})$ is a rigid surjection. Then f is a rigid surjection $(A, \sqsubset) \to (A', \sqsubset')$ and a quotient map $(A, \varrho) \to (A', \varrho')$.*

Proof. (a) Assume that $\hat{f}^{-1}(u, u, \ldots, u) \neq \emptyset$ and let

$$\min \hat{f}^{-1}(u, u, \ldots, u) = (x_1, x_2, \ldots, x_r),$$

where $|\{x_1, x_2, \dots, x_r\}| \geqslant 2$. Then $\hat{f}(x_1, x_2, \dots, x_r) = (u, u, \dots, u)$. So, $f(x_1) = u$, whence $\hat{f}(x_1, x_1, \dots, x_1) = (u, u, \dots, u)$. But $(x_1, x_1, \dots, x_1) \sqsubset_{\mathrm{sal}} (x_1, x_2, \dots, x_r)$ because $\mathrm{tp}(x_1, x_1, \dots, x_1) \lhd \mathrm{tp}(x_1, x_2, \dots, x_r)$. This contradicts the fact that $\min \hat{f}^{-1}(u, u, \dots, u) = (x_1, x_2, \dots, x_r)$.

Thus, $\min \hat{f}^{-1}(u, u, \dots, u) = (x, x, \dots, x)$ for some $x \in A$. Then $f(x) = u$ whence $\min f^{-1}(u) \sqsubseteq x$. Assume that $\min f^{-1}(u) = t \sqsubset x$. Then $(t, t, \dots, t) \sqsubset_{\mathrm{sal}} (x, x, \dots, x) = \min \hat{f}^{-1}(u, u, \dots, u)$, which contradicts the fact that $(t, t, \dots, t) \in \hat{f}^{-1}(u, u, \dots, u)$. Therefore, $\min f^{-1}(u) = x$.

(b) Let us start by showing that f is surjective. Take any $u \in A'$. Then $(u, u, \dots, u) \in \varrho'$ because ϱ' is reflexive, so there is an $(x_1, x_2, \dots, x_r) \in \varrho$ such that $\hat{f}(x_1, x_2, \dots, x_r) = (u, u, \dots, u)$ because $\hat{f}$ is surjective. But then $f(x_1) = u$.

Since f is a homomorphism and $\hat{f}\colon \varrho \to \varrho'$ is a surjective map it immediately follows that f is a quotient map.

Finally, let us prove that f is a rigid surjection. Take any $u, v \in A'$ such that $u \sqsubset' v$ and let us show that $\min f^{-1}(u) \sqsubset \min f^{-1}(v)$. From $u \sqsubset' v$ it follows that $(u, u, \dots, u) \sqsubset'_{\mathrm{sal}} (v, v, \dots, v)$, whence $\min \hat{f}^{-1}(u, u, \dots, u) \sqsubset_{\mathrm{sal}} \min \hat{f}^{-1}(v, v, \dots, v)$ because $\hat{f}$ is a rigid surjection. The conclusion now follows from (a). $\square$

For $r \geqslant 2$, let $\mathrm{ERst}_{\mathrm{srq}}(r)$ be the *category whose objects are finite r-erst's and whose morphisms are strong rigid quotient maps.* Our goal in this section is to prove that $\mathrm{ERst}_{\mathrm{srq}}(r)$ has the dual Ramsey property for every $r \geqslant 2$. In order to do so, we use the main idea of [6]. A pair of *maps*

$$F\colon \mathrm{Ob}(\mathbf{D}) \rightleftarrows \mathrm{Ob}(\mathbf{C})\colon G$$

is a *pre-adjunction between the categories* $\mathbf{C}$ *and* $\mathbf{D}$ provided there is a family of maps

$$\Phi_{Y,X}\colon \hom_{\mathbf{C}}(F(Y), X) \to \hom_{\mathbf{D}}(Y, G(X))$$

indexed by the family $\{(Y, X) \in \mathrm{Ob}(\mathbf{D}) \times \mathrm{Ob}(\mathbf{C})\colon \hom_{\mathbf{C}}(F(Y), X) \neq \emptyset\}$ and satisfying the following:

(PA) For every $C \in \mathrm{Ob}(\mathbf{C})$, every $D, E \in \mathrm{Ob}(\mathbf{D})$, every $u \in \hom_{\mathbf{C}}(F(D), C)$ and every $f \in \hom_{\mathbf{D}}(E, D)$ there is a $v \in \hom_{\mathbf{C}}(F(E), F(D))$ satisfying $\Phi_{D,C}(u) \cdot f = \Phi_{E,C}(u \cdot v)$.

$$\begin{tikzcd} F(D) \arrow[r, "u"] & C \\ F(E) \arrow[u, "v"] \arrow[ur, "u\cdot v"'] & \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "\Phi_{D,C}(u)"] & G(C) \\ E \arrow[u, "f"] \arrow[ur, "\Phi_{E,C}(u\cdot v)"'] & \end{tikzcd}$$

(Note that in a pre-adjunction F and G are *not* required to be functors, just maps from the class of objects of one of the two categories into the class of objects of the other category; also Φ is just a family of maps between hom-sets satisfying the requirement above.)

Theorem 4.3 ([6])**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories such that* $\mathbf{C}$ *has the Ramsey property and there is a pre-adjunction* $F\colon \mathrm{Ob}(\mathbf{D}) \rightleftarrows \mathrm{Ob}(\mathbf{C})\colon G$*. Then* $\mathbf{D}$ *has the Ramsey property.*

For a finite chain $\mathcal{L} = \{l_1 \sqsubset l_2 \sqsubset \ldots \sqsubset l_N\}$ let

$$r \otimes \mathcal{L} = (\{1, 2, \ldots, r\} \times \{l_1, l_2, \ldots, l_N\}, \varrho_{r\otimes\mathcal{L}}, \prec)$$

denote an r-erst on $r \cdot N$ elements (written il instead of (i, l)), where

$$\varrho_{r\otimes\mathcal{L}} = \Delta_{\{1,2,\ldots,r\}\times\{l_1,l_2,\ldots,l_N\},r} \cup \{(1l, 2l, \ldots, rl)\colon\ l \in \mathcal{L}\},$$

and $\prec$ is the anti-lexicographic ordering of $\{1, 2, \ldots, r\} \times \{l_1, l_2, \ldots, l_N\}$ induced by the respective linear orderings: $ik \prec jl$ if and only if $k \sqsubset l$, or $k = l$ and $i < j$.

Theorem 4.4. *Let* $r \geqslant 2$ *be an integer. Let* $\mathbf{C}$ *be a full subcategory of* $\mathrm{ERst}_{\mathrm{srq}}(r)$ *such that for every finite chain* $\mathcal{L}$ *there is an object in* $\mathrm{Ob}(\mathbf{C})$ *isomorphic to* $r \otimes \mathcal{L}$*. Then* $\mathbf{C}$ *has the dual Ramsey property. In particular,* $\mathrm{ERst}_{\mathrm{srq}}(r)$ *has the dual Ramsey property.*

P r o o f. Without loss of generality we may assume that $r \otimes \mathcal{L} \in \mathrm{Ob}(\mathbf{C})$ for every finite chain $\mathcal{L}$. In order to prove the theorem we are going to show that there is a pre-adjunction

$$F\colon\ \mathrm{Ob}(\mathbf{C}^{\mathrm{op}}) \rightleftarrows \mathrm{Ob}(\mathbf{Ch}_{\mathrm{rs}}^{\mathrm{op}})\colon\ G.$$

The result then follows from Theorem 4.3 and the fact that the category $\mathbf{Ch}_{\mathrm{rs}}^{\mathrm{op}}$ has the Ramsey property (see Example 3.7). Explicitly, unpacking the definition of pre-adjunction in case of opposite categories, we have to show the following:

(PA) For every finite chain $\mathcal{L} \in \mathrm{Ob}(\mathbf{Ch}_{\mathrm{rs}})$, all $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C})$ for every $u \in \hom_{\mathbf{Ch}_{\mathrm{rs}}}(\mathcal{L}, F(\mathcal{A}))$ and every $f \in \hom_{\mathbf{C}}(\mathcal{A}, \mathcal{B})$ there is a $v \in \hom_{\mathbf{Ch}_{\mathrm{rs}}}(F(\mathcal{A}), F(\mathcal{B}))$ satisfying $f \circ \Phi_{\mathcal{A},\mathcal{L}}(u) = \Phi_{\mathcal{B},\mathcal{L}}(v \circ u)$.

$$\begin{tikzcd}
F(\mathcal{A}) \arrow[d, "v"'] & \mathcal{L} \arrow[l, "u"'] \arrow[dl, "v\circ u"] \\
F(\mathcal{B}) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, "f"'] & G(\mathcal{L}) \arrow[l, "\Phi_{\mathcal{A},\mathcal{L}}(u)"'] \arrow[dl, "\Phi_{\mathcal{B},\mathcal{L}}(v\circ u)"] \\
\mathcal{B} &
\end{tikzcd}$$

Take a finite chain $\mathcal{L}$ and a finite r-erst $\mathcal{A}$. Without loss of generality we can assume that $\mathcal{L} = \{1 < 2 < \ldots < N\}$ and $\mathcal{A} = (\{1, 2, \ldots, n\}, \varrho_{\mathcal{A}}, <)$, where $<$

is the usual ordering of the integers and $\varrho_{\mathcal{A}} = \{e_1 <_{\text{sal}} e_2 <_{\text{sal}} \ldots <_{\text{sal}} e_{q(\mathcal{A})}\}$. For each i let $e_i = (p_i^1, p_i^2, \ldots, p_i^r)$, where, as stipulated by the definition of r-erst, either $p_i^1 = p_i^2 = \ldots = p_i^r$ or $p_i^1 < p_i^2 < \ldots < p_i^r$.

Define F and G by $F(\mathcal{A}) = (\varrho_{\mathcal{A}}, <_{\text{sal}})$ and $G(\mathcal{L}) = r \otimes \mathcal{L}$. Next, let us define $\Phi_{\mathcal{A},\mathcal{L}}\colon \hom_{\mathbf{Ch}_{\text{rs}}}(\mathcal{L}, F(\mathcal{A})) \to \hom_{\mathbf{C}}(G(\mathcal{L}), \mathcal{A})$. For a rigid surjection

$$u\colon \{1 < 2 < \ldots < N\} \to \{e_1 <_{\text{sal}} e_2 <_{\text{sal}} \ldots <_{\text{sal}} e_{q(\mathcal{A})}\} \tag{4.1}$$

define $\varphi_u\colon r \otimes \mathcal{L} \to \mathcal{A}$ by $\varphi_u(is) = \pi_i \circ u(s)$ and put $\Phi_{\mathcal{A},\mathcal{L}}(u) = \varphi_u$. Here, π_i stands for the ith projection $\pi_i(x_1, x_2, \ldots, x_r) = x_i$.

To show that the definition of Φ is correct we have to show that for every rigid surjection u as in (4.1) the mapping φ_u is a strong rigid quotient map $r \otimes \mathcal{L} \to \mathcal{A}$.

Let us first show that $\varphi_u\colon (\{1, 2, \ldots, r\} \times \{1, 2, \ldots, N\}, \varrho_{r\otimes\mathcal{L}}) \to (A, \varrho_{\mathcal{A}})$ is a homomorphism. Take any $(x_1, x_2, \ldots, x_r) \in \varrho_{r\otimes\mathcal{L}}$. The case $x_1 = \ldots = x_r$ is trivial, so let us consider the case, where $(x_1, x_2, \ldots, x_r) = (1s, 2s, \ldots, rs)$ for some $s \in \mathcal{L}$:

$$\begin{aligned}(\varphi_u(x_1), \varphi_u(x_2), \ldots, \varphi_u(x_r)) &= (\varphi_u(1s), \varphi_u(2s), \ldots, \varphi_u(rs))\\ &= (\pi_1 \circ u(s), \pi_2 \circ u(s), \ldots, \pi_r \circ u(s)) = u(s) \in \varrho_{\mathcal{A}}.\end{aligned}$$

Let us now show that $\widehat{\varphi}_u\colon (\varrho_{r\otimes\mathcal{L}}, \prec_{\text{sal}}) \to (\varrho_{\mathcal{A}}, <_{\text{sal}})$ is a rigid surjection. Note that $\widehat{\varphi}_u(i_1 s, i_2 s, \ldots, i_r s) = (\pi_{i_1} \circ u(s), \pi_{i_2} \circ u(s), \ldots, \pi_{i_r} \circ u(s))$.

Claim 1. Take any $e_k = (p_k^1, p_k^2, \ldots, p_k^r) \in \varrho_{\mathcal{A}}$, $1 \leqslant k \leqslant F(\mathcal{A})$, and let $t = \min u^{-1}(e_k)$. Then the following holds:
1° if $p_k^1 < p_k^2 < \ldots < p_k^r$ then $\min \widehat{\varphi}_u^{-1}(e_k) = (1t, 2t, \ldots, rt)$,
2° if $p_k^1 = p_k^2 = \ldots = p_k^r$ then $\min \widehat{\varphi}_u^{-1}(e_k) = (1t, 1t, \ldots, 1t)$.

P r o o f. 1° Assume that $p_k^1 < p_k^2 < \ldots < p_k^r$. From

$$\widehat{\varphi}_u((1t, 2t, \ldots, rt)) = u(t) = e_k$$

it follows that $(1t, 2t, \ldots, rt) \in \widehat{\varphi}_u^{-1}(e_k)$, so it suffices to show that

$$(1t, 2t, \ldots, rt) \preccurlyeq_{\text{sal}} (i_1 s, i_2 s, \ldots, i_r s)$$

for every $(i_1 s, i_2 s, \ldots, i_r s) \in \widehat{\varphi}_u^{-1}(e_k)$. Take any $(i_1 s, i_2 s, \ldots, i_r s) \in \varrho_{r\otimes\mathcal{L}}$ such that $\widehat{\varphi}_u(i_1 s, i_2 s, \ldots, i_r s) = e_k$. Then

$$(\pi_{i_1} \circ u(s), \pi_{i_2} \circ u(s), \ldots, \pi_{i_r} \circ u(s)) = (p_k^1, p_k^2, \ldots, p_k^r). \tag{4.2}$$

Let $u(s) = e_m$. Since $p_k^1 < p_k^2 < \ldots < p_k^r$, the only possibility to achieve (4.2) is $i_1 = 1, i_2 = 2, \ldots, i_r = r$ and $m = k$. Therefore, $u(s) = e_k$ whence $s \in u^{-1}(e_k)$, so $t \leqslant s$. Now

$$(1t, 2t, \ldots, rt) \preccurlyeq_{\mathrm{sal}} (1s, 2s, \ldots, rs) = (i_1 s, i_2 s, \ldots, i_r s).$$

2° Assume that $p_k^1 = p_k^2 = \ldots = p_k^r$. From

$$\widehat{\varphi}_u(1t, 1t, \ldots, 1t) = (p_k^1, p_k^1, \ldots, p_k^1) = e_k$$

it follows that $(1t, 1t, \ldots, 1t) \in \widehat{\varphi}_u^{-1}(e_k)$, so it suffices to show that

$$(1t, 1t, \ldots, 1t) \preccurlyeq_{\mathrm{sal}} (i_1 s, i_2 s, \ldots, i_r s)$$

for every $(i_1 s, i_2 s, \ldots, i_r s) \in \widehat{\varphi}_u^{-1}(e_k)$. Take any $(i_1 s, i_2 s, \ldots, i_r s) \in \varrho_{r \otimes \mathcal{L}}$ such that $\widehat{\varphi}_u(i_1 s, i_2 s, \ldots, i_r s) = e_k$. Then

$$(\pi_{i_1} \circ u(s), \pi_{i_2} \circ u(s), \ldots, \pi_{i_r} \circ u(s)) = (p_k^1, p_k^1, \ldots, p_k^1), \tag{4.3}$$

whence

$$\pi_{i_\alpha} \circ u(s) = \pi_{i_\beta} \circ u(s) \quad \text{for all } \alpha \text{ and } \beta. \tag{4.4}$$

Let $u(s) = e_m = (p_m^1, p_m^2, \ldots, p_m^r)$.

(a) Assume that $p_m^1 = p_m^2 = \ldots = p_m^r$. Then

$$(p_m^1, p_m^1, \ldots, p_m^1) = (\pi_{i_1} \circ u(s), \pi_{i_2} \circ u(s), \ldots, \pi_{i_r} \circ u(s)) = (p_k^1, p_k^1, \ldots, p_k^1)$$

whence $u(s) = e_m = e_k$. Then $s \in u^{-1}(e_k)$, so $t \leqslant s$. Hence,

$$(1t, 1t, \ldots, 1t) \preccurlyeq_{\mathrm{sal}} (i_1 s, i_2 s, \ldots, i_r s)$$

for any choice $(i_1, i_2, \ldots, i_r) \in \{(1, 1, \ldots, 1), \ldots, (r, r, \ldots, r), (1, 2, \ldots, r)\}$.

(b) Assume now that $p_m^1 < p_m^2 < \ldots < p_m^r$. Then (4.4) implies that $i_1 = i_2 = \ldots = i_r$. Let $i_1 = \ldots = i_r = \alpha$. Since $\mathrm{tp}(p_k^1, p_k^1, \ldots, p_k^1) \lhd \mathrm{tp}(p_m^1, p_m^2, \ldots, p_m^r)$ it follows that

$$e_k = (p_k^1, p_k^1, \ldots, p_k^1) \prec_{\mathrm{sal}} (p_m^1, p_m^2, \ldots, p_m^r) = e_m.$$

Therefore,

$$t = \min u^{-1}(e_k) < \min u^{-1}(e_m) \leqslant s$$

whence $(1t, 1t, \ldots, 1t) \prec_{\mathrm{sal}} (\alpha s, \alpha s, \ldots, \alpha s) = (i_1 s, i_2 s, \ldots, i_r s)$.

This concludes the proof of Claim 1. □

We are now ready to show that $\widehat{\varphi}_u\colon (\varrho_{r\otimes\mathcal{L}}, \prec_{\mathrm{sal}}) \to (\varrho_{\mathcal{A}}, <_{\mathrm{sal}})$ is a rigid surjection. Take any $e_i = (p_i^1, p_i^2, \ldots, p_i^r), e_j = (p_j^1, p_j^2, \ldots, p_j^r) \in \varrho_{\mathcal{A}}$ such that $e_i <_{\mathrm{sal}} e_j$. Then $\min u^{-1}(e_i) < \min u^{-1}(e_j)$ because u is a rigid surjection. Let $s = \min u^{-1}(e_i)$ and $t = \min u^{-1}(e_j)$.

- ▷ If $p_i^1 = p_i^2 = \ldots = p_i^r$ and $p_j^1 = p_j^2 = \ldots = p_j^r$ then, by Claim 1, $\min \widehat{\varphi}_u^{-1}(e_i) = (1s, 1s, \ldots, 1s) \prec_{\mathrm{sal}} (1t, 1t, \ldots, 1t) = \min \widehat{\varphi}_u^{-1}(e_j)$.
- ▷ If $p_i^1 = p_i^2 = \ldots = p_i^r$ and $p_j^1 < p_j^2 < \ldots < p_j^r$ then, by Claim 1, $\min \widehat{\varphi}_u^{-1}(e_i) = (1s, 1s, \ldots, 1s) \prec_{\mathrm{sal}} (1t, 2t, \ldots, rt) = \min \widehat{\varphi}_u^{-1}(e_j)$.
- ▷ If $p_i^1 < p_i^2 < \ldots < p_i^r$ and $p_j^1 < p_j^2 < \ldots < p_j^r$ then, by Claim 1, $\min \widehat{\varphi}_u^{-1}(e_i) = (1s, 2s, \ldots, rs) \prec_{\mathrm{sal}} (1t, 2t, \ldots, rt) = \min \widehat{\varphi}_u^{-1}(e_j)$.

This proves that $\widehat{\varphi}_u$ is a rigid surjection and the definition of Φ is correct.

We still have to show that this family of maps satisfies the requirement (PA). But this is easy. Let $\mathcal{B} = (\{1, 2, \ldots, l\}, \varrho_{\mathcal{B}}, <)$ be a finite r-erst, where $<$ is the usual ordering of the integers, and let $f\colon \mathcal{A} \to \mathcal{B}$ be a strong rigid quotient map. Then $\hat{f}\colon (\varrho_{\mathcal{A}}, <_{\mathrm{sal}}) \to (\varrho_{\mathcal{B}}, <_{\mathrm{sal}})$ is a rigid surjection by definition. Let us show that $f \circ \varphi_u = \varphi_{\hat{f}\circ u}$:

$$f \circ \varphi_u(is) = f \circ \pi_i \circ u(s) = \pi_i \circ \hat{f} \circ u(s) = \varphi_{\hat{f}\circ u}(is).$$

This calculation relies on the fact that $f \circ \pi_i = \pi_i \circ \hat{f}$, which is clearly true: $f \circ \pi_i(x_1, x_2, \ldots, x_r) = f(x_i) = \pi_i(f(x_1), f(x_2), \ldots, f(x_r)) = \pi_i \circ \hat{f}(x_1, x_2, \ldots, x_r)$. □

Specializing the above result for $r = 2$ we get the following corollary.

Corollary 4.5. *The following categories have the dual Ramsey property:*

- ▷ *the category* $\mathrm{EDig}_{\mathrm{srq}}$ *whose objects are finite reflexive digraphs with linear extensions and morphisms are strong rigid quotient maps,*
- ▷ *the category* $\mathrm{EPos}_{\mathrm{srq}}$ *whose objects are finite posets with linear extensions and morphisms are strong rigid quotient maps.*

Proof. For the first item it suffices to note that $\mathrm{EDig}_{\mathrm{srq}} = \mathrm{ERst}_{\mathrm{srq}}(2)$. For the second item it suffices to note that $\mathrm{EPos}_{\mathrm{srq}}$ is a full subcategory of $\mathrm{EDig}_{\mathrm{srq}}$ such that $2 \otimes \mathcal{L} \in \mathrm{Ob}(\mathrm{EPos}_{\mathrm{srq}})$ for every finite chain $\mathcal{L}$. □

As another corollary of Theorem 4.4 we now prove a dual Ramsey theorem for reflexive graphs and hypergraphs together with special quotient maps.

Definition 4.6. For a chain $\mathcal{A} = (A, \sqsubset)$ let us define $\sqsubset_{\mathrm{sal}}$ on $\mathcal{P}(A)$ as follows ("sal" in the subscript stands for "special anti-lexicographic"). Take any $X, Y \in \mathcal{P}(A)$.

▷ $\emptyset$ is the least element of $\mathcal{P}(A)$ with respect to $\sqsubset_{\mathrm{sal}}$,
▷ if $X = \{x\}$ and $Y = \{y\}$ then $X \sqsubset_{\mathrm{sal}} Y$ if and only if $x \sqsubset y$,
▷ if $|X| = 1$ and $|Y| > 1$ then $X \sqsubset_{\mathrm{sal}} Y$,
▷ if $|X| > 1$ and $|Y| > 1$ then $X \sqsubset_{\mathrm{sal}} Y$ if and only if $X \sqsubset_{\mathrm{alex}} Y$.

For a set A and an integer k let $\binom{A}{k}$ denote the set of all the k-element subsets of A. Let $r \geqslant 2$ be an integer. A *linearly ordered reflexive r-uniform hypergraph* is a triple $(V, E, \sqsubset)$, where $E = \binom{V}{1} \cup S$ for some $S \subseteq \binom{V}{r}$ and $\sqsubset$ is a linear order on V. A mapping $f\colon V \to V'$ between (unordered) reflexive r-uniform hypergraphs $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$ is a *hypergraph homomorphism* if $e \in E$ implies $\{f(x)\colon x \in e\} \in E'$.

Definition 4.7. Let $\mathcal{G} = (V, E, \sqsubset)$ and $\mathcal{G}' = (V', E', \sqsubset')$ be linearly ordered reflexive r-uniform hypergraphs. Each hypergraph homomorphism $f\colon (V, E) \to (V', E')$ induces a mapping $\tilde{f}\colon E \to E'$ straightforwardly: $\tilde{f}(e) = \{f(x)\colon x \in e\}$. A hypergraph homomorphism $f\colon (V, E) \to (V', E')$ is a *strong rigid quotient map of hypergraphs* if

▷ $\tilde{f}\colon (E, \sqsubset_{\mathrm{sal}}) \to (E', \sqsubset'_{\mathrm{sal}})$ is a rigid surjection and
▷ for every $e = \{x_1, \ldots, x_r\} \in E$, if $f\restriction_e$ is not a constant map then $x_i \sqsubset x_j \Rightarrow f(x_i) \sqsubset' f(x_j)$ for all i and j.

Example 4.8. Let $C_3 = (\{1, 2, 3\}, E_3, <)$ and $C_4 = (\{1, 2, 3, 4\}, E_4, <)$ be the reflexive 3-cycle and the reflexive 4-cycle, respectively, where $E_3 = \{1, 2, 3, 12, 23, 31\}$, $E_4 = \{1, 2, 3, 4, 12, 23, 34, 14\}$, and $<$ is the usual ordering of the integers. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}$$

be two quotient maps $(\{1, 2, 3, 4\}, E_4) \to (\{1, 2, 3\}, E_3)$. Then f is a strong rigid quotient map and g is not. Namely,

$$\tilde{f} = \begin{pmatrix} 1 & 2 & 3 & 4 & 12 & 23 & 14 & 34 \\ 1 & 1 & 2 & 3 & 1 & 12 & 13 & 23 \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} 1 & 2 & 3 & 4 & 12 & 23 & 14 & 34 \\ 1 & 2 & 3 & 3 & 12 & 23 & 13 & 3 \end{pmatrix}$$

and we can now easily see that $\tilde{f}\colon (E_4, <_{\mathrm{sal}}) \to (E_3, <_{\mathrm{sal}})$ is rigid while $\tilde{g}\colon (E_4, <_{\mathrm{sal}}) \to (E_3, <_{\mathrm{sal}})$ is not.

Let $\mathrm{OHgr}_{\mathrm{srq}}(r)$ be the category whose objects are finite linearly ordered reflexive r-uniform hypergraphs and whose morphisms are strong rigid quotient maps of hypergraphs. Note that $\mathrm{OHgr}_{\mathrm{srq}}(2)$ is the category whose objects are finite linearly ordered reflexive graphs and whose morphisms are strong rigid quotient maps of graphs. Let us denote this category by $\mathrm{OGra}_{\mathrm{srq}}$.

For a finite chain $\mathcal{L} = \{l_1 \sqsubset l_2 \sqsubset \ldots \sqsubset l_N\}$ and $r \geqslant 2$ let

$$r \boxtimes \mathcal{L} = (\{1, 2, \ldots, r\} \times \{l_1, l_2, \ldots, l_N\}, E_{r\boxtimes\mathcal{L}}, \prec)$$

denote a linearly ordered reflexive r-uniform hypergraph on $r \cdot N$ vertices (written il instead of (i, l)), where

$$E_{r\boxtimes\mathcal{L}} = \{\{il_j\}\colon\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant N\} \cup \{\{1l_j, 2l_j, \ldots, rl_j\}\colon\ 1 \leqslant j \leqslant N\}$$

and $\prec$ is the anti-lexicographic ordering of $\{1, 2, \ldots, r\} \times \{l_1, l_2, \ldots, l_N\}$ induced by the respective linear orderings: $il \prec jk$ if and only if $l \sqsubset k$, or $l = k$ and $i < j$.

Corollary 4.9. *Let* $\mathbf{C}$ *be a full subcategory of* $\mathrm{OHgr}_{\mathrm{srq}}(r)$*,* $r \geqslant 2$*, such that for every finite chain* $\mathcal{L}$ *there is an object in* $\mathrm{Ob}(\mathbf{C})$ *isomorphic to* $r \boxtimes \mathcal{L}$*. Then* $\mathbf{C}$ *has the dual Ramsey property. In particular, the following categories have the dual Ramsey property:*

- ▷ *the category* $\mathrm{OHgr}_{\mathrm{srq}}(r)$ *for every* $r \geqslant 2$*,*
- ▷ *the category* $\mathrm{OGra}_{\mathrm{srq}}$*,*
- ▷ *the full subcategory of* $\mathrm{OGra}_{\mathrm{srq}}$ *spanned by bipartite graphs,*
- ▷ *the full subcategory of* $\mathrm{OGra}_{\mathrm{srq}}$ *spanned by* K_n*-free graphs (where* $n \geqslant 3$ *is fixed).*

Proof. Let us start by proving that $\mathrm{OHgr}_{\mathrm{srq}}(r)$ and $\mathrm{ERst}_{\mathrm{srq}}(r)$ are isomorphic. Define a functor

$$F\colon\ \mathrm{OHgr}_{\mathrm{srq}}(r) \to \mathrm{ERst}_{\mathrm{srq}}(r)\colon\ (V, E, \sqsubset) \mapsto (V, \varrho, \sqsubset)\colon\ f \mapsto f,$$

where

$$\varrho = \Delta_{V,r} \cup \{(x_1, x_2, \ldots, x_r) \in V^r\colon\ x_1 \sqsubset x_2 \sqsubset \ldots \sqsubset x_r \text{ and } \{x_1, x_2, \ldots, x_r\} \in E\}.$$

On the other hand, define a functor

$$G\colon\ \mathrm{ERst}_{\mathrm{srq}}(r) \to \mathrm{OHgr}_{\mathrm{srq}}(r)\colon\ (A, \varrho, \sqsubset) \mapsto (A, E, \sqsubset)\colon\ f \mapsto f,$$

where

$$E = \{\{x_1, x_2, \ldots, x_r\}\colon\ (x_1, x_2, \ldots, x_r) \in \varrho\}.$$

By construction, F and G are mutually inverse functors, so the categories $\mathrm{OHgr}_{\mathrm{srq}}(r)$ and $\mathrm{ERst}_{\mathrm{srq}}(r)$ are isomorphic. However, we still have to show that the functors F and G are well defined. It is easy to see that both F and G are well defined on objects. Let us show that both F and G are well defined on morphisms.

Let $f\colon (V, E, \sqsubset) \to (V', E', \sqsubset')$ be a morphism in $\mathrm{OHgr}_{\mathrm{srq}}(r)$ and let us show that $f\colon (V, \varrho) \to (V', \varrho')$ is a homomorphism. Take any $(x_1, x_2, \ldots, x_r) \in \varrho \setminus \Delta_{V,r}$. Then

$$\{x_1, x_2, \ldots, x_r\} \in E \quad \text{and} \quad x_1 \sqsubset x_2 \sqsubset \ldots \sqsubset x_r.$$

If $f(x_1) = f(x_2) = \ldots = f(x_r)$ we are done. Assume, therefore, that $f\restriction_{\{x_1,x_2,\ldots,x_r\}}$ is not a constant map. Then, by the definition of morphisms in $\mathrm{OHgr}_{\mathrm{srq}}(r)$ it follows that

$$\{f(x_1), f(x_2), \ldots, f(x_r)\} \in E \quad \text{and} \quad f(x_1) \sqsubset' f(x_2) \sqsubset' \ldots \sqsubset' f(x_r).$$

Therefore, $(f(x_1), f(x_2), \ldots, f(x_r)) \in \varrho'$.

Conversely, let $f\colon (V, \varrho, \sqsubset) \to (V', \varrho', \sqsubset')$ be a morphism in $\mathrm{ERst}_{\mathrm{srq}}(r)$ and let us show that $f\colon (V, E) \to (V', E')$ is a homomorphism of hypergraphs. Take any $\{x_1, x_2, \ldots, x_r\} \in E$. If $f(x_1) = f(x_2) = \ldots = f(x_r)$ we are done. Assume, therefore, that $f\restriction_{\{x_1,x_2,\ldots,x_r\}}$ is not a constant map. Without loss of generality we may assume that $x_1 \sqsubset x_2 \sqsubset \ldots \sqsubset x_r$. Then $(x_1, x_2, \ldots, x_r) \in \varrho$, so $(f(x_1), f(x_2), \ldots, f(x_r)) \in \varrho'$ because f is a morphism in $\mathrm{ERst}_{\mathrm{srq}}(r)$. Therefore, $\{f(x_1), f(x_2), \ldots, f(x_r)\} \in E'$ and $f(x_1) \sqsubset' f(x_2) \sqsubset' \ldots \sqsubset' f(x_r)$. So, f is a homomorphism satisfying the additional requirement that for every $e = \{x_1, \ldots, x_r\} \in E$, if $f\restriction_e$ is not a constant map then $x_i \sqsubset x_j \Rightarrow f(x_i) \sqsubset' f(x_j)$ for all i and j.

In order to complete the proof that F and G are well defined on morphisms we still have to show that $\hat{f}\colon (\varrho, \sqsubset_{\mathrm{sal}}) \to (\varrho', \sqsubset'_{\mathrm{sal}})$ is a rigid surjection if and only if $\tilde{f}\colon (E, \sqsubset_{\mathrm{sal}}) \to (E', \sqsubset'_{\mathrm{sal}})$ is a rigid surjection. But this follows straightforwardly from the following facts:

- ▷ the mapping $\xi\colon \varrho \to E\colon (x_1, x_2, \ldots, x_r) \mapsto \{x_1, x_2, \ldots, x_r\}$ is an isomorphism from $(\varrho, \sqsubset_{\mathrm{sal}})$ to $(E, \sqsubset_{\mathrm{sal}})$,
- ▷ the mapping $\xi'\colon \varrho' \to E'\colon (x_1, x_2, \ldots, x_r) \mapsto \{x_1, x_2, \ldots, x_r\}$ is an isomorphism from $(\varrho', \sqsubset'_{\mathrm{sal}})$ to $(E', \sqsubset'_{\mathrm{sal}})$, and
- ▷ $\tilde{f} = \xi' \circ \hat{f} \circ \xi^{-1}$.

Therefore, F and G are well defined functors.

The first item in the statement of the theorem now follows immediately from Theorem 4.4, having in mind that $F(r \boxtimes \mathcal{L}) = r \otimes \mathcal{L}$ for every chain $\mathcal{L}$. As for the remaining items, note that $\mathrm{OGra}_{\mathrm{srq}} = \mathrm{OHgr}_{\mathrm{srq}}(2)$ and that both subcategories of $\mathrm{OGra}_{\mathrm{srq}}$ mentioned in the third and the fourth item contain $2 \boxtimes \mathcal{L}$ for every finite chain $\mathcal{L}$. □

5. The Dual Nešetřil-Rödl Theorem

In this section we prove a dual form of the Nešetřil-Rödl Theorem, in its restricted form which does not account for subclasses defined by forbidden "quotients". Let Θ be a relational language and let $\sqsubset \notin \Theta$ be a new binary relational symbol. A *reflexive Θ-structure with a linear extension* (or Θ-*erst* for short) is a linearly ordered reflexive Θ-structure $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ such that $(A, \theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ is an $\mathrm{ar}(\theta)$-erst for every $\theta \in \Theta$.

Definition 5.1. Let $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ and $\mathcal{B} = (B, \Theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}})$ be two Θ-erst's. A homomorphism $f\colon (A, \Theta^{\mathcal{A}}) \to (B, \Theta^{\mathcal{B}})$ is a *strong rigid quotient map* from $\mathcal{A}$ to $\mathcal{B}$ if $\hat{f}\colon (\theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}}_{\mathrm{sal}}) \to (\theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}}_{\mathrm{sal}})$ is a rigid surjection for every $\theta \in \Theta$.

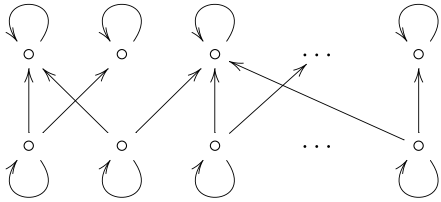
Let $\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset)$ be the category whose objects are all finite Θ-erst's and whose morphisms are strong rigid quotient maps. In order to prove that $\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset)$ has the dual Ramsey property we employ a strategy devised in [5]. Let us recall two technical statements from [5].

A *diagram* in a category $\mathbf{C}$ is a functor $F\colon \Delta \to \mathbf{C}$, where the category Δ is referred to as the *shape of the diagram*. We say that a diagram $F\colon \Delta \to \mathbf{C}$ is *consistent in* $\mathbf{C}$ if there exist a $C \in \mathrm{Ob}(\mathbf{C})$ and a family of morphisms $(h_\delta\colon F(\delta) \to C)_{\delta \in \mathrm{Ob}(\Delta)}$ such that for every morphism $g\colon \delta \to \gamma$ in Δ we have $h_\gamma \cdot F(g) = h_\delta$:

$$\begin{tikzcd} & C & \\ F(\delta) \arrow[ur, "h_\delta"] \arrow[rr, "F(g)"'] & & F(\gamma) \arrow[ul, "h_\gamma"'] \end{tikzcd}$$

We say that C together with the family of morphisms $(h_\delta)_{\delta \in \mathrm{Ob}(\Delta)}$ forms a *compatible cone in* $\mathbf{C}$ *over the diagram* F.

A *binary category* is a finite, acyclic, bipartite digraph with loops, where all the arrows go from one class of vertices into the other and the out-degree of all the vertices in the first class is 2 (modulo loops):



A *binary diagram* in a category $\mathbf{C}$ is a functor $F\colon \Delta \to \mathbf{C}$, where Δ is a binary category, F takes the bottom row of Δ onto the same object and takes the top row of Δ onto the same object, see Figure 2. A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is *closed for binary diagrams* if every binary diagram $F\colon \Delta \to \mathbf{D}$ which is consistent in $\mathbf{C}$ is also consistent in $\mathbf{D}$.

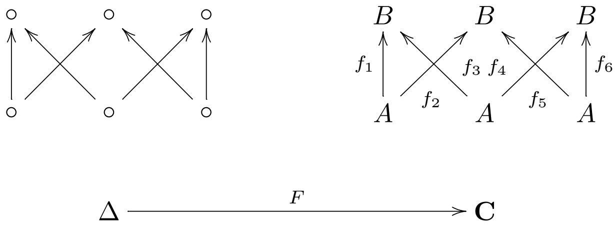


Figure 2. A binary diagram in $\mathbf{C}$ (of shape Δ)

Theorem 5.2 ([5])**.** *Let $\mathbf{C}$ be a category such that every morphism in $\mathbf{C}$ is monic and such that $\hom_{\mathbf{C}}(A, B)$ is finite for all $A, B \in \mathrm{Ob}(\mathbf{C})$, and let $\mathbf{D}$ be a (not necessarily full) subcategory of $\mathbf{C}$. If $\mathbf{C}$ has the Ramsey property and $\mathbf{D}$ is closed for binary diagrams, then $\mathbf{D}$ has the Ramsey property.*

We are now ready to prove the main technical result of this section.

Proposition 5.3. *The category $\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset)$ has the dual Ramsey property for every relational language Θ.*

Proof. Part I. Assume, first, that $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is a finite relational language and let $r_i = \mathrm{ar}(\theta_i)$, $1 \leqslant i \leqslant n$. Let $\mathbf{C}_i$ denote the category $\mathrm{ERst}_{\mathrm{srq}}(\{\theta_i\}, \sqsubset)$, $1 \leqslant i \leqslant n$. For each i we have that $\mathrm{ERst}_{\mathrm{srq}}(\{\theta_i\}, \sqsubset) = \mathrm{ERst}_{\mathrm{srq}}(r_i)$, so $\mathrm{ERst}_{\mathrm{srq}}(\{\theta_i\}, \sqsubset)$ has the dual Ramsey property (see Theorem 4.4).

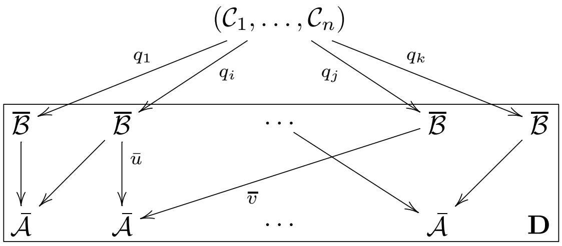
For an object $\mathcal{A} = (A, \theta_1^{\mathcal{A}}, \ldots, \theta_n^{\mathcal{A}}, \sqsubset^{\mathcal{A}}) \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset))$ let $\mathcal{A}^{(i)} = (A, \theta_i^{\mathcal{A}}, \sqsubset^{\mathcal{A}}) \in \mathrm{Ob}(\mathbf{C}_i)$. As we have just seen each $\mathbf{C}_i$ has the dual Ramsey property, so the product category $\mathbf{C}_1 \times \ldots \times \mathbf{C}_n$ has the dual Ramsey property by Corollary 3.10. Let $\mathbf{D}$ be the following subcategory of $\mathbf{C}_1 \times \ldots \times \mathbf{C}_n$:

- ▷ every $\mathcal{A} = (A, \theta_1^{\mathcal{A}}, \ldots, \theta_n^{\mathcal{A}}, \sqsubset^{\mathcal{A}}) \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset))$ gives rise to an object $\bar{\mathcal{A}} = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(n)})$ of $\mathbf{D}$, and these are the only objects in $\mathbf{D}$,
- ▷ every morphism $f\colon \mathcal{A} \to \mathcal{B}$ in $\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset)$ gives rise to a morphism $\overline{f} = (f, \ldots, f)\colon \bar{\mathcal{A}} \to \overline{\mathcal{B}}$ in $\mathbf{D}$, and these are the only morphisms in $\mathbf{D}$.

Clearly, the categories $\mathbf{D}$ and $\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset)$ are isomorphic, so in order to complete the proof of the lemma it suffices to show that $\mathbf{D}$ has the dual Ramsey property.

As $\mathbf{D}$ is a subcategory of $\mathbf{C}_1 \times \ldots \times \mathbf{C}_n$ and the latter one has the dual Ramsey property, following Theorem 5.2 it suffices to show that $\mathbf{D}^{\mathrm{op}}$ is closed for binary diagrams in $(\mathbf{C}_1 \times \ldots \times \mathbf{C}_n)^{\mathrm{op}}$.

Let $F\colon \Delta \to \mathbf{D}^{\mathrm{op}}$ be a binary diagram in $\mathbf{D}^{\mathrm{op}}$, where the top row consists of copies of $\overline{\mathcal{B}} \in \mathrm{Ob}(\mathbf{D})$ and the bottom row consists of copies of $\bar{\mathcal{A}} \in \mathrm{Ob}(\mathbf{D})$ for some $\mathcal{A} = (A, \theta_1^{\mathcal{A}}, \dots, \theta_n^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ and $\mathcal{B} = (B, \theta_1^{\mathcal{B}}, \dots, \theta_n^{\mathcal{B}}, \sqsubset^{\mathcal{B}})$. Assume that F is consistent in $(\mathbf{C}_1 \times \dots \times \mathbf{C}_n)^{\mathrm{op}}$ and let $(\mathcal{C}_1, \dots, \mathcal{C}_n)$ together with the morphisms $q_1, \dots, q_k$ be a compatible cone in $(\mathbf{C}_1 \times \dots \times \mathbf{C}_n)^{\mathrm{op}}$ over F:



Let $\mathcal{C}_i = (C_i, \theta_i^{\mathcal{C}_i}, \sqsubset^i)$ and $q_i = (q_i^1, \dots, q_i^n)$, where $q_i^s\colon \mathcal{C}_s \to \mathcal{B}^{(s)}$ is a strong rigid quotient map. Without loss of generality we can assume that $C_1, C_2, \dots, C_n$ are pairwise disjoint sets. Let $\mathcal{D} = (D, \theta_1^{\mathcal{D}}, \dots, \theta_n^{\mathcal{D}}, \sqsubset^{\mathcal{D}})$, where

- ▷ $D = C_1 \cup C_2 \cup \dots \cup C_n$,
- ▷ $\theta_i^{\mathcal{D}} = \Delta_{D,r_i} \cup \theta_i^{\mathcal{C}_i}$, $1 \leqslant i \leqslant n$, and
- ▷ $\sqsubset^{\mathcal{D}}$ is the linear order on D obtained by concatenating the linear orders $\sqsubset^1, \sqsubset^2, \dots, \sqsubset^n$; in other words, $\sqsubset^{\mathcal{D}}$ is the unique linear order on D such that $\sqsubset^{\mathcal{D}} \restriction_{C_i} = \sqsubset^i$, $1 \leqslant i \leqslant n$, and if $x \in C_i$ and $y \in C_j$, where $i < j$ then $x \sqsubset^{\mathcal{D}} y$.

Clearly, $\mathcal{D} \in \mathrm{Ob}(\mathrm{ERst_{srq}}(\Theta, \sqsubset))$, so $\overline{\mathcal{D}} \in \mathrm{Ob}(\mathbf{D})$.

For each morphism $q_i = (q_i^1, \dots, q_i^n)$ let $\varphi_i\colon D \to B$ be the mapping

$$\varphi_i(x) = \begin{cases} q_i^1(x), & x \in C_1, \\ q_i^2(x), & x \in C_2, \\ \quad\vdots & \\ q_i^n(x), & x \in C_n. \end{cases}$$

Let us show that $\varphi_i\colon \mathcal{D} \to \mathcal{B}$ is a strong rigid quotient map, $1 \leqslant i \leqslant k$. It is easy to see that each φ_i is a homomorphism $(D, \theta_1^{\mathcal{D}}, \dots, \theta_n^{\mathcal{D}}) \to (B, \theta_1^{\mathcal{B}}, \dots, \theta_n^{\mathcal{B}})$. Take any s and any $(x_1, x_2, \dots, x_{r_s}) \in \theta_s^{\mathcal{D}} \setminus \Delta_{D,r_s}$. Then $(x_1, x_2, \dots, x_{r_s}) \in \theta_s^{\mathcal{C}_s}$ whence $\{x_1, x_2, \dots, x_{r_s}\} \subseteq C_s$. But then

$$(\varphi_i(x_1), \varphi_i(x_2), \dots, \varphi_i(x_{r_s})) = (q_i^s(x_1), q_i^s(x_2), \dots, q_i^s(x_{r_s})) \in \theta_s^{\mathcal{B}}$$

because $q_i^s\colon \mathcal{C}_s \to \mathcal{B}^{(s)}$ is a homomorphism.

Next, take any $s \in \{1, 2, \dots, n\}$ and let us show that $\widehat{\varphi}_i \colon (\theta_s^{\mathcal{D}}, \sqsubset_{\text{sal}}^{\mathcal{D}}) \to (\theta_s^{\mathcal{B}}, \sqsubset_{\text{sal}}^{\mathcal{B}})$ is a rigid surjection.

Case 1°: $s = 1$. The construction of $\sqsubset^{\mathcal{D}}$ then ensures that

$$\min \widehat{\varphi}_i^{-1}(x_1, x_2, \dots, x_{r_1}) = \min(\hat{q}_i^1)^{-1}(x_1, x_2, \dots, x_{r_1})$$

for every $(x_1, x_2, \dots, x_{r_1}) \in \theta_1^{\mathcal{B}}$, so

$$\begin{aligned}\min \widehat{\varphi}_i^{-1}(x_1, x_2, \dots, x_{r_1}) &= \min(\hat{q}_i^1)^{-1}(x_1, x_2, \dots, x_{r_1}) \sqsubset_{\text{sal}}^{\mathcal{D}} \min(\hat{q}_i^1)^{-1}(y_1, y_2, \dots, y_{r_1}) \\ &= \min \widehat{\varphi}_i^{-1}(y_1, y_2, \dots, y_{r_1})\end{aligned}$$

for all $(x_1, x_2, \dots, x_{r_1}), (y_1, y_2, \dots, y_{r_1}) \in \theta_1^{\mathcal{B}}$ satisfying $(x_1, x_2, \dots, x_{r_1}) \sqsubset_{\text{sal}}^{\mathcal{B}} (y_1, y_2, \dots, y_{r_1})$ because q_i^1 is a strong rigid quotient map.

Case 2°: $s > 1$. Take $(x_1, x_2, \dots, x_{r_s}), (y_1, y_2, \dots, y_{r_s}) \in \theta_s^{\mathcal{B}}$ such that $(x_1, x_2, \dots, x_{r_s}) \sqsubset_{\text{sal}}^{\mathcal{B}} (y_1, y_2, \dots, y_{r_s})$.

Case 2.1°: $(x_1, x_2, \dots, x_{r_s}) \in \Delta_{B, r_s}$. Then, by the construction of $\sqsubset^{\mathcal{D}}$,

$$\begin{aligned}\min \widehat{\varphi}_i^{-1}(x_1, x_2, \dots, x_{r_s}) &= \min(\hat{q}_i^1)^{-1}(x_1, x_2, \dots, x_{r_s}), \\ \min \widehat{\varphi}_i^{-1}(y_1, y_2, \dots, y_{r_s}) &= \min(\hat{q}_i^t)^{-1}(y_1, y_2, \dots, y_{r_s}),\end{aligned}$$

where $t = 1$ if $(y_1, y_2, \dots, y_{r_s}) \in \Delta_{B, r_s}$, or $t = s$ if $(y_1, y_2, \dots, y_{r_s}) \in \theta_s^{\mathcal{C}_s} \setminus \Delta_{B, r_s}$. If $t = 1$ we are done because q_i^1 is a strong rigid quotient map. If, however, $t = s$, we are done by the definition of $\sqsubset_{\text{sal}}^{\mathcal{D}}$.

Case 2.2°: $(x_1, x_2, \dots, x_{r_s}) \notin \Delta_{B, r_s}$. Then $(y_1, y_2, \dots, y_{r_s}) \notin \Delta_{B, r_s}$ by definition of $\sqsubset_{\text{sal}}^{\mathcal{D}}$. Therefore,

$$\begin{aligned}\min \widehat{\varphi}_i^{-1}(x_1, x_2, \dots, x_{r_s}) &= \min(\hat{q}_i^s)^{-1}(x_1, x_2, \dots, x_{r_s}), \\ \min \widehat{\varphi}_i^{-1}(y_1, y_2, \dots, y_{r_s}) &= \min(\hat{q}_i^s)^{-1}(y_1, y_2, \dots, y_{r_s}),\end{aligned}$$

and the claim follows because q_i^s is a strong rigid quotient map.

Therefore, $\varphi_i \colon \mathcal{D} \to \mathcal{B}$ is a strong rigid quotient map for each i, whence follows that $\bar{\varphi}_i \colon \overline{\mathcal{D}} \to \overline{\mathcal{B}}$ is a morphism in $\mathbf{D}$ for each i. To complete the proof we still have to show that $\bar{u} \circ \bar{\varphi}_i = \overline{v} \circ \bar{\varphi}_j$ whenever $\bar{u} \circ q_i = \overline{v} \circ q_j$. Assume that $\bar{u} \circ q_i = \overline{v} \circ q_j$. Take any $x \in D$. Then $x \in C_s$ for some s, so

$$u \circ \varphi_i(x) = u \circ q_i^s(x) = v \circ q_j^s(x) = v \circ \varphi_j(x),$$

because $\bar{u} \circ q_i = \overline{v} \circ q_j$ means that $u \circ q_i^t = v \circ q_j^t$ for each t. Therefore, $\bar{u} \circ \bar{\varphi}_i = \overline{v} \circ \bar{\varphi}_j$.

This concludes the proof in case Θ is a finite relational language.

Part II. Assume now that Θ is an arbitrary relational language satisfying $\sqsubset \notin \Theta$, and take any $k \geqslant 2$ and $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset))$ such that there is a strong rigid quotient map $\mathcal{B} \to \mathcal{A}$.

Since $\mathcal{B}$ is a finite Θ-erst, $\theta^{\mathcal{B}} = \emptyset$ for every $\theta \in \Theta$ such that $\mathrm{ar}(\theta) > |B|$. Moreover, on a finite set there are only finitely many relations whose arities do not exceed $|B|$. Therefore, there exists a finite $\Sigma \subseteq \Theta$ such that for every $\theta \in \Theta \setminus \Sigma$ we have $\theta^{\mathcal{B}} = \emptyset$ or $\theta^{\mathcal{B}} = \sigma^{\mathcal{B}}$ for some $\sigma \in \Sigma$. Since there is a strong rigid quotient map $\mathcal{B} \to \mathcal{A}$, we have the following:

▷ if $\theta^{\mathcal{B}} = \emptyset$ for some $\theta \in \Theta \setminus \Sigma$ then $\theta^{\mathcal{A}} = \emptyset$,
▷ if $\theta^{\mathcal{B}} = \sigma^{\mathcal{B}}$ for some $\theta \in \Theta \setminus \Sigma$ and $\sigma \in \Sigma$ then $\theta^{\mathcal{A}} = \sigma^{\mathcal{A}}$.

The category $\mathrm{ERst}_{\mathrm{srq}}(\Sigma, \sqsubset)$ has the dual Ramsey property because Σ is finite (Part I), so there is a $\mathcal{C} = (C, \Sigma^{\mathcal{C}}, \sqsubset^{\mathcal{C}}) \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(\Sigma, \sqsubset))$ such that

$$\mathcal{C} \to (\mathcal{B}|_{\Sigma \cup \{\sqsubset\}})_k^{\mathcal{A}|_{\Sigma \cup \{\sqsubset\}}}$$

in $\mathrm{ERst}_{\mathrm{srq}}(\Sigma, \sqsubset)^{\mathrm{op}}$. Define $\mathcal{C}^* = (C, \Theta^{\mathcal{C}^*}, \sqsubset^{\mathcal{C}^*}) \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset))$ as follows:

▷ $\sqsubset^{\mathcal{C}^*} = \sqsubset^{\mathcal{C}}$,
▷ if $\sigma \in \Sigma$ let $\sigma^{\mathcal{C}^*} = \sigma^{\mathcal{C}}$,
▷ if $\theta \in \Theta \setminus \Sigma$ and $\theta^{\mathcal{B}} = \emptyset$ let $\theta^{\mathcal{C}^*} = \emptyset$,
▷ if $\theta \in \Theta \setminus \Sigma$ and $\theta^{\mathcal{B}} = \sigma^{\mathcal{B}}$ for some $\sigma \in \Sigma$, let $\theta^{\mathcal{C}^*} = \sigma^{\mathcal{C}^*}$.

Clearly, $\mathcal{C}^*$ is a Θ-erst and $\mathcal{C}^* \to (\mathcal{B})_k^{\mathcal{A}}$ in $\mathrm{ERst}_{\mathrm{srq}}(\Theta, \sqsubset)^{\mathrm{op}}$ because

$$\hom_{\mathrm{ERst}_{\mathrm{srq}}(\Sigma,\sqsubset)}(\mathcal{C}, \mathcal{A}|_{\Sigma} \cup \{\sqsubset\}) = \hom_{\mathrm{ERst}_{\mathrm{srq}}(\Theta,\sqsubset)}(\mathcal{C}^*, \mathcal{A}),$$
$$\hom_{\mathrm{ERst}_{\mathrm{srq}}(\Sigma,\sqsubset)}(\mathcal{C}, \mathcal{B}|_{\Sigma} \cup \{\sqsubset\}) = \hom_{\mathrm{ERst}_{\mathrm{srq}}(\Theta,\sqsubset)}(\mathcal{C}^*, \mathcal{B}).$$

This concludes the proof. □

Definition 5.4. Let $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ and $\mathcal{B} = (B, \Theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}})$ be two linearly ordered reflexive Θ-structures. A homomorphism $f\colon (A, \Theta^{\mathcal{A}}) \to (B, \Theta^{\mathcal{B}})$ is a *strong rigid quotient map of structures* if the following holds for every $\theta \in \Theta$:

▷ $\hat{f}\colon (\theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}}_{\mathrm{sal}}) \to (\theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}}_{\mathrm{sal}})$ is a rigid surjection; and
▷ for every $(x_1, x_2, \ldots, x_r) \in \theta^{\mathcal{A}}$, if $f\restriction_{\{x_1, x_2, \ldots, x_r\}}$ is not a constant map then $x_i \sqsubset^{\mathcal{A}} x_j \Rightarrow f(x_i) \sqsubset^{\mathcal{B}} f(x_j)$ for all i and j.

Let $\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)$ be the category whose objects are all finite linearly ordered reflexive Θ-structures and whose morphisms are strong rigid quotient maps of structures. Our final result is a dual version of the Nešetřil-Rödl Theorem.

Theorem 5.5 (Dual Nešetřil-Rödl Theorem (restricted form))**.** *Let Θ be a relational language and let $\sqsubset \notin \Theta$ be a binary relational symbol. Then $\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)$ has the dual Ramsey property.*

Proof. Fix a relational language Θ such that $\sqsubset \notin \Theta$. Let

$$X_\Theta = \{\theta/\sigma\colon\ \theta \in \Theta \text{ and } \sigma \text{ is a total quasiorder on } \{1, 2, \dots, \mathrm{ar}(\theta)\}\}$$

be a relational language where θ/σ is a new relational symbol (formally, a pair (θ, σ)) such that

$$\mathrm{ar}(\theta/\sigma) = |\{1, 2, \dots, \mathrm{ar}(\theta)\}/\equiv_\sigma|.$$

We are going to show that the categories $\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)$ and $\mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset)$ are isomorphic. The dual Ramsey property for $\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)$ then follows directly from Proposition 5.3.

For $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}}) \in \mathrm{Ob}(\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset))$ define an $\mathcal{A}^\dagger = (A, X_\Theta^{\mathcal{A}^\dagger}, \sqsubset^{\mathcal{A}^\dagger})$ as follows:

$$\begin{aligned} \sqsubset^{\mathcal{A}^\dagger} &= \sqsubset^{\mathcal{A}}, \\ (\theta/\sigma)^{\mathcal{A}^\dagger} &= \Delta_{A, \mathrm{ar}(\theta/\sigma)} \cup \{\mathrm{mat}(\bar{a})\colon\ \bar{a} \in \theta^{\mathcal{A}} \text{ and } \mathrm{tp}(\bar{a}) = \sigma\}. \end{aligned}$$

On the other hand, take any $\mathcal{B} = (B, X_\Theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}}) \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset))$ and define $\mathcal{B}^* = (B, \Theta^{\mathcal{B}^*}, \sqsubset^{\mathcal{B}^*}) \in \mathrm{Ob}(\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset))$ as follows:

$$\begin{aligned} &\sqsubset^{\mathcal{B}^*} = \sqsubset^{\mathcal{B}}, \\ &\theta^{\mathcal{B}^*} = \{\mathrm{tup}(\sigma, \bar{a})\colon\ \sigma \text{ is a total quasiorder on } \{1, 2, \dots, \mathrm{ar}(\theta)\} \text{ and } \bar{a} \in (\theta/\sigma)^{\mathcal{B}}\}. \end{aligned}$$

Consider the functors

$$\begin{aligned} F&\colon\ \mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset) \to \mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset)\colon\ \mathcal{A} \mapsto \mathcal{A}^\dagger\colon\ f \mapsto f, \\ G&\colon\ \mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset) \to \mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)\colon\ \mathcal{B} \mapsto \mathcal{B}^*\colon\ f \mapsto f. \end{aligned}$$

Because of (2.1) we have that $(\mathcal{A}^\dagger)^* = \mathcal{A}$ and $(\mathcal{B}^*)^\dagger = \mathcal{B}$ for all $\mathcal{A} \in \mathrm{Ob}(\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset))$ and all $\mathcal{B} \in \mathrm{Ob}(\mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset))$. Hence, F and G are mutually inverse functors, so $\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)$ and $\mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset)$ are isomorphic categories. However, we still have to show that F and G are well defined. Clearly, both functors are well defined on objects.

Let us show that F is well defined on morphisms. Take any morphism $f\colon\ \mathcal{A} \to \mathcal{B}$ in $\mathrm{Rel}_{\mathrm{srq}}(\Theta, \sqsubset)$, where $\mathcal{A} = (A, \Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ and $\mathcal{B} = (B, \Theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}})$.

To see that $f\colon\ (A, X_\Theta^{\mathcal{A}^\dagger}) \to (B, X_\Theta^{\mathcal{B}^\dagger})$ is a homomorphism, take any $\theta \in \Theta$, any total quasiorder σ on $\{1, 2, \dots, \mathrm{ar}(\theta)\}$ and any $(x_1, x_2, \dots, x_r) \in (\theta/\sigma)^{\mathcal{A}^\dagger} \setminus \Delta_{A,r}$, where $r = \mathrm{ar}(\theta/\sigma)$. Then there exists an $\bar{a} \in \theta^{\mathcal{A}}$ such that $\sigma = \mathrm{tp}(\bar{a})$ and $(x_1, x_2, \dots, x_r) = \mathrm{mat}(\bar{a})$. If $f(x_1) = f(x_2) = \ldots = f(x_r)$ we are done. Assume, therefore, that $f\restriction_{\{x_1, x_2, \dots, x_r\}}$ is not a constant map. Because f is a homomorphism of Θ-structures, $\hat{f}(\bar{a}) \in \theta^{\mathcal{B}}$. We also know that $\mathrm{tp}(\hat{f}(\bar{a})) = \mathrm{tp}(\bar{a}) = \sigma$ (see

Lemma 2.4), so $\mathrm{mat}(\hat{f}(\bar{a})) \in (\theta/\sigma)^{\mathcal{B}^\dagger}$. By using Lemma 2.4 again we have that $\mathrm{mat}(\hat{f}(\bar{a})) = \hat{f}(\mathrm{mat}(\bar{a})) = \hat{f}(x_1, x_2, \ldots, x_r) = (f(x_1), f(x_2), \ldots, f(x_r))$.

Next, let us show that $\hat{f}\restriction_{(\theta/\sigma)^{\mathcal{A}^\dagger}} : ((\theta/\sigma)^{\mathcal{A}^\dagger}, \sqsubset_{\mathrm{sal}}^{\mathcal{A}^\dagger}) \to ((\theta/\sigma)^{\mathcal{B}^\dagger}, \sqsubset_{\mathrm{sal}}^{\mathcal{B}^\dagger})$ is a rigid surjection for every $\theta \in \Theta$ and every total quasiorder σ on $\{1, 2, \ldots, \mathrm{ar}(\theta)\}$. For notational convenience we let $\hat{f}_\theta = \hat{f}\restriction_{\theta^\mathcal{A}}$ and $\hat{f}_{\theta/\sigma} = \hat{f}\restriction_{(\theta/\sigma)^{\mathcal{A}^\dagger}}$. Take any $\theta \in \Theta$, any total quasiorder σ on $\{1, 2, \ldots, \mathrm{ar}(\theta)\}$ and let $r = \mathrm{ar}(\theta/\sigma)$. Note, first, that $\hat{f}_{\theta/\sigma}$ is surjective because f is a quotient map (see Lemma 4.2) and $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\hat{f}(\bar{a}))$ whenever $\bar{a} = (a_1, a_2, \ldots, a_n) \in \theta^\mathcal{A}$ and $f\restriction_{\{a_1, a_2, \ldots, a_n\}}$ is not a constant map (see Lemma 2.4). Take any $(x_1, x_2, \ldots, x_r)$, $(y_1, y_2, \ldots, y_r) \in (\theta/\sigma)^{\mathcal{B}^\dagger}$ such that $(x_1, x_2, \ldots, x_r) \sqsubset_{\mathrm{sal}}^{\mathcal{B}^\dagger} (y_1, y_2, \ldots, y_r)$.

Case 1°: $|\{x_1, x_2, \ldots, x_r\}| = |\{y_1, y_2, \ldots, y_r\}| = 1$.

By Lemma 4.2 (a) we have that $\min \hat{f}_{\theta/\sigma}^{-1}(x_1, x_1, \ldots, x_1) = (s, s, \ldots, s)$, where $s = \min f^{-1}(x_1)$, and $\min \hat{f}_{\theta/\sigma}^{-1}(y_1, y_1, \ldots, y_1) = (t, t, \ldots, t)$, where $t = \min f^{-1}(y_1)$. Since $(x_1, x_1, \ldots, x_1) \sqsubset_{\mathrm{sal}}^{\mathcal{B}^\dagger} (y_1, y_1, \ldots, y_1)$, we know that $x_1 \sqsubset^\mathcal{B} y_1$, so $s \sqsubset^\mathcal{A} t$ because f is a rigid surjection $(A, \sqsubset^\mathcal{A}) \to (B, \sqsubset^\mathcal{B})$. But then

$$\min \hat{f}_{\theta/\sigma}^{-1}(x_1, x_1, \ldots, x_1) = (s, s, \ldots, s) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^\dagger} (t, t, \ldots, t) = \min \hat{f}_{\theta/\sigma}^{-1}(y_1, y_1, \ldots, y_1).$$

Case 2°: $|\{x_1, x_2, \ldots, x_r\}| = 1$ and $|\{y_1, y_2, \ldots, y_r\}| > 1$.

Then $\min \hat{f}_{\theta/\sigma}^{-1}(x_1, x_1, \ldots, x_1) = (s, s, \ldots, s)$ and $\min \hat{f}_{\theta/\sigma}^{-1}(y_1, y_2, \ldots, y_r) = (t_1, t_2, \ldots, t_r)$, where $|\{t_1, t_2, \ldots, t_r\}| > 1$, so

$$\mathrm{tp}(\min \hat{f}_{\theta/\sigma}^{-1}(x_1, x_1, \ldots, x_1)) \lhd \mathrm{tp}(\min \hat{f}_{\theta/\sigma}^{-1}(y_1, y_2, \ldots, y_r)),$$

whence $\min \hat{f}_{\theta/\sigma}^{-1}(x_1, x_1, \ldots, x_1) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^\dagger} \min \hat{f}_{\theta/\sigma}^{-1}(y_1, y_2, \ldots, y_r)$.

Case 3°: $|\{x_1, x_2, \ldots, x_r\}| > 1$ and $|\{y_1, y_2, \ldots, y_r\}| > 1$.

Let $(x_1, x_2, \ldots, x_r) = \mathrm{mat}(\bar{p})$ and $(y_1, y_2, \ldots, y_r) = \mathrm{mat}(\bar{q})$ for some $\bar{p}, \bar{q} \in \theta^\mathcal{B}$ such that $\mathrm{tp}(\bar{p}) = \mathrm{tp}(\bar{q}) = \sigma$. Since

$$\mathrm{mat}(\bar{p}) = (x_1, x_2, \ldots, x_r) \sqsubset_{\mathrm{sal}}^{\mathcal{B}^\dagger} (y_1, y_2, \ldots, y_r) = \mathrm{mat}(\bar{q})$$

and $\mathrm{tp}(\bar{p}) = \mathrm{tp}(\bar{q})$ we have that $\bar{p} \sqsubset_{\mathrm{sal}}^\mathcal{B} \bar{q}$ (see Lemma 2.3). Since $\hat{f}_\theta \colon (\theta^\mathcal{A}, \sqsubset_{\mathrm{sal}}^\mathcal{A}) \to (\theta^\mathcal{B}, \sqsubset_{\mathrm{sal}}^\mathcal{B})$ is a rigid surjection, we know that $\min \hat{f}_\theta^{-1}(\bar{p}) \sqsubset_{\mathrm{sal}}^\mathcal{A} \min \hat{f}_\theta^{-1}(\bar{q})$. On the other hand, $\mathrm{tp}(\min \hat{f}_\theta^{-1}(\bar{p})) = \mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{p}) = \mathrm{tp}(\bar{q}) = \mathrm{tp}(\bar{b}) = \mathrm{tp}(\min \hat{f}_\theta^{-1}(\bar{q}))$ for some $\bar{a} \in \hat{f}_\theta^{-1}(\bar{p})$ and $\bar{b} \in \hat{f}_\theta^{-1}(\bar{q})$, where the minimum is achieved, so by Lemma 2.3 we conclude that

$$\mathrm{mat}(\min \hat{f}_\theta^{-1}(\bar{p})) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^\dagger} \mathrm{mat}(\min \hat{f}_\theta^{-1}(\bar{q})).$$

By Lemma 2.5 we finally get $\min \hat{f}_{\theta/\sigma}^{-1}(\mathrm{mat}(\bar{p})) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^\dagger} \min \hat{f}_{\theta/\sigma}^{-1}(\mathrm{mat}(\bar{q}))$, that is, $\min \hat{f}_{\theta/\sigma}^{-1}(x_1, x_2, \ldots, x_r) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^\dagger} \min \hat{f}_{\theta/\sigma}^{-1}(y_1, y_2, \ldots, y_r)$. This concludes the proof of Case 3° and the proof that the functor F is well defined on morphisms.

Let us show that G is well defined on morphisms. Take any morphism $f\colon \mathcal{A} \to \mathcal{B}$ in $\mathrm{ERst}_{\mathrm{srq}}(X_\Theta, \sqsubset)$, where $\mathcal{A} = (A, X_\Theta^{\mathcal{A}}, \sqsubset^{\mathcal{A}})$ and $\mathcal{B} = (B, X_\Theta^{\mathcal{B}}, \sqsubset^{\mathcal{B}})$.

Let us first show that $f\colon (A, \Theta^{\mathcal{A}^*}) \to (B, \Theta^{\mathcal{B}^*})$ is a homomorphism. Take any $\theta \in \Theta$ and any $\bar{x} = (x_1, x_2, \dots, x_n) \in \theta^{\mathcal{A}^*}$. If $f\restriction_{\{x_1,x_2,\dots,x_n\}}$ is a constant map we are done. Assume, therefore, that this is not the case. Then $\bar{x} = \mathrm{tup}(\sigma, \bar{a})$ for some σ and some $\bar{a} \in (\theta/\sigma)^{\mathcal{A}}$. Because $f\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism, $\hat{f}(\bar{a}) \in (\theta/\sigma)^{\mathcal{B}}$, whence $\mathrm{tup}(\sigma, \hat{f}(\bar{a})) \in \theta^{\mathcal{B}^*}$. Therefore, $\hat{f}(\bar{x}) = \hat{f}(\mathrm{tup}(\sigma, \bar{a})) = \mathrm{tup}(\sigma, \hat{f}(\bar{a})) \in \theta^{\mathcal{B}^*}$, using Lemma 2.4 for the second equality.

Next, let us show that for every $\theta \in \Theta$ and every $(x_1, x_2, \dots, x_n) \in \theta^{\mathcal{A}^*}$, if $f\restriction_{\{x_1,x_2,\dots,x_n\}}$ is not a constant map then $x_i \sqsubset^{\mathcal{A}^*} x_j \Rightarrow f(x_i) \sqsubset^{\mathcal{B}^*} f(x_j)$ for all i and j. Take any $\theta \in \Theta$, any $\bar{x} = (x_1, x_2, \dots, x_n) \in \theta^{\mathcal{A}^*}$ and assume that $f\restriction_{\{x_1,x_2,\dots,x_n\}}$ is not a constant map. By definition of $\theta^{\mathcal{A}^*}$ we have that $\bar{x} = \mathrm{tup}(\sigma, \bar{a})$ for some σ and some $\bar{a} \in (\theta/\sigma)^{\mathcal{A}}$. Clearly, $\mathrm{mat}(\bar{x}) = \bar{a}$. Assume that $x_i \sqsubset^{\mathcal{A}^*} x_j$. Then $x_i \sqsubset^{\mathcal{A}} x_j$, so

$$\bar{a} = \mathrm{mat}(\bar{x}) = (\dots, x_i, \dots, x_j, \dots) \in (\theta/\sigma)^{\mathcal{A}}.$$

Because $f\colon \mathcal{A} \to \mathcal{B}$ is a homomorphism,

$$(\dots, f(x_i), \dots, f(x_j), \dots) \in (\theta/\sigma)^{\mathcal{B}},$$

so $f(x_i) \sqsubset^{\mathcal{B}} f(x_j)$, or, equivalently, $f(x_i) \sqsubset^{\mathcal{B}^*} f(x_j)$.

Finally, let us show that $\hat{f}\restriction_{\theta^{\mathcal{A}^*}}\colon (\theta^{\mathcal{A}^*}, \sqsubset_{\mathrm{sal}}^{\mathcal{A}^*}) \to (\theta^{\mathcal{B}^*}, \sqsubset_{\mathrm{sal}}^{\mathcal{B}^*})$ is a rigid surjection for every $\theta \in \Theta$. For notational convenience, this time we let $\hat{f}_\theta = \hat{f}\restriction_{\theta^{\mathcal{A}^*}}$ and $\hat{f}_{\theta/\sigma} = \hat{f}\restriction_{(\theta/\sigma)^{\mathcal{A}}}$.

Note, first, that $\hat{f}_\theta\colon \theta^{\mathcal{A}^*} \to \theta^{\mathcal{B}^*}$ is surjective because so is $\hat{f}_{\theta/\sigma}\colon (\theta/\sigma)^{\mathcal{A}} \to (\theta/\sigma)^{\mathcal{B}}$ for every σ. Take any $\bar{x}, \overline{y} \in \theta^{\mathcal{B}^*}$ such that $\bar{x} \sqsubset_{\mathrm{sal}}^{\mathcal{B}^*} \overline{y}$. Let $\bar{x} = (x_1, x_2, \dots, x_n)$ and $\overline{y} = (y_1, y_2, \dots, y_n)$.

Case 1°: $|\{x_1, x_2, \dots, x_n\}| = |\{y_1, y_2, \dots, y_n\}| = 1$.

By Lemma 4.2 (a) we have that $\min \hat{f}_\theta^{-1}(x_1, x_1, \dots, x_1) = (s, s, \dots, s)$, where $s = \min f^{-1}(x_1)$, and $\min \hat{f}_\theta^{-1}(y_1, y_1, \dots, y_1) = (t, t, \dots, t)$, where $t = \min f^{-1}(y_1)$. Since $(x_1, x_1, \dots, x_1) \sqsubset_{\mathrm{sal}}^{\mathcal{B}^*} (y_1, y_1, \dots, y_1)$, we know that $x_1 \sqsubset^{\mathcal{B}} y_1$, so $s \sqsubset^{\mathcal{A}} t$ because f is a rigid surjection $(A, \sqsubset^{\mathcal{A}}) \to (B, \sqsubset^{\mathcal{B}})$. But then

$$\min \hat{f}_\theta^{-1}(x_1, x_1, \dots, x_1) = (s, s, \dots, s) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^*} (t, t, \dots, t) = \min \hat{f}_\theta^{-1}(y_1, y_1, \dots, y_1).$$

Case 2°: $|\{x_1, x_2, \dots, x_n\}| = 1$ and $|\{y_1, y_2, \dots, y_n\}| > 1$.

Then $\min \hat{f}_\theta^{-1}(x_1, x_1, \dots, x_1) = (s, s, \dots, s)$ and $\min \hat{f}_\theta^{-1}(y_1, y_2, \dots, y_n) = (t_1, t_2, \dots, t_n)$, where $|\{t_1, t_2, \dots, t_n\}| > 1$, so

$$\mathrm{tp}(\min \hat{f}_\theta^{-1}(x_1, x_1, \dots, x_1)) \lhd \mathrm{tp}(\min \hat{f}_\theta^{-1}(y_1, y_2, \dots, y_n)),$$

whence $\min \hat{f}_\theta^{-1}(x_1, x_1, \dots, x_1) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^*} \min \hat{f}_\theta^{-1}(y_1, y_2, \dots, y_n)$.

Case 3°: $|\{x_1, x_2, \dots, x_n\}| > 1$ and $|\{y_1, y_2, \dots, y_n\}| > 1$.

By the definition of $\theta^{\mathcal{B}^*}$ we have that $\bar{x} = \mathrm{tup}(\sigma, \bar{a})$ for some σ, and some $\bar{a} \in (\theta/\sigma)^{\mathcal{B}}$ and $\overline{y} = \mathrm{tup}(\tau, \bar{b})$ for some τ and some $\bar{b} \in (\theta/\tau)^{\mathcal{B}}$.

Assume, first, that $\sigma \neq \tau$. Then $\bar{x} \sqsubset_{\mathrm{sal}}^{\mathcal{B}^*} \overline{y}$ actually means that $\mathrm{tp}(\bar{x}) \lhd \mathrm{tp}(\overline{y})$. Let $\min \hat{f}_\theta^{-1}(\bar{x}) = \bar{u} \in \theta^{\mathcal{A}^*}$ and $\min \hat{f}_\theta^{-1}(\overline{y}) = \overline{v} \in \theta^{\mathcal{A}^*}$. Lemma 2.4 then yields that $\mathrm{tp}(\bar{u}) = \mathrm{tp}(\bar{x}) \lhd \mathrm{tp}(\overline{y}) = \mathrm{tp}(\overline{v})$, so $\bar{u} \sqsubset_{\mathrm{sal}}^{\mathcal{A}^*} \overline{v}$.

Assume, now, that $\sigma = \tau$. Then $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\overline{y})$, so Lemma 2.3 implies that $\bar{a} = \mathrm{mat}(\bar{x}) \sqsubset_{\mathrm{sal}}^{\mathcal{B}} \mathrm{mat}(\overline{y}) = \bar{b}$. Since $\hat{f}_{\theta/\sigma} \colon ((\theta/\sigma)^{\mathcal{A}}, \sqsubset_{\mathrm{sal}}^{\mathcal{A}}) \to ((\theta/\sigma)^{\mathcal{B}}, \sqsubset_{\mathrm{sal}}^{\mathcal{B}})$ is a rigid surjection, it follows that $\min \hat{f}_{\theta/\sigma}^{-1}(\mathrm{mat}(\bar{x})) \sqsubset_{\mathrm{sal}}^{\mathcal{A}} \min \hat{f}_{\theta/\sigma}^{-1}(\mathrm{mat}(\overline{y}))$. Lemma 2.5 yields that

$$\min \hat{f}_{\theta/\sigma}^{-1}(\mathrm{mat}(\bar{x})) = \mathrm{mat}(\min \hat{f}_\theta^{-1}(\bar{x})) \quad \text{and} \quad \min \hat{f}_{\theta/\sigma}^{-1}(\mathrm{mat}(\overline{y})) = \mathrm{mat}(\min \hat{f}_\theta^{-1}(\overline{y})).$$

Therefore, $\mathrm{mat}(\min \hat{f}_\theta^{-1}(\bar{x})) \sqsubset_{\mathrm{sal}}^{\mathcal{A}} \mathrm{mat}(\min \hat{f}_\theta^{-1}(\overline{y}))$. On the other hand, $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\overline{y})$ implies that $\mathrm{tp}(\min \hat{f}_\theta^{-1}(\bar{x})) = \mathrm{tp}(\min \hat{f}_\theta^{-1}(\overline{y}))$. Lemma 2.3 then ensures that $\min \hat{f}_\theta^{-1}(\bar{x}) \sqsubset_{\mathrm{sal}}^{\mathcal{A}^*} \min \hat{f}_\theta^{-1}(\overline{y})$. This concludes the proof of Case 3°, the proof that the functor G is well defined on morphisms, and the proof of the theorem. □

6. Tournaments – a non-example

In this section we prove that the category whose objects are finite linearly ordered reflexive tournaments and whose morphisms are rigid surjective homomorphisms does not have the dual Ramsey property.

A linearly ordered reflexive tournament is a structure $(A, \to, \sqsubset)$, where $\sqsubset$ is a linear order on A and $\to \subseteq A^2$ is a reflexive relation such that for all $x \neq y$, either $x \to y$ or $y \to x$. A mapping $f \colon A \to A'$ is a *rigid surjective homomorphism* from $\mathcal{A} = (A, \to, \sqsubset)$ to $\mathcal{A}' = (A', \to', \sqsubset')$ if $f \colon (A, \to) \to (A', \to')$ is a homomorphism and $f \colon (A, \sqsubset) \to (A', \sqsubset')$ is a rigid surjection.

A reflexive tournament $(T, \to)$ is an *inflation* of a reflexive tournament $(S, \to)$ if there exists a surjective homomorphism $(T, \to) \to (S, \to)$. Finite reflexive tournaments $(S_1, \to)$ and $(S_2, \to)$ are *siblings* if there exists a finite reflexive tournament $(T, \to)$ which is an inflation of $(S_1, \to)$ and an inflation of $(S_2, \to)$. Let C_3 denote the reflexive tournament $(\{1, 2, 3\}, \to)$ whose nontrivial edges are $1 \to 2$, $2 \to 3$ and $3 \to 1$, and let C_3^+ denote the reflexive tournament $(\{1, 2, 3, 4\}, \to)$ whose nontrivial edges are $1 \to 2$, $2 \to 3$, $3 \to 1$, $1 \to 4$, $2 \to 4$ and $3 \to 4$.

Lemma 6.1. *C_3 and C_3^+ are not siblings.*

Suppose, to the contrary, that there is a finite reflexive tournament T and surjective homomorphisms $f\colon T \to C_3$ and $g\colon T \to C_3^+$. Let $A_i = f^{-1}(i)$, $1 \leqslant i \leqslant 3$, and $B_j = f^{-1}(j)$, $1 \leqslant j \leqslant 4$. Let $D_{ij} = A_i \cap B_j$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 4$. For each j, it is not possible that all of the sets D_{1j}, D_{2j} and D_{3j} are empty because $B_j = D_{1j} \cup D_{2j} \cup D_{3j}$ is nonempty. Analogously, for each i, it is not possible that all of the sets D_{i1}, D_{i2}, D_{i3} and D_{i4} are empty.

Now, consider D_{ij} and D_{uv} for some $1 \leqslant i$, $u \leqslant 3$ and $1 \leqslant j$, $v \leqslant 4$, and note that if $i \to u$ in C_3 and $v \to j$ in C_3^+ then $D_{ij} = \emptyset$ or $D_{uv} = \emptyset$. (If this is not the case, take arbitrary $x \in D_{ij}$ and $y \in D_{uv}$. If $x \to y$ in T then $g(x) \to g(y)$ in C_3^+. But $g(x) = j$ because $x \in D_{ij} \subseteq B_j$ and $g(y) = v$ because $y \in D_{uv} \subseteq B_v$. Hence, $j \to v$, which contradicts the assumption. The other possibility, $y \to x$ in T, leads analogously to the contradiction with $i \to u$ in C_3.) We say that (ij, uv) is a critical pair if $i \to u$ in C_3 and $v \to j$ in C_3^+. It is easy to list all the critical pairs:

$$\begin{array}{llllll}
(11,23), & (11,32), & (11,34), & (12,21), & (12,33), & (12,34),\\
(13,22), & (13,31), & (13,34), & (14,21), & (14,22), & (14,23),\\
(21,33), & (22,31), & (23,32), & (24,31), & (24,32), & (24,33).
\end{array}$$

Let $M = [m_{ij}]_{3\times 4}$ be a 01-matrix such that

$$m_{ij} = \begin{cases} 0, & D_{ij} = \emptyset,\\ 1, & D_{ij} \neq \emptyset, \end{cases}$$

where $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 4$. Then, as we have just seen, M has the following properties:

(i) each row contains at least one occurrence of 1,

(ii) each column contains at least one occurrence of 1,

(iii) for every critical pair (ij, uv) we have that $m_{ij} = 0$ or $m_{uv} = 0$ (or both).

Let us show that no 01-matrix $M = [m_{ij}]_{3\times 4}$ satisfies all the three properties. There are only seven possibilities to fill the first column by 0's and 1's (the option 000 is excluded by (ii)). Let us consider only the case 100, see Figure 3 (a), as the other cases follow by analogous arguments. The entries 23, 32 and 34 have to be 0 because of (iii), and the critical pairs $(11, 23)$, $(11, 32)$ and $(11, 34)$, see Figure 3 (b).

1			
0			
0			

(a)

1			0
0		0	
0	0		0

(b)

1	0		0
0		0	0
0	0	1	0

(c)

1	0		0
0	1	0	0
0	0	1	0

(d)

Figure 3. C_3 and C_3^+ are not siblings.

Then the entry 33 has to be 1 because of (i), so (iii), and the critical pairs $(12,33)$ and $(24,33)$ force the entries 12 and 24 to be 0, see Figure 3 (c). Using (i) once more, the entry 22 has to be 1, and the critical pair $(14,22)$ forces the entry 14 to be 0, see Figure 3 (d). Now, the last column of the matrix is 000, which contradicts (ii).

Therefore, the assumption that there are a finite reflexive tournament T and surjective homomorphisms $f\colon T\to C_3$ and $g\colon T\to C_3^+$ leads to a contradiction. $\square$

Theorem 6.2. *Let T be a category whose objects are finite linearly ordered reflexive tournaments and whose morphisms are rigid surjective homomorphisms. Then T does not have the dual Ramsey property.*

Proof. Let $\mathcal{A}=(A,\to,<)$ and $\mathcal{B}=(B,\to,<)$ be linearly ordered tournaments depicted in Figure 4, where $A=\{1,2\}$, $B=\{1,2,3,4,5,6,7\}$ and $<$ is the ordering of the integers. Let us show that no finite linearly ordered reflexive tournament $\mathcal{T}$ satisfies $\mathcal{T}\to(\mathcal{B})^{\mathcal{A}}_2$ in $T^{\rm op}$. Take any finite linearly ordered reflexive tournament $\mathcal{T}=(T,\to,\sqsubset)$ and define the coloring $\chi\colon \hom_T(\mathcal{T},\mathcal{A})\to\{1,2\}$ as

$$\chi(f)=\begin{cases}1, & \text{the subtournament of } (T,\to) \text{ induced by } f^{-1}(1) \text{ is an inflation of } C_3,\\ 2, & \text{otherwise.}\end{cases}$$

Let $\varphi,\psi\colon \{1,2,3,4,5,6,7\}\to\{1,2\}$ be the maps

$$\varphi=\begin{pmatrix}1&2&3&4&5&6&7\\1&1&1&2&2&2&2\end{pmatrix}\quad\text{and}\quad\psi=\begin{pmatrix}1&2&3&4&5&6&7\\1&1&1&1&2&2&2\end{pmatrix}.$$

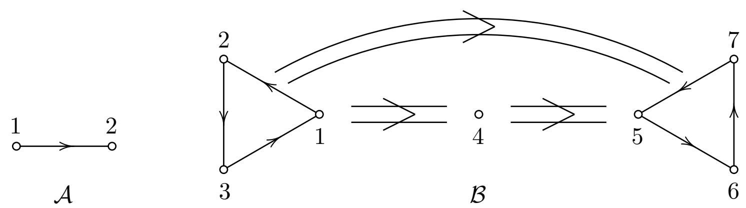


Figure 4. The tournaments in the proof of Theorem 6.2.

Clearly, $\varphi,\psi\in\hom_T(\mathcal{B},\mathcal{A})$. Now, take any $w\in\hom_T(\mathcal{T},\mathcal{B})$. Since

$$(\varphi\circ w)^{-1}(1)=w^{-1}(\varphi^{-1}(1))=w^{-1}(\{1,2,3\}),$$

it follows that $(\varphi\circ w)^{-1}$ induces an inflation of C_3 in $(T,\to)$, so $\chi(\varphi\circ w)=1$. Let us show that $\chi(\psi\circ w)=2$. Suppose this is not the case. Then $\chi(\psi\circ w)=1$, whence follows that $(\psi\circ w)^{-1}=w^{-1}(\{1,2,3,4\})$ induces a subtournament $(S,\to)$ of $(T,\to)$ which is an inflation of C_3. But the subtournament of $(B,\to)$ induced by $\{1,2,3,4\}$ is C_3^+, whence follows that $(S,\to)$ is at the same time an inflation of C_3—contradiction with Lemma 6.1. $\square$

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