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BLOW-UP FOR THE COMPRESSIBLE ISENTROPIC NAVIER-STOKES-POISSON EQUATIONS

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Abstract. We will show the blow-up of smooth solutions to the Cauchy problems for compressible unipolar isentropic Navier-Stokes-Poisson equations with attractive forcing and compressible bipolar isentropic Navier-Stokes-Poisson equations in arbitrary dimensions under some restrictions on the initial data. The key of the proof is finding the relations between the physical quantities and establishing some differential inequalities.

Keywords: compressible isentropic Navier-Stokes-Poisson equations; unipolar; bipolar; smooth solution; blow-up

MSC 2010: 35Q35, 35B44

1. Introduction

This paper is concerned with the Cauchy problems of the following two compressible isentropic Navier-Stokes-Poisson equations in $\mathbb{R}^d$:

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho u) = 0, \\ (\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla P(\varrho) = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u + a \varrho \nabla \Phi, \\ -\Delta \Phi = \varrho, \quad x \in \mathbb{R}^d, \ t > 0, \\ (\varrho, u)|_{t=0} = (\varrho_0(x), u_0(x)), \end{cases} \tag{1.1}$$

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$$
(1.2) \quad \begin{cases} \partial_t \varrho_i + \operatorname{div}(\varrho_i u_i) = 0, \\ (\varrho_i u_i)_t + \operatorname{div}(\varrho_i u_i \otimes u_i) + \nabla P_i(\varrho_i) = \varrho_i \nabla \Phi + \mu \Delta u_i + (\mu + \lambda) \nabla \operatorname{div} u_i, \\ \partial_t \varrho_e + \operatorname{div}(\varrho_e u_e) = 0, \\ (\varrho_e u_e)_t + \operatorname{div}(\varrho_e u_e \otimes u_e) + \nabla P_e(\varrho_e) = -\varrho_e \nabla \Phi + \mu \Delta u_e + (\mu + \lambda) \nabla \operatorname{div} u_e, \\ \Delta \Phi = \varrho_i - \varrho_e, \quad x \in \mathbb{R}^d, \ t > 0, \\ (\varrho_i, u_i, \varrho_e, u_e)|_{t=0} = (\varrho_{i0}(x), u_{i0}(x), \varrho_{e0}(x), u_{e0}(x)). \end{cases}
$$

Here $(1.1)_1$–$(1.1)_3$ and $(1.2)_1$–$(1.2)_5$ are called the compressible unipolar isentropic Navier-Stokes-Poisson equations and compressible bipolar isentropic Navier-Stokes-Poisson equations, respectively. The unknown functions ϱ (ϱ_i, ϱ_e), u (u_i, u_e) and Φ denote the density, velocity field and potential of underlying force, respectively. P, P_i, P_e are the pressures, the typical expressions are

$$
(1.3) \qquad P(\varrho) = \varrho^\gamma, \quad P_i(\varrho_i) = \varrho_i^\gamma, \quad P_e(\varrho_e) = \varrho_e^\gamma, \quad \gamma > 1.
$$

The coefficients μ and λ represent shear coefficient viscosity of the fluid and the second viscosity coefficient, respectively, the two Lamé viscosity coefficients satisfy

$$
(1.4) \qquad \mu > 0, \quad 2\mu + d\lambda > 0.
$$

The coefficient a in $(1.1)_2$ signifies the property of the forcing, which is repulsive if $a > 0$ and attractive if $a < 0$. The compressible unipolar isentropic N-S-P system can be used to describe many models if we consider different potential force. For example, $(1.1)_1$–$(1.1)_3$ is the self-gravitation model if Φ is the gravitational potential force, and the semiconductor model if Φ is the electrostatic potential force. The compressible bipolar isentropic N-S-P system $(1.2)_1$–$(1.2)_5$ is often used to describe the semiconductor device in the case that the interplay interaction of charged particles of different types (ions and electrons) is taken into consideration.

There have been many works about the existence and stability of stationary solutions and the global existence and long-time behavior of transient solutions to the unipolar and bipolar N-S-P systems; we can refer to [1], [2], [5], [6], [14], [18], [19] and the references therein. In this paper, we are interested in the blow-up phenomena of smooth solutions to the Cauchy problems (1.1) and (1.2). There is a lot of important progress made for the blow-up of smooth solutions to the compressible Navier-Stokes equations. To our knowledge, Xin in [16] first proves that when the initial densities are compactly supported, any smooth solutions to the compressible Navier-Stokes equations for nonbarotropic flows in the absence of heat conduction will blow up in finite time for any spatial dimension, and this feature also holds for the isentropic flows in one dimensional case. Cho and Jin in [3] extend Xin's work (see [16]) to the case of fluids with positive heat conduction, later Tan and Wang in [11] gave a much simpler proof of the result of Cho and Jin (see [3]) under an additional assumption that one of the components of initial momentum is not zero. If the initial data do not have compact support but rapidly decrease, for $d \geqslant 3$ and $\gamma \geqslant 2d/(d+2)$, Rozanova in [10] proves that any smooth solutions to the compressible Navier-Stokes equations for the nonbarotropic flows with positive heat conduction still blow up in finite time. Du, Li and Zhang in [4] show that the one-dimensional or two-dimensional radially symmetric isothermal compressible Navier-Stokes system has no nontrivial global smooth solutions if the initial density is compactly supported. Xin and Yan in [17] prove that any classical solutions of viscous compressible fluids without heat conduction will blow up in finite time as long as the initial data has an isolated mass group. Lai in [9] establishes a blow-up result for the isentropic compressible Navier-Stokes equations in three space dimensions by assuming the gradient of the velocity satisfies some decay constraint and the initial total momentum does not vanish. Recently, Jiu, Wang and Xin in [8] showed the blow-up of smooth solutions to the Cauchy problem for the full compressible Navier-Stokes equations and isentropic compressible Navier-Stokes equations with constant and degenerate viscosities in arbitrary dimensions under some restrictions on the initial data, but they do not require that the initial data has compact support or contains vacuum in any finite regions. For further generalization of the blow-up results of [8] about the full compressible Navier-Stokes equations and isentropic compressible Navier-Stokes equations with constant viscosities, we can refer to [13].

Compared with the compressible Navier-Stokes equations, the compressible Navier-Stokes-Poisson equations are much more complicated due to the coupling between the flow field and the potential field. Therefore, only a few blow-up results of the compressible Navier-Stokes equations have been transferred to the compressible N-S-P system. For example, motivated by [16], under the assumption that the initial density has compact support, Xie in [15] showed blow-up result of smooth solutions to the full compressible N-S-P system in $\mathbb{R}^3$, Jiang and Tan in [7] obtained blow-up result of the compressible reactive self-gravitating gas with chemical kinetics equations in $\mathbb{R}^3$, Tang and Zhang in [12] established the blow-up result for both isentropic and isothermal N-S-P system in $\mathbb{R}^2$. The aim of this paper is to extend the work (see [13]) to the compressible unipolar isentropic N-S-P system with attractive forcing and compressible bipolar isentropic N-S-P system. To our best knowledge, this paper is the first work about the blow-up of smooth solutions to the compressible bipolar N-S-P system.

Before we state our main results, we give some physical quantities:

$$F(t) := \int_{\mathbb{R}^d} \varrho u \cdot x \,\mathrm{d}x, \tag{1.5}$$

$$F_b(t) := \int_{\mathbb{R}^d} \varrho_i u_i \cdot x \, \mathrm{d}x + \int_{\mathbb{R}^d} \varrho_e u_e \cdot x \, \mathrm{d}x = F_i(t) + F_e(t), \tag{1.6}$$

$$G(t) := \frac{1}{2} \int_{\mathbb{R}^d} \varrho |x|^2 \, \mathrm{d}x, \tag{1.7}$$

$$G_b(t) := \frac{1}{2} \int_{\mathbb{R}^d} \varrho_i |x|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} \varrho_e |x|^2 \, \mathrm{d}x = G_i(t) + G_e(t), \tag{1.8}$$

$$\begin{aligned} E(t) &:= \frac{1}{2} \int_{\mathbb{R}^d} \varrho |u|^2 \, \mathrm{d}x + \frac{1}{\gamma - 1} \int_{\mathbb{R}^d} \varrho^\gamma \, \mathrm{d}x - \frac{a}{2} \int_{\mathbb{R}^d} |\nabla \Phi|^2 \, \mathrm{d}x \\ &= E_k(t) + E_i(t) + E_p(t), \end{aligned} \tag{1.9}$$

$$\begin{aligned} E_b(t) &:= \frac{1}{2} \int_{\mathbb{R}^d} \varrho_i |u_i|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} \varrho_e |u_e|^2 \, \mathrm{d}x + \frac{1}{\gamma - 1} \int_{\mathbb{R}^d} \varrho_i^\gamma \, \mathrm{d}x \\ &\quad + \frac{1}{\gamma - 1} \int_{\mathbb{R}^d} \varrho_e^\gamma \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \Phi|^2 \, \mathrm{d}x \\ &= E_k^i(t) + E_k^e(t) + E_I^i(t) + E_I^e(t) + E_p^b(t), \end{aligned} \tag{1.10}$$

where $F(t), G(t)$, $E(t)$, $E_k(t)$, $E_i(t)$ and $E_p(t)$ represent the momentum weight, the momentum of inertia, the total energy, the kinetic energy, the internal energy and the potential energy for the compressible unipolar isentropic N-S-P system with attractive forcing ($a < 0$), respectively; $F_b(t)$, $F_i(t)$, $F_e(t)$, $G_b(t)$, $G_i(t)$, $G_e(t)$, $E_b(t)$, $E_k^i(t)$, $E_k^e(t)$, $E_I^i(t)$, $E_I^e(t)$ and $E_p^b(t)$ represent the total momentum weight, the momentum weight of ions, the momentum weight of electrons, the momentum of inertia of ions, the momentum of inertia of electrons, the total energy, the kinetic energy of ions, the kinetic energy of electrons, the internal energy of ions, the internal energy of electrons and the potential energy for the compressible bipolar isentropic N-S-P system, respectively. We always assume that all the initial data on the above physical quantities are finite and

$$F(0), \quad G(0), \quad E_i(0) + E_p(0) > 0 \tag{1.11}$$

for the compressible unipolar isentropic N-S-P system with attractive forcing and

$$F_b(0), \quad G_b(0), \quad E_I^i(0) + E_I^e + E_p^b(0) > 0 \tag{1.12}$$

for the compressible bipolar isentropic N-S-P system. We are concerned with the smooth solutions with decay at far fields. To be more precise, for any $T > 0$ we require that the solutions of (1.1) satisfy

$$\varrho |x|^2, \ \ P(\varrho)|x|, \ \ \varrho |u|^2 |x|, \ \ |\nabla u||x|, \ \ \varrho |u||\Phi|, \ \ |\nabla \Phi|^2 |x| \in L^\infty((0,T), L^1(\mathbb{R}^d)), \tag{1.13}$$

and the solutions of (1.2) satisfy

$$\begin{aligned} &\varrho_i |x|^2, \ \ \varrho_e |x|^2, \ \ P_i(\varrho_i)|x|, \ \ P_e(\varrho_e)|x|, \ \ \varrho_i |u_i|^2 |x|, \ \ \varrho_e |u_e|^2 |x|, \ \ |\nabla u_i||x|, \\ &|\nabla u_e||x|, \ \ \varrho_i |u_i||\Phi|, \ \ \varrho_e |u_e||\Phi|, \ \ |\nabla \Phi|^2 |x| \in L^\infty((0,T), L^1(\mathbb{R}^d)). \end{aligned} \tag{1.14}$$

It should be remarked that conditions (1.13) and (1.14) guarantee that integration by parts in our calculations makes sense (see also [8], [10], [13]).

Our main results are stated as follows:

Theorem 1.1. *Assume that* $a < 0$ *and let the initial data* $(1.1)_4$ *satisfy* (1.11) *and* (1.13). *Then there is no smooth solution to the Cauchy problem* (1.1) *such that* (1.13) *holds. Moreover, the life span* T_1 *of the smooth solution to* (1.1) *satisfies that*

$$T_1 < \frac{C_2}{C_1 E(0)} \tan\Big(\frac{C_2}{F(0)} + \arctan\frac{F(0)}{C_2}\Big) - \frac{F(0)}{C_1 E(0)}, \tag{1.15}$$

where

$$C_1 = \max\{2, d(\gamma-1)\}, \quad C_2 = \sqrt{2C_1E(0)G(0) - F(0)^2}. \tag{1.16}$$

Theorem 1.2. *Let the initial data* $(1.2)_6$ *satisfy* (1.12) *and* (1.14). *Then there is no smooth solution to the Cauchy problem* (1.2) *such that* (1.14) *holds. Moreover, the life span* T_2 *of the smooth solution to* (1.2) *satisfies that*

$$T_2 < \frac{C_3}{C_1 E_b(0)} \tan\Big(\frac{2C_3}{F_b(0)} + \arctan\frac{F_b(0)}{C_3}\Big) - \frac{F_b(0)}{C_1 E_b(0)}, \tag{1.17}$$

where

$$C_3 = \sqrt{2C_1E_b(0)G_b(0) - F_b(0)^2}. \tag{1.18}$$

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following two lemmas.

Lemma 2.1. *Under the assumptions of Theorem* 1.1, *it holds*

$$E(t) \leqslant E(0). \tag{2.1}$$

Proof. Multiplying $(1.1)_2$ by u and integrating it with respect to x in $\mathbb{R}^d$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} (\varrho u)_t \cdot u \,\mathrm{d}x + \int_{\mathbb{R}^d} \operatorname{div}(\varrho u \otimes u) \cdot u \,\mathrm{d}x + \int_{\mathbb{R}^d} \nabla P(\varrho) \cdot u \,\mathrm{d}x \\ &\quad = \mu \int_{\mathbb{R}^d} \Delta u \cdot u \,\mathrm{d}x + (\mu+\lambda) \int_{\mathbb{R}^d} \nabla \operatorname{div} u \cdot u \,\mathrm{d}x + a \int_{\mathbb{R}^d} \varrho \nabla\Phi \cdot u \,\mathrm{d}x. \end{aligned} \tag{2.2}$$

Using $(1.1)_1$, (1.3), $(1.1)_3$ and integrating by parts, we get

$$\int_{\mathbb{R}^d} (\varrho u)_t \cdot u \,\mathrm{d}x + \int_{\mathbb{R}^d} \operatorname{div}(\varrho u \otimes u) \cdot u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \frac{1}{2} \varrho |u|^2 \,\mathrm{d}x, \tag{2.3}$$

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla P(\varrho) \cdot u \,\mathrm{d}x &= \gamma \int_{\mathbb{R}^d} \varrho^{\gamma-1} \nabla \varrho \cdot u \,\mathrm{d}x = \frac{\gamma}{\gamma-1} \int_{\mathbb{R}^d} \nabla(\varrho^{\gamma-1}) \cdot (\varrho u) \,\mathrm{d}x \\ &= -\frac{\gamma}{\gamma-1} \int_{\mathbb{R}^d} \varrho^{\gamma-1} \operatorname{div}(\varrho u) \,\mathrm{d}x = \frac{\gamma}{\gamma-1} \int_{\mathbb{R}^d} \varrho^{\gamma-1} \varrho_t \,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \frac{\varrho^\gamma}{\gamma-1} \,\mathrm{d}x, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mu \int_{\mathbb{R}^d} & \Delta u \cdot u \,\mathrm{d}x + (\mu+\lambda) \int_{\mathbb{R}^d} \nabla \operatorname{div} u \cdot u \,\mathrm{d}x \\ &= -\mu \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - (\mu+\lambda) \int_{\mathbb{R}^d} |\operatorname{div} u|^2 \,\mathrm{d}x, \end{aligned} \tag{2.5}$$

$$\begin{aligned} a \int_{\mathbb{R}^d} \varrho \nabla \Phi \cdot u \,\mathrm{d}x &= -a \int_{\mathbb{R}^d} \Phi \operatorname{div}(\varrho u) \,\mathrm{d}x = a \int_{\mathbb{R}^d} \Phi \varrho_t \,\mathrm{d}x \\ &= -a \int_{\mathbb{R}^d} \Phi \Delta \Phi_t \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \frac{a}{2} |\nabla \Phi|^2 \,\mathrm{d}x. \end{aligned} \tag{2.6}$$

Combining (2.2)–(2.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\mu \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - (\mu+\lambda) \int_{\mathbb{R}^d} |\operatorname{div} u|^2 \,\mathrm{d}x \leqslant 0, \tag{2.7}$$

this implies that (2.1) holds. □

Lemma 2.2. *Under the assumptions of Theorem* 1.1, *we have*

$$G'(t) = F(t), \tag{2.8}$$

$$G''(t) = F'(t) = 2E_k(t) + d(\gamma-1)E_i(t), \tag{2.9}$$

$$F(t), \quad G(t) > 0. \tag{2.10}$$

P r o o f. Multiplying $(1.1)_1$ by $|x|^2$ and integrating it over $\mathbb{R}^d$, we get (2.8). Using (2.8), multiplying $(1.1)_2$ by x and integrating it over $\mathbb{R}^d$, we obtain

$$G''(t) = F'(t) = \int_{\mathbb{R}^d} \varrho |u|^2 \,\mathrm{d}x + d \int_{\mathbb{R}^d} \varrho^\gamma \,\mathrm{d}x = 2E_k(t) + d(\gamma-1)E_i(t), \tag{2.11}$$

where we have used

$$
\begin{aligned}
(2.12)\quad a\int_{\mathbb{R}^d} \varrho\nabla\Phi\cdot x\,\mathrm{d}x &= a\int_{\mathbb{R}^d}(-\Delta\Phi)\nabla\Phi\cdot x\,\mathrm{d}x = -a\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\partial_{ii}\Phi(\partial_j\Phi\cdot x_j)\,\mathrm{d}x\\
&= a\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\partial_i\Phi\partial_{ji}\Phi\cdot x_j\,\mathrm{d}x + a\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\partial_i\Phi\partial_j\Phi\cdot\partial_i x_j\,\mathrm{d}x\\
&= a\sum_{i,j=1}^{d}\frac{1}{2}\int_{\mathbb{R}^d}\partial_j|\partial_i\Phi|^2\cdot x_j\,\mathrm{d}x + a\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\partial_i\Phi\partial_j\Phi\delta_{ij}\,\mathrm{d}x\\
&= -a\int_{\mathbb{R}^d}|\nabla\Phi|^2\,\mathrm{d}x + a\int_{\mathbb{R}^d}|\nabla\Phi|^2\,\mathrm{d}x = 0,
\end{aligned}
$$

which can be found in [12] for $d=2$. From (2.9) we see that $F'(t)\geqslant 0$, then by (1.11) we know that $F(t)>0$, this together with (2.8) and (1.11) lead to $G(t)>0$. $\square$

P r o o f of Theorem 1.1. By (2.9), (1.9), (2.1) and (1.16), we get

$$
(2.13)\qquad G''(t)\leqslant \max\{2, d(\gamma-1)\}E(t)\leqslant C_1E(0),
$$

where we have used $a<0$. Integrating (2.13) over $[0,t]$, we obtain

$$
(2.14)\qquad G(t)\leqslant \frac{C_1}{2}E(0)t^2+F(0)t+G(0).
$$

Using Hölder's inequality, we have

$$
(2.15)\quad F(t)^2=\Big(\int_{\mathbb{R}^d}\varrho u\cdot x\,\mathrm{d}x\Big)^2\leqslant\Big(\int_{\mathbb{R}^d}\varrho|u|^2\,\mathrm{d}x\Big)\cdot\Big(\int_{\mathbb{R}^d}\varrho|x|^2\,\mathrm{d}x\Big)=4E_k(t)G(t).
$$

In view of (2.9), (2.15) and (2.14), it follows

$$
\begin{aligned}
(2.16)\qquad F'(t)\geqslant 2E_k(t)\geqslant\frac{F(t)^2}{2G(t)}&\geqslant\frac{F(t)^2}{C_1E(0)t^2+2F(0)t+2G(0)}\\
&=\frac{F(t)^2}{C_1E(0)[(t+F(0)/C_1E(0))^2+C_2^2/C_1^2E(0)^2]},
\end{aligned}
$$

where C_1, C_2 are defined in (1.16).

By (2.15) and (1.11), we know that

$$
(2.17)\qquad F(0)^2\leqslant 4E_k(0)G(0)<4E(0)G(0),
$$

this together with (1.16) lead to

$$
(2.18)\qquad \frac{C_2^2}{C_1^2E(0)^2}>0.
$$

Dividing (2.16) by $F(t)^2$ and integrating the resultant inequality over $[0, T_1]$, we obtain

$$\begin{aligned}\frac{1}{F(0)} &> \frac{1}{F(0)} - \frac{1}{F(T_1)} \\ &\geqslant \frac{1}{C_2}\left[\arctan\frac{C_1E(0)(T_1 + F(0)/C_1E(0))}{C_2} - \arctan\frac{F(0)}{C_2}\right],\end{aligned} \tag{2.19}$$

where we have used (2.10). We can solve out T_1 by (2.19) as (1.15). We complete the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

Similar to the proof of Theorem 1.1, in order to prove Theorem 1.2, we need the following two lemmas.

Lemma 3.1. *Under the assumptions of Theorem* 1.2, *we have*

$$E_b(t) \leqslant E_b(0). \tag{3.1}$$

Proof. Multiplying $(1.2)_2$, $(1.2)_4$ by u_i, u_e, respectively, integrating them over $\mathbb{R}^d$ and using $(1.2)_1$, $(1.2)_3$, (1.3), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}\varrho_i|u_i|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{\varrho_i^\gamma}{\gamma-1}\,\mathrm{d}x \\ = -\mu\int_{\mathbb{R}^d}|\nabla u_i|^2\,\mathrm{d}x - (\mu+\lambda)\int_{\mathbb{R}^d}|\operatorname{div} u_i|^2\,\mathrm{d}x + \int_{\mathbb{R}^d}\varrho_i\nabla\Phi\cdot u_i\,\mathrm{d}x,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}\varrho_e|u_e|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{\varrho_e^\gamma}{\gamma-1}\,\mathrm{d}x \\ = -\mu\int_{\mathbb{R}^d}|\nabla u_e|^2\,\mathrm{d}x - (\mu+\lambda)\int_{\mathbb{R}^d}|\operatorname{div} u_e|^2\,\mathrm{d}x - \int_{\mathbb{R}^d}\varrho_e\nabla\Phi\cdot u_e\,\mathrm{d}x.\end{aligned} \tag{3.3}$$

Using integration by parts, $(1.2)_1$, $(1.2)_3$ and $(1.2)_5$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\varrho_i\nabla\Phi\cdot u_i\,\mathrm{d}x - \int_{\mathbb{R}^d}\varrho_e\nabla\Phi\cdot u_e\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d}\Phi\operatorname{div}(\varrho_i u_i)\,\mathrm{d}x + \int_{\mathbb{R}^d}\Phi\operatorname{div}(\varrho_e u_e)\,\mathrm{d}x = \int_{\mathbb{R}^d}\Phi(\partial_t\varrho_i - \partial_t\varrho_e)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\Phi\Delta\Phi_t\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|\nabla\Phi|^2\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

Combining (3.2)–(3.4), it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}E_b(t) = -\mu\int_{\mathbb{R}^d}|\nabla u_i|^2\,\mathrm{d}x - (\mu+\lambda)\int_{\mathbb{R}^d}|\operatorname{div} u_i|^2\,\mathrm{d}x - \mu\int_{\mathbb{R}^d}|\nabla u_e|^2\,\mathrm{d}x - (\mu+\lambda)\int_{\mathbb{R}^d}|\operatorname{div} u_e|^2\,\mathrm{d}x \leqslant 0, \tag{3.5}$$

this implies that (3.1) holds. □

Lemma 3.2. *Under the assumptions of Theorem* 1.2, *we have*

$$G_b'(t) = F_b(t), \tag{3.6}$$

$$G_b''(t) = F_b'(t) = 2E_k^i(t) + 2E_k^e(t) + d(\gamma-1)E_I^i(t) + d(\gamma-1)E_I^e(t), \tag{3.7}$$

$$F_b(t),\ G_b(t) > 0. \tag{3.8}$$

P r o o f. Multiplying $(1.2)_1$, $(1.2)_3$ by $|x|^2$ and integrating them over $\mathbb{R}^d$, we get $G_i'(t) = F_i(t)$, $G_e'(t) = F_e(t)$, so (3.6) holds. Multiplying $(1.2)_2$, $(1.2)_4$ by x and integrating them over $\mathbb{R}^d$, we obtain

$$G_i''(t) = F_i'(t) = 2E_k^i(t) + d(\gamma-1)E_I^i(t) + \int_{\mathbb{R}^d}\varrho_i\nabla\Phi\cdot x\,\mathrm{d}x, \tag{3.9}$$

$$G_e''(t) = F_e'(t) = 2E_k^e(t) + d(\gamma-1)E_I^e(t) - \int_{\mathbb{R}^d}\varrho_e\nabla\Phi\cdot x\,\mathrm{d}x. \tag{3.10}$$

Using $(1.2)_5$ and (2.12), we have

$$\int_{\mathbb{R}^d}\varrho_i\nabla\Phi\cdot x\,\mathrm{d}x - \int_{\mathbb{R}^d}\varrho_e\nabla\Phi\cdot x\,\mathrm{d}x = \int_{\mathbb{R}^d}\Delta\Phi\nabla\Phi\cdot x\,\mathrm{d}x = 0. \tag{3.11}$$

Combining (3.9)–(3.11), we get (3.7). The proof of (3.8) is similar to the one of (2.10). □

P r o o f of Theorem 1.2. Similarly to (2.14) and (2.15), we have

$$G_b(t) \leqslant \frac{C_1}{2}E_b(0)t^2 + F_b(0)t + G_b(0), \tag{3.12}$$

$$F_i(t)^2 \leqslant 4E_k^i(t)G_i(t), \quad F_e(t)^2 \leqslant 4E_k^e(t)G_e(t). \tag{3.13}$$

Consequently, it follows from (3.7) that

$$\begin{aligned} F_b'(t) &\geqslant 2E_k^i(t) + 2E_k^e(t) \geqslant \frac{F_i(t)^2}{2G_i(t)} + \frac{F_e(t)^2}{2G_e(t)} \geqslant \frac{F_i(t)^2+F_e(t)^2}{2G_b(t)} \geqslant \frac{F_b(t)^2}{4G_b(t)} \\ &\geqslant \frac{F_b(t)^2}{2C_1E_b(0)t^2 + 4F_b(0)t + 4G_b(0)} \\ &= \frac{F_b(t)^2}{2C_1E_b(0)[(t+F_b(0)/C_1E_b(0))^2 + C_3^2/C_1^2E_b(0)^2]}, \end{aligned} \tag{3.14}$$

where C_3 is defined in (1.18).

By (1.6), (3.13), (1.8), (1.10) and (1.12), we know that

$$
\begin{aligned}
F_b(0)^2 &= [F_i(0)+F_e(0)]^2 = F_i(0)^2+F_e(0)^2+2F_i(0)F_e(0) \\
&\leqslant 4\Big[E_k^i(0)G_i(0)+E_k^e(0)G_e(0)+2\sqrt{E_k^i(0)G_i(0)E_k^e(0)G_e(0)}\Big] \\
&\leqslant 4[E_k^i(0)G_i(0)+E_k^e(0)G_e(0)+E_k^i(0)G_e(0)+E_k^e(0)G_i(0)] \\
&= 4[E_k^i(0)+E_k^e(0)][G_i(0)+G_e(0)] < 4E_b(0)G_b(0),
\end{aligned} \tag{3.15}
$$

this together with (1.18) imply that

$$
\frac{C_3^2}{C_1^2E(0)^2} > 0. \tag{3.16}
$$

Dividing (3.14) by $F_b(t)^2$ and integrating the resultant inequality over $[0,T_2]$, we obtain

$$
\begin{aligned}
\frac{1}{F_b(0)} &> \frac{1}{F_b(0)} - \frac{1}{F_b(T_2)} \\
&\geqslant \frac{1}{2C_3}\Big[\arctan\frac{C_1E_b(0)(T_2+F_b(0)/C_1E_b(0))}{C_3} - \arctan\frac{F_b(0)}{C_3}\Big],
\end{aligned} \tag{3.17}
$$

where we have used (3.8). We can solve out T_2 by (3.17) as (1.17). We complete the proof of Theorem 1.2. $\square$

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