TRUNCATED SPECTRAL REGULARIZATION FOR AN ILL-POSED NON-LINEAR PARABOLIC PROBLEM

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Abstract. It is known that the nonlinear nonhomogeneous backward Cauchy problem $u_t(t) + Au(t) = f(t, u(t))$, $0 \leqslant t < \tau$ with $u(\tau) = \phi$, where A is a densely defined positive self-adjoint unbounded operator on a Hilbert space, is ill-posed in the sense that small perturbations in the final value can lead to large deviations in the solution. We show, under suitable conditions on ϕ and f, that a solution of the above problem satisfies an integral equation involving the spectral representation of A, which is also ill-posed. Spectral truncation is used to obtain regularized approximations for the solution of the integral equation, and error analysis is carried out with exact and noisy final value ϕ. Also stability estimates are derived under appropriate parameter choice strategies. This work extends and generalizes many of the results available in the literature, including the work by Tuan (2010) for linear homogeneous final value problem and the work by Jana and Nair (2016b) for linear nonhomogeneous final value problem.

Keywords: ill-posed problem; nonlinear parabolic equation; regularization; parameter choice; semigroup; contraction principle

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1. Introduction

Let H be a Hilbert space over the real or complex field, and let $A\colon D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator. For $\tau > 0$ and $\phi \in H$, consider the problem of solving the *nonlinear final value problem*, denoted briefly as nonlinear FVP,

$$u_t(t) + Au(t) = f(t, u(t)), \quad 0 \leqslant t < \tau \tag{1.1}$$

$$u(\tau) = \phi, \tag{1.2}$$

where $f(\cdot,\cdot)$ is an H valued function defined on $[0,\tau] \times H$. It is well known that the above problem is ill-posed (cf. Goldstein [6]) in the sense that a small perturbation

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in the final value ϕ can lead to a large deviation in solution. Some regularization method has to be employed to get stable approximate solutions.

There are many regularization methods for parabolic FVP. Here are some of the methods in the literature.

(a) *Quasi-reversibility method:* This method is based on considering a perturbation of the the operator A and it was introduced by Lattès and Lions (cf. [10]) for linear homogeneous FVP. Other authors also used this method for linear homogeneous FVP (see e.g., Boussetila and Rebbani [1], Miller [11], and Showlter [15]). In [7], Jana and Nair used this method for linear nonhomogeneous FVP. In [5], Fury used this method for nonautonomous semilinear problems. In [18], Tuan, Trong and Quan considered a similar approach for nonlinear nonhomogeneous FVP with nonhomogeneous term $f(u(t))$.

(b) *Quasi-boundary value method:* This method is based on considering a perturbation in the final value and it was used by Clark and Oppenheimer (see [2]), Denche and Bessila (see [3]), Denche and Djezzar (see [4]) for linear homogeneous FVP.

(c) *Truncated spectral regularization method:* This method is based on truncation of the spectral representation of an operator. In [16], Tuan considered this method for linear homogeneous FVP. This method was considered for linear nonhomogeneous FVP (see e.g., Jana and Nair [8] and Tuan and Trong [17]).

In this paper, we will consider the *truncated spectral regularization* method for nonlinear nonhomogeneous FVP (1.1)–(1.2).

We may recall that a function $u\colon [0,\tau] \to H$ is a solution of the FVP (1.1)–(1.2) if u is differentiable on $[0,\tau]$ and satisfies (1.1)–(1.2). We shall see that if $u(\cdot)$ is a solution of (1.1)–(1.2), then it satisfies the integral equation

$$(1.3)\qquad u(t) = \int_0^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi - \int_t^\tau\int_0^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau$$

under some suitable conditions on ϕ and $f(\cdot,\cdot)$, where $\{E_\lambda\colon\ \lambda \geqslant 0\}$ is the resolution of identity of the operator A. We define the *mild solution* of the nonlinear FVP given by (1.1)–(1.2) as the solution of the integral equation (1.3) (cf. Theorem 3.9). Note that due to the presence of the unbounded operator $\varphi \mapsto \int_0^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\varphi$, the problem of finding a *mild solution* is ill-posed. Therefore some regularization method has to be employed to obtain stable approximate solutions. In this paper, we consider regularized solution $u_\beta(t,\phi)$ as the solution of the integral equation obtained from (1.3) by truncation, that is, $u_\beta(t,\phi)$ is a solution of

$$(1.4)\quad u_\beta(t,\phi) = \int_0^\beta \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi - \int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u_\beta(s,\phi))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau$$

for $\beta > 0$. Under suitable conditions on ϕ and f, the existence, regularity, and convergence of the regularized solutions are proved when the final value is noisy as well as exact.

In Section 2, we give preliminary results required for our analysis. In Section 3, we define the *mild solution* for nonliner FVP and prove the existence of a *mild solution* under certain condition. In Section 4, we define the regularized solutions. In Section 5, we show convergence of the regularized solutions to the *mild solution*, and derive an error estimate when the final value ϕ is noisy as well as exact, and deduce many results as special cases.

2. Preliminaries

Throughout the paper, $C([0,\tau];H)$ represents the Banach space of all H valued continuous functions on $[0,\tau]$ with the norm

$$\|v\|_\infty := \sup_{0\leqslant t\leqslant \tau} \|v(t)\|, \quad v \in C([0,\tau];H).$$

Also $L^1([0,\tau];H)$ denotes the space of all H-valued Lebesgue measurable functions h on $[0,\tau]$ such that

$$\int_0^\tau \|h(t)\| \,\mathrm{d}t < \infty,$$

where integration is in the sense of Lebesgue. We denote the domain and the range of an operator T by $D(T)$ and $R(T)$, respectively.

2.1. Some consequences of the spectral theorem. Let $A\colon D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator on the Hilbert space H. Recall from the spectral theorem (cf. Yosida [19]) that

$$A\varphi := \int_0^\infty \lambda \,\mathrm{d}E_\lambda \varphi, \quad \varphi \in D(A)$$

where

$$D(A) := \left\{\varphi \in H\colon \int_0^\infty \lambda^2 \,\mathrm{d}\|E_\lambda\varphi\|^2 < \infty\right\}.$$

Also, for any continuous or piecewise continuous function $h\colon [0,\infty) \to [0,\infty)$, the operator $h(A)$ is defined by

$$h(A)\varphi := \int_0^\infty h(\lambda) \,\mathrm{d}E_\lambda \varphi, \quad \varphi \in D(h(A)),$$

where

$$D(h(A)) := \left\{\varphi \in H\colon \int_0^\infty h(\lambda)^2 \,\mathrm{d}\|E_\lambda \varphi\|^2 < \infty\right\}$$

and in that case

$$\|h(A)\varphi\|^2 = \int_0^\infty h(\lambda)^2 \,\mathrm{d}\|E_\lambda \varphi\|^2, \quad \varphi \in D(h(A)).$$

In particular, for $t \geqslant 0$, the operator e^{tA} is given by

$$\mathrm{e}^{tA}\varphi = \int_0^\infty \mathrm{e}^{\lambda t} \,\mathrm{d}E_\lambda \varphi \quad \forall\, \varphi \in D(\mathrm{e}^{tA})$$

where

$$D(\mathrm{e}^{tA}) = \left\{\varphi \in H\colon \int_0^\infty \mathrm{e}^{2\lambda t} \,\mathrm{d}\|E_\lambda \varphi\|^2 < \infty\right\}.$$

Also, we can see that for $t \geqslant 0$

$$\mathrm{e}^{-tA} = \int_0^\infty \mathrm{e}^{-t\lambda} \,\mathrm{d}E_\lambda,$$

where

$$D(\mathrm{e}^{-tA}) := \left\{\varphi \in H\colon \int_0^\infty \mathrm{e}^{-2\lambda t} \,\mathrm{d}\|E_\lambda \varphi\|^2 < \infty\right\} = H.$$

Also, we note that $\{\mathrm{e}^{-tA}\colon\, t \geqslant 0\}$ is a family of bounded linear operators on H which is a strongly continuous (or C_0) semigroup generated by $-A$ (cf. Pazy [13]), and $\|\mathrm{e}^{-tA}\| \leqslant 1$ for all $t \geqslant 0$. By using spectral representation, the following can be proved easily. For $t > 0$,

$$R(\mathrm{e}^{-tA}) \subseteq D(\mathrm{e}^{tA}) \quad \text{and} \tag{2.1}$$

$$\mathrm{e}^{-tA}\mathrm{e}^{tA} = I \quad \text{on } D(\mathrm{e}^{tA}) \quad \text{and} \quad \mathrm{e}^{tA}\mathrm{e}^{-tA} = I \quad \text{on } H. \tag{2.2}$$

Observe that for $t > 0$, e^{-tA} is an injective operator with range space $D(\mathrm{e}^{tA})$, and e^{tA} is a bijective closed operator with its inverse e^{-tA}.

3. Mild solution and it existence

We will make use of the following lemma for the next results.

Lemma 3.1. *Let $F\colon [0,\tau]\times H\to H$ be a Borel measurable function satisfying the following conditions:*

(i) *There exists $\varphi_0\in H$ such that the function $s\mapsto F(s,\varphi_0)$ belongs to $L^1([0,\tau];H)$.*

(ii) *There exists $c>0$ such that $\|F(s,\varphi_1)-F(s,\varphi_2)\|\leqslant c\|\varphi_1-\varphi_2\|$ for all $\varphi_1,\varphi_2\in H$ and for all $s\in[0,\tau]$.*

Then, for each $w\in L^1([0,\tau];H)$, the function $s\mapsto F(s,w(s))$ belongs to $L^1([0,\tau];H)$.

Proof. Let $w\in L^1([0,\tau];H)$. Since F is Borel measurable, it follows that the function $s\mapsto F(s,w(s))$ is measurable. Hence, using (i)–(ii), we have

$$\begin{aligned}\int_0^\tau \|F(s,w(s))\|\,\mathrm{d}s &\leqslant \int_0^\tau \|F(s,\varphi_0)\|\,\mathrm{d}s+\int_0^\tau \|F(s,w(s))-F(s,\varphi_0)\|\,\mathrm{d}s\\ &\leqslant \int_0^\tau \|F(s,\varphi_0)\|\,\mathrm{d}s+c\int_0^\tau \|w(s)-\varphi_0\|\,\mathrm{d}s<\infty.\end{aligned}$$

Thus, $s\mapsto F(s,w(s))$ belongs to $L^1([0,\tau],H)$. □

Theorem 3.2. *Let $u\colon [0,\tau]\to H$ be a solution of the FVP* (1.1)–(1.2). *Suppose $f(\cdot,\cdot)$ is a Borel measurable function satisfying one of the following conditions:*

(i) *The function $s\mapsto f(s,\varphi)$ is in $L^1([0,\tau];H)$ for some $\varphi\in H$, and*

$$\|f(s,\varphi_1)-f(s,\varphi_2)\|\leqslant\kappa\|\varphi_1-\varphi_2\|$$

for all $\varphi_1,\varphi_2\in H$ and for all $s\in[0,\tau]$, where $\kappa>0$.

(ii) *The function $f\colon [0,\tau]\times H\to H$ is continuous.*

Then

(1) *$s\mapsto f(s,u(s))$ belongs to $L^1([0,\tau];H)$,*

(2) *for each $t\in[0,\tau]$, $\phi-\int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s$ belongs to $D(\mathrm{e}^{(\tau-t)A})$ and*

(3) *u satisfies*

$$u(t)=\mathrm{e}^{(\tau-t)A}\Big(\phi-\int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s\Big)$$

for all $t\in[0,\tau]$.

Further, if $\phi\in D(\mathrm{e}^{\tau A})$, $f(s,u(s))\in D(\mathrm{e}^{sA})$ for all $s\in[0,\tau]$ and $s\mapsto \mathrm{e}^{sA}f(s,u(s))$ belongs to $L^1([0,\tau],H)$, then

$$u(t)=\mathrm{e}^{(\tau-t)A}\phi-\int_t^\tau \mathrm{e}^{(s-t)A}f(s,u(s))\,\mathrm{d}s \tag{3.1}$$

for all $t\in[0,\tau]$.

P r o o f. We use the procedure as in Pazy [13] for finding the mild solution for the initial value linear nonhomogeneous abstract Cauchy problem. Let

$$w(s) = \mathrm{e}^{-(\tau-s)A}u(s), \quad 0 \leqslant s \leqslant \tau.$$

Taking the derivative of w with respect to s, we get

$$w'(s) = A\mathrm{e}^{-(\tau-s)A}u(s) + \mathrm{e}^{-(\tau-s)A}u'(s).$$

Now using (1.1), we get

$$\begin{aligned} w'(s) &= A\mathrm{e}^{-(\tau-s)A}u(s) - A\mathrm{e}^{-(\tau-s)A}u(s) + \mathrm{e}^{-(\tau-s)A}f(s,u(s)) \\ &= \mathrm{e}^{-(\tau-s)A}f(s,u(s)). \end{aligned}$$

If $f(\cdot,\cdot)$ satisfies condition (i), by Lemma 3.1, $s \mapsto f(s,u(s))$ belongs to $L^1([0,\tau];H)$. If $f(\cdot,\cdot)$ satisfies condition (ii), then $s \mapsto f(s,u(s))$ is continuous. In either case $s \mapsto f(s,u(s))$ belongs to $L^1([0,\tau];H)$. Therefore $s \mapsto w'(s) := \mathrm{e}^{-(\tau-s)A}f(s,u(s))$ belongs to $L^1([0,\tau];H)$. Now, integrating w' from t to τ, we get

$$w(\tau) - w(t) = \int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s,$$

i.e.,

$$\mathrm{e}^{-(\tau-t)A}u(t) = \phi - \int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s.$$

From the above equation, it is clear that $\phi - \int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s$ belongs to $D(\mathrm{e}^{(\tau-t)A})$ and

$$u(t) = \mathrm{e}^{(\tau-t)A}\Big(\phi - \int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s\Big). \tag{3.2}$$

Since $\mathrm{e}^{(\tau-t)A}$ is a closed operator (see (2.2)) and $\int_t^\tau \mathrm{e}^{(s-t)A}f(s,u(s))\,\mathrm{d}s$ exists under the assumptions $f(s,u(s)) \in D(\mathrm{e}^{sA})$ for all $s \in [0,\tau]$ and $s \mapsto \mathrm{e}^{sA}f(s,u(s))$ belongs to $L^1([0,\tau],H)$, by Hille's Theorem for $t \in [0,\tau]$ we have

$$\mathrm{e}^{(\tau-t)A}\int_t^\tau \mathrm{e}^{-(\tau-s)A}f(s,u(s))\,\mathrm{d}s = \int_t^\tau \mathrm{e}^{(s-t)A}f(s,u(s))\,\mathrm{d}s.$$

Thus, if $\phi \in D(\mathrm{e}^{\tau A})$, $f(s,u(s)) \in D(\mathrm{e}^{sA})$ for all $s \in [0,\tau]$ and $s \mapsto \mathrm{e}^{sA}f(s,u(s))$ belongs to $L^1([0,\tau],H)$, then from (3.2) we get

$$u(t) = \mathrm{e}^{(\tau-t)A}\phi - \int_t^\tau \mathrm{e}^{(s-t)A}f(s,u(s))\,\mathrm{d}s.$$

This completes the proof. □

Remark 3.3. In [18], Tuan, Trong and Quan mention that if $u(\cdot)$ is a solution of (1.1)–(1.2), then $u(\cdot)$ satisfies (3.1) for $f(u(s))$ in place of $f(s,u(s))$ but no justification is given. Thus, Theorem 3.2 specifies a certain condition under which (3.1) holds. In particular, Theorem 3.2 justifies the expression (3.1) given in [18], under certain conditions on ϕ and f, whenever $f(s,u(s))$ is of the form $f(u(s))$.

In view of the last part of Theorem 3.2, we define the *mild solution* of (1.1)–(1.2) as follows.

Definition 3.4. Given $\phi \in D(\mathrm{e}^{\tau A})$ and a Borel measurable function $f\colon [0,\tau] \times H \to H$, a function $u\colon [0,\tau] \to H$ is called a *mild solution* of the nonlinear FVP given by (1.1)–(1.2) if

(i) $f(s,u(s)) \in D(\mathrm{e}^{sA})$ for all $s \in [0,\tau]$,
(ii) $s \mapsto \mathrm{e}^{sA} f(s,u(s))$ belongs to $L^1([0,\tau],H)$ and
(iii) $u(t) = \mathrm{e}^{(\tau-t)A}\phi - \int_t^\tau \mathrm{e}^{(s-t)A} f(s,u(s))\,\mathrm{d}s$, $0 \leqslant t \leqslant \tau$.

Remark 3.5. Definition of a *mild solution* for the linear nonhomogeneous parabolic problem given in Jana and Nair [8] coincides with the above definition when $f(t,u(t))$ is replaced by $f(t)$.

Remark 3.6. By Theorem 3.2, if $u(\cdot)$ is a solution of the nonlinear FVP given by (1.1)–(1.2), then it is a mild solution if

(i) $\phi \in D(\mathrm{e}^{\tau A})$,
(ii) $f(s,u(s)) \in D(\mathrm{e}^{sA})$ for all $s \in [0,\tau]$ and
(iii) the function $s \mapsto \mathrm{e}^{sA} f(s,u(s))$ belongs to $L^1([0,\tau],H)$.

However, a mild solution need not be a solution. For an example, let us chose $f(\cdot,\cdot) = 0$ and $\phi \in D(\mathrm{e}^{\tau A})$ but $\phi \notin D(A\mathrm{e}^{\tau A})$. Then the mild solution is of the form $u(t) = \mathrm{e}^{(\tau-t)A}\phi$. Note that

$$\lim_{h\to 0} \frac{u(h)-u(0)}{h} = -A\mathrm{e}^{\tau A}\phi.$$

Since $\phi \notin D(A\mathrm{e}^{\tau A})$, $u(\cdot)$ is not differentiable at 0. Therefore, $u(\cdot)$ is not a solution.

Now we prove the existence of a mild solution under some conditions on $f(\cdot,\cdot)$. For this we will make use of the following lemmas.

Lemma 3.7. *Let $T\colon C([0,\tau];H) \to C([0,\tau];H)$ be such that there exists $c > 0$ satisfying*

$$\|T(v)(t) - T(w)(t)\| \leqslant c\int_t^\tau \|v(s)-w(s)\|\,\mathrm{d}s \quad 0 \leqslant t \leqslant \tau \tag{3.3}$$

for all $v, w \in C([0,\tau];H)$. Then T has a unique fixed point.

P r o o f. We show that the operator T is a contraction with respect to a new complete norm on $C([0,\tau];H)$, so that by the *contraction mapping principle* T has a unique fixed point. By assumption,

$$\|T(v)(t)-T(w)(t)\| \leqslant c\int_t^\tau \|v(s)-w(s)\|\,\mathrm{d}s.$$

Hence, for any $\eta>0$,

$$\begin{aligned}\mathrm{e}^{t\eta}\|T(v)(t)-T(w)(t)\| &\leqslant c\int_t^\tau \mathrm{e}^{t\eta}\|v(s)-w(s)\|\,\mathrm{d}s\\ &\leqslant c\int_t^\tau \mathrm{e}^{(t-s)\eta}\mathrm{e}^{s\eta}\|v(s)-w(s)\|\,\mathrm{d}s\\ &\leqslant K_\eta \sup_{0\leqslant s\leqslant\tau}\mathrm{e}^{s\eta}\|v(s)-w(s)\|,\end{aligned}$$

where

$$K_\eta := c\int_t^\tau \mathrm{e}^{(t-s)\eta}\,\mathrm{d}s = \frac{c}{\eta}(1-\mathrm{e}^{-(\tau-t)\eta}).$$

Now, let

$$\|v\|_\eta := \sup_{0\leqslant s\leqslant\tau}\mathrm{e}^{s\eta}\|v(s)\|, \quad v\in C([0,\tau];H).$$

Note that $\|\cdot\|_\eta$ is a norm on $C([0,\tau];H)$, and it satisfies

$$\|v\|_\infty \leqslant \|v\|_\eta \leqslant \mathrm{e}^{\tau\eta}\|v\|_\infty \quad \forall\, v\in C([0,\tau];H).$$

Thus, $C([0,\tau];H)$ is a Banach space with respect to $\|\cdot\|_\eta$, and $K_\eta<1$ whenever $\eta>c$. Thus, for $\eta>c$, T is a contraction with respect to the complete norm $\|\cdot\|_\eta$ on $C([0,\tau];H)$. □

Lemma 3.8. *If $h\in C([0,\tau];H)$, then the function $t\mapsto \mathrm{e}^{-tA}h(t)$ is continuous.*

P r o o f. Let $t,t_0\in[0,\tau]$ and $\psi(t)=\mathrm{e}^{-tA}h(t)$. By using the fact that $\|\mathrm{e}^{-tA}\|\leqslant 1$,

$$\begin{aligned}\|\psi(t)-\psi(t_0)\| &\leqslant \|\mathrm{e}^{-tA}(h(t)-h(t_0))\| + \|(\mathrm{e}^{-tA}-\mathrm{e}^{-t_0A})h(t_0)\|\\ &\leqslant \|h(t)-h(t_0)\| + \|(\mathrm{e}^{-tA}-\mathrm{e}^{-t_0A})h(t_0)\|.\end{aligned}$$

Since for each $\varphi\in H$, $t\mapsto \mathrm{e}^{-tA}\varphi$ (cf. Pazy [13]), $\lim\limits_{t\to t_0}\psi(t)=\psi(t_0)$. □

Theorem 3.9. *Let $\phi\in D(\mathrm{e}^{\tau A})$ and $f\colon [0,\tau]\times H\to H$ satisfy the following conditions:*

(i) *For each $\varphi\in H$, $f(s,\varphi)\in D(\mathrm{e}^{sA})$ for all $s\in[0,\tau]$ and the function $(s,\varphi)\mapsto \mathrm{e}^{sA}f(s,\varphi)$ is Borel measurable.*

(ii) *For some* $\varphi_0 \in H$, *the function* $s \mapsto \mathrm{e}^{sA} f(s, \varphi_0)$ *belongs to* $L^1([0, \tau]; H)$.

(iii) *There exists* $\kappa > 0$ *such that*

$$\|\mathrm{e}^{sA}(f(s, \varphi_1) - f(s, \varphi_2))\| \leqslant \kappa \|\varphi_1 - \varphi_2\|$$

for all $\varphi_1, \varphi_2 \in H$ *and for all* $s \in [0, \tau]$.

Then the function $s \mapsto \mathrm{e}^{sA} f(s, w(s))$ *belongs to* $L^1([0, \tau]; H)$ *for any* $w \in C([0, \tau]; H)$, *and there exists a unique* $u \in C([0, \tau]; H)$ *such that*

$$u(t) = \mathrm{e}^{(\tau - t)A}\phi - \int_t^\tau \mathrm{e}^{(s-t)A} f(s, u(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau.$$

Proof. By Lemma 3.1, for each $w \in C([0, \tau]; H)$, the function $s \mapsto \mathrm{e}^{sA} f(s, w(s))$ belongs to $L^1([0, \tau]; H)$. For $w \in C([0, \tau]; H)$, let

$$G(w)(t) = \mathrm{e}^{(\tau - t)A}\phi - \int_t^\tau \mathrm{e}^{(s-t)A} f(s, w(s))\,\mathrm{d}s,$$

which can also be written as

$$G(w)(t) = \mathrm{e}^{-tA}\mathrm{e}^{\tau A}\phi - \mathrm{e}^{-tA} \int_t^\tau \mathrm{e}^{sA} f(s, w(s))\,\mathrm{d}s.$$

Since $\{\mathrm{e}^{-tA} \colon t \geqslant 0\}$ is a C_0 semigroup, $t \mapsto \mathrm{e}^{-tA}\mathrm{e}^{\tau A}\phi$ is continuous. Also, since the function $s \mapsto \mathrm{e}^{sA} f(s, w(s))$ belongs to $L^1([0, \tau]; H)$, $t \mapsto \int_t^\tau \mathrm{e}^{sA} f(s, w(s))\,\mathrm{d}s$ is continuous. Hence, by Lemma 3.8, $G(w) \in C([0, \tau]; H)$ so that the map $G\colon C([0, \tau]; H) \to C([0, \tau]; H)$ is well defined. We prove that G has a unique fixed point. For this purpose, we observe that

$$\|G(v)(t) - G(w)(t)\| \leqslant \kappa \int_t^\tau \|v(s) - w(s)\|\,\mathrm{d}s \tag{3.4}$$

for all $v, w \in C([0, \tau]; H)$ and for all $t \in [0, \tau]$. Indeed, using the fact that $\|\mathrm{e}^{-tA}\| \leqslant 1$ and condition (iii), we have

$$\begin{aligned} \|G(v)(t) - G(w)(t)\| &\leqslant \int_t^\tau \|\mathrm{e}^{-tA}\|\, \|\mathrm{e}^{sA}\big(f(s, v(s)) - f(s, w(s))\big)\|\,\mathrm{d}s \\ &\leqslant \kappa \int_t^\tau \|v(s) - w(s)\|\,\mathrm{d}s. \end{aligned}$$

Hence, by Lemma 3.7, G has a unique fixed point, and thereby the conclusion of the theorem holds. $\square$

4. Regularization

Let $u(\cdot)$ be a mild solution of the nonlinear FVP given by (1.1)–(1.2) as per Definition 3.4. That is,

$$(4.1)\qquad u(t)=\int_0^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi-\int_t^\tau\int_0^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\,\mathrm{d}s,\quad 0\leqslant t\leqslant\tau,$$

where $\phi\in D(\mathrm{e}^{\tau A})$ and $f\colon [0,\tau]\times H\to H$ is a Borel measurable function satisfying the conditions (i)–(iii) in Definition 3.4. Since $\varphi\mapsto\int_0^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\varphi$ is an unbounded operator, it is clear from (4.1) that the dependence of $u(\cdot)$ on ϕ is not continuous. To obtain a stable approximation for $u(t)$, some regulation method has to be employed. For this purpose, we define $u_\beta(t,\phi)$ as the solution of the integral equation obtained from (4.1) by truncation, that is, $u_\beta(t,\phi)$ is a solution of

$$(4.2)\quad u_\beta(t,\phi)=\int_0^\beta \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi-\int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u_\beta(s,\phi))\,\mathrm{d}s,\quad 0\leqslant t\leqslant\tau$$

for $\beta>0$. We show that the nonlinear integral equation (4.2) has a unique solution, and the solution is, indeed, a regularized solution under some assumptions on $f(\cdot,\cdot)$. For this, we shall make use of the following lemma.

Lemma 4.1. *Let $\varphi\in H$ and $0\leqslant t\leqslant s\leqslant\tau$. Then for $\beta>0$*

$$\Big\|\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda\varphi\Big\|\leqslant \mathrm{e}^{(s-t)\beta}\|\varphi\|.$$

P r o o f. By using the fact that $\mathrm{e}^{2(s-t)\lambda}\leqslant \mathrm{e}^{2(s-t)\beta}$ for $0\leqslant\lambda\leqslant\beta$, we get

$$\begin{aligned}\Big\|\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda\varphi\Big\|^2&=\int_0^\beta \mathrm{e}^{2(s-t)\lambda}\,\mathrm{d}\|E_\lambda\varphi\|^2\\&\leqslant \mathrm{e}^{2(s-t)\beta}\int_0^\beta \mathrm{d}\|E_\lambda\varphi\|^2\leqslant \mathrm{e}^{2(s-t)\beta}\|\varphi\|^2.\end{aligned}$$

Hence the result holds. □

Theorem 4.2. *Let $\phi\in H$ and let $f\colon [0,\tau]\times H\to H$ be a Borel measurable function satisfying the following conditions:*

(i) *For some $\varphi_0\in H$, the function $s\mapsto f(s,\varphi_0)$ belongs to $L^1([0,\tau];H)$.*

(ii) *There exists $\kappa>0$ such that $\big\|f(s,\varphi_1)-f(s,\varphi_2)\big\|\leqslant\kappa\|\varphi_1-\varphi_2\|$ for all $\varphi_1,\varphi_2\in H$ and for all $s\in[0,\tau]$.*

Then for each $\beta>0$, there exists a unique $u_\beta\in C([0,\tau];H)$ such that

$$u_\beta(t)=\int_0^\beta \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi-\int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u_\beta(s))\,\mathrm{d}s,\quad 0\leqslant t\leqslant\tau.$$

P r o o f. Let $v \in C([0,\tau];H)$. By Lemma 3.1, $s \mapsto f(s,v(s))$ belongs to $L^1([0,\tau];H)$. Therefore, by using Lemma 4.1 and the fact that $e^{s\beta} \leqslant e^{\tau\beta}$ for $\beta > 0$ and $0 \leqslant s \leqslant \tau$, we get

$$\int_0^\tau \left\| \int_0^\beta e^{s\lambda} \, dE_\lambda f(s,v(s)) \right\| ds \leqslant \int_0^\tau e^{s\beta} \| f(s,v(s)) \| \, ds$$
$$\leqslant e^{\tau\beta} \int_0^\tau \| f(s,v(s)) \| \, ds < \infty.$$

Hence, the integral

$$\int_t^\tau \int_0^\beta e^{(s-t)\lambda} \, dE_\lambda f(s,v(s)) \, ds$$

is well defined for every $v \in C([0,\tau];H)$ and for $0 \leqslant t \leqslant \tau$. Now, for $\beta > 0$, let

$$G_\beta(v) := \int_0^\beta e^{(\tau-t)\lambda} \, dE_\lambda \phi - \int_t^\tau \int_0^\beta e^{(s-t)\lambda} \, dE_\lambda f(s,v(s)) \, ds, \quad v \in C([0,\tau];H).$$

We observe that

$$\begin{aligned} G_\beta(v)(t) &= \int_0^\beta e^{(\tau-t)\lambda} \, dE_\lambda \phi - \int_t^\tau \int_0^\beta e^{(s-t)\lambda} \, dE_\lambda f(s,v(s)) \, ds \\ &= e^{(\tau-t)A} \chi_{[0,\beta]}(A)\phi - \int_t^\tau e^{(s-t)A} \chi_{[0,\beta]}(A) f(s,v(s)) \, ds \\ &= e^{-tA}\phi_\beta - e^{-tA} \int_t^\tau f_\beta(s,v(s)) \, ds, \end{aligned}$$

where $\phi_\beta = e^{\tau A}\chi_{[0,\beta]}(A)\phi$. Since $\{e^{-tA} \colon t \geqslant 0\}$ is a C_0 semigroup, $t \mapsto e^{-tA}\phi_\beta$ is continuous. Also, as $s \mapsto f_\beta(s,v(s))$ belongs to $L^1([0,\tau];H)$, $t \mapsto \int_t^\tau f_\beta(s,w(s)) \, ds$ is continuous. Hence, by Lemma 3.8, $G_\beta(v) \in C([0,\tau];H)$.

Next we show that $G_\beta \colon C([0,\tau];H) \to C([0,\tau];H)$ has a unique fixed point. For this, we observe that,

$$\| G_\beta(v)(t) - G_\beta(w)(t) \| \leqslant \kappa e^{\tau\beta} \int_t^\tau \| v(s) - w(s) \| \, ds \tag{4.3}$$

for all $v, w \in C([0,\tau];H)$ and for all $t \in [0,\tau]$. This is seen as follows:

Let $v, w \in C([0,\tau]; H)$ and $t \in [0,\tau]$. Using the fact that $\mathrm{e}^{2(s-t)\lambda} \leqslant \mathrm{e}^{2(\tau-t)\beta}$ for $0 \leqslant t \leqslant s \leqslant \tau$; $0 \leqslant \lambda \leqslant \beta$ and the condition (ii), we have

$$\begin{aligned}\|G_\beta(v)(t) - G_\beta(w)(t)\| &\leqslant \int_t^\tau \left\| \int_0^\beta \mathrm{e}^{(s-t)\lambda} \,\mathrm{d}E_\lambda(f(s,v(s)) - f(s,w(s))) \right\| \mathrm{d}s \\ &= \int_t^\tau \left(\int_0^\beta \mathrm{e}^{2(s-t)\lambda} \,\mathrm{d}\|E_\lambda(f(s,v(s)) - f(s,w(s)))\|^2 \right)^{1/2} \mathrm{d}s \\ &\leqslant \int_t^\tau \mathrm{e}^{(\tau-t)\beta} \left(\int_0^\beta \mathrm{d}\|E_\lambda(f(s,v(s)) - f(s,w(s)))\|^2 \right)^{1/2} \mathrm{d}s \\ &\leqslant \mathrm{e}^{(\tau-t)\beta} \int_t^\tau \|f(s,v(s)) - f(s,w(s))\| \,\mathrm{d}s \\ &\leqslant \kappa \mathrm{e}^{\tau\beta} \int_t^\tau \|v(s) - w(s)\| \,\mathrm{d}s.\end{aligned}$$

Thus, (4.3) holds for all $v, w \in C([0,\tau]; H)$ and for all $t \in [0,\tau]$. Hence, by Lemma 3.7, G_β has a unique fixed point, thereby the conclusion of the theorem holds. $\square$

Now, we prove that the solution of the integral equation (4.2) is continuously dependent on ϕ. For that, we will make use of the following lemma which is a consequence of the well-known Gronwall's inequality (cf. Perko [14]).

Lemma 4.3. *If $h\colon [0,\tau] \to \mathbb{R}$ is a non-negative continuous function satisfying*

$$h(t) \leqslant c_0 + \kappa \int_t^\tau h(s) \,\mathrm{d}s,$$

for some $c_0 > 0$, then $h(t) \leqslant c_0 \mathrm{e}^{(\tau-t)\kappa}$.

Theorem 4.4. *Let $f\colon [0,\tau] \times H \to H$ be a Borel measurable function satisfying the conditions* (i)–(ii) *in Theorem* 4.2*, and let $\beta > 0$ and $\phi_1, \phi_2 \in H$. Let $u_\beta(t,\phi_1)$ and $u_\beta(t,\phi_2)$ be the solutions of the integral equation* (4.2) *with ϕ replaced by ϕ_1 and ϕ_2, respectively. Then, for $0 \leqslant t \leqslant \tau$,*

$$\|u_\beta(t,\phi_1) - u_\beta(t,\phi_2)\| \leqslant \mathrm{e}^{(\tau-t)\kappa} \mathrm{e}^{(\tau-t)\beta} \|\phi_1 - \phi_2\|.$$

P r o o f. For $0 \leqslant t \leqslant \tau$, let

$$u_\beta^{(1)}(t) := u_\beta(t,\phi_1), \quad u_\beta^{(2)}(t) := u_\beta(t,\phi_2), \quad w_\beta^{1,2}(t) := f(s, u_\beta^{(1)}(t)) - f(s, u_\beta^{(1)}(t)).$$

Now

$$u_\beta^{(1)}(t) - u_\beta^{(2)}(t) = \int_0^\beta \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda(\phi_1-\phi_2) - \int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda(w_\beta^{1,2}(s))\,\mathrm{d}s.$$

By using Lemma 4.1, we have

$$\Big\|\int_0^\beta \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda(\phi_1-\phi_2)\Big\| \leqslant \mathrm{e}^{(\tau-t)\beta}\|\phi_1-\phi_2\|,$$
$$\Big\|\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda(w_\beta^{1,2}(s))\Big\| \leqslant \mathrm{e}^{(s-t)\beta}\|w^{1,2}(s)\|.$$

Hence,

$$\|u_\beta^{(1)}(t) - u_\beta^{(2)}(t)\| \leqslant \mathrm{e}^{(\tau-t)\beta}\|\phi_1-\phi_2\| + \int_t^\tau \mathrm{e}^{(s-t)\beta}\|w_\beta^{1,2}(s)\|\,\mathrm{d}s.$$

By condition (ii) in Theorem 4.2, we get

$$\|w_\beta^{1,2}(s)\| \leqslant \kappa\|u_\beta^{(1)}(s) - u_\beta^{(2)}(s)\|.$$

Therefore

$$\mathrm{e}^{t\beta}\|u_\beta^{(1)}(t) - u_\beta^{(2)}(t)\| \leqslant \mathrm{e}^{\tau\beta}\|\phi_1-\phi_2\| + \kappa\int_t^\tau \mathrm{e}^{s\beta}\|u_\beta^{(1)}(s) - u_\beta^{(2)}(s)\|\,\mathrm{d}s.$$

Hence, by Lemma 4.3,

$$\mathrm{e}^{t\beta}\|u_\beta^{(1)}(t) - u_\beta^{(2)}(t)\| \leqslant \mathrm{e}^{\tau\beta}\|\phi_1-\phi_2\|\mathrm{e}^{(\tau-t)\kappa}.$$

Thus, we obtain the required inequality. □

5. Convergence and error estimates

Let $u(\cdot)$ be the *mild solution* of the nonliner FVP given by (1.1)–(1.2). Note that the condition on ϕ and $f(\cdot,\cdot)$ in Definition 3.4 imply

$$\int_0^\infty \mathrm{e}^{2\lambda\tau}\,\mathrm{d}\|E_\lambda\phi\|^2 < \infty \quad \text{and} \quad \int_0^\tau \Big\|\int_0^\infty \mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\Big\|\,\mathrm{d}s < \infty. \tag{5.1}$$

We shall prove the convergence of the regularized solutions to the mild solution and estimate the errors assuming also that ϕ and $f(\cdot,\cdot)$ satisfy the following assumption.

Assumption (A). There exists a continuous or piecewise continuous and monotonically increasing function $g\colon (0,\infty)\to(0,\infty)$ such that

(1) $\displaystyle\int_0^\infty g^2(\lambda)e^{2\tau\lambda}\,d\|E_\lambda\phi\|^2 \leqslant \varrho_g < \infty,$

(2) $\displaystyle\int_0^\tau \Big(\int_0^\infty g^2(\lambda)e^{2s\lambda}\,d\|E_\lambda f(s,u(s))\|^2\Big)^{1/2}\,ds \leqslant \eta_g < \infty$

where ϱ_g and η_g are positive constants.

It is to be observed that the assumption (2) implies that

$$\int_0^\infty g(\lambda)e^{s\lambda}\,dE_\lambda f(s,u(s))$$

is well defined almost everywhere for $s\in[0,\tau]$ with

$$\Big\|\int_0^\infty g(\lambda)e^{s\lambda}\,dE_\lambda f(s,u(s))\Big\|^2 = \int_0^\infty g^2(\lambda)e^{2s\lambda}\,d\|E_\lambda f(s,u(s))\|^2,$$

and hence the condition in (2) is equivalent to the assumption that the function

$$s\mapsto \Big\|\int_0^\infty g(\lambda)e^{s\lambda}\,dE_\lambda f(s,u(s))\Big\|$$

belongs to $L^1([0,\tau],H)$.

Clearly, if $|g|\leqslant 1$, then the conditions in (5.1) imply the conditions (1) and (2) in Assumption (A). We will see that the case when g is an unbounded function, is more relevant for our analysis. For the choices (i) $g(\lambda)=e^{p\lambda}$ for some $p\geqslant 0$ and (ii) $g(\lambda)=\lambda^q$ for some $q>0$, the conditions in Assumption (A) take the form

$$\text{(i)}\quad \int_0^\infty e^{2(p+\tau)\lambda}\,d\|E_\lambda\phi\|^2<\infty \quad\text{and}\quad \int_0^\tau\Big\|\int_0^\infty e^{(p+s)\lambda}\,dE_\lambda f(s,u(s))\Big\|\,ds<\infty$$

and

$$\text{(ii)}\quad \int_0^\infty \lambda^{2q}e^{2\tau\lambda}\,d\|E_\lambda\phi\|^2<\infty \quad\text{and}\quad \int_0^\tau\Big\|\int_0^\infty \lambda^{q}e^{s\lambda}\,dE_\lambda f(s,u(s))\Big\|\,ds<\infty,$$

respectively.

5.1. Error analysis with exact data. For proving the results on convergence and error estimates, we will make use of the following two lemmas.

Lemma 5.1. *Let $w \in L^1([0,\tau];H)$ and let $g\colon (0,\infty) \to (0,\infty)$ be a continuous or piecewise continuous and monotonically increasing function such that*

$$\int_0^\tau \Bigl(\int_0^\infty g^2(\lambda)\mathrm{e}^{2s\lambda}\,\mathrm{d}\|E_\lambda w(s)\|^2\Bigr)^{1/2}\,\mathrm{d}s < \infty.$$

Then

$$\lim_{\beta\to\infty}\int_0^\tau \Bigl\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda w(s)\Bigr\|\,\mathrm{d}s = 0.$$

P r o o f. By the assumption, for almost all $s \in [0,\tau]$,

$$\begin{aligned}\Bigl\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda w(s)\Bigr\|^2 &= \int_\beta^\infty g^2(\lambda)\mathrm{e}^{2s\lambda}\,\mathrm{d}\|E_\lambda w(s)\|^2\\ &\leqslant \int_0^\infty g^2(\lambda)\mathrm{e}^{2s\lambda}\,\mathrm{d}\|E_\lambda w(s)\|^2 < \infty\end{aligned}$$

so that

$$\Bigl\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda w(s)\Bigr\|^2 \to 0 \quad \text{as} \quad \beta\to\infty$$

for almost all $s \in [0,\tau]$. Therefore, by DCT, we get

$$\lim_{\beta\to\infty}\int_0^\tau \Bigl\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda w(s)\Bigr\|\,\mathrm{d}s = \int_0^\tau \lim_{\beta\to\infty}\Bigl\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda w(s)\Bigr\|\,\mathrm{d}s = 0.$$

□

Lemma 5.2. *Let $g\colon (0,\infty)\to(0,\infty)$ be a continuous or piecewise continuous and monotonically increasing function. Let $s \geqslant 0$ and $\varphi \in H$ be such that*

$$\int_0^\infty g^2(\lambda)\mathrm{e}^{2s\lambda}\,\mathrm{d}\|E_\lambda\varphi\|^2 < \infty.$$

Then, for $\beta > 0$,

$$\Bigl\|\int_\beta^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda\varphi\Bigr\| \leqslant \frac{\mathrm{e}^{-t\beta}}{g(\beta)}\Bigl\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda\varphi\Bigr\|.$$

P r o o f. By using the fact that

$$\frac{\mathrm{e}^{-t\lambda}}{g(\lambda)} \leqslant \frac{\mathrm{e}^{-t\beta}}{g(\beta)}$$

for all $\lambda \geqslant \beta$, we get

$$\begin{aligned}\Big\|\int_\beta^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda\varphi\Big\|^2 &= \int_\beta^\infty \frac{\mathrm{e}^{-2t\lambda}}{g^2(\lambda)} g^2(\lambda)\mathrm{e}^{2s\lambda}\,\mathrm{d}\|E_\lambda\varphi\|^2 \\ &\leqslant \frac{\mathrm{e}^{-2t\beta}}{g^2(\beta)}\int_\beta^\infty g^2(\lambda)\mathrm{e}^{2s\lambda}\,\mathrm{d}\|E_\lambda\varphi\|^2 = \frac{\mathrm{e}^{-2t\beta}}{g^2(\beta)}\Big\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda\varphi\Big\|^2.\end{aligned}$$

Hence the result holds. □

Theorem 5.3. *Let $\phi \in H$ and let $f\colon [0,\tau]\times H \to H$ be a Borel measurable function satisfying the conditions* (i)–(ii) *in Theorem 4.2. Let $u(\cdot)$ be the mild solution of the nonlinear FVP given by* (1.1)–(1.2) *and let $u_\beta(\cdot) := u_\beta(\cdot,\phi)$ be the solution of the integral equation* (4.2) *for $\beta > 0$. Let g, ϕ and $f(\cdot,\cdot)$ satisfy Assumption* (A). *Then for $0 \leqslant t < \tau$,*

$$\|u(t) - u_\beta(t)\| \leqslant \mathrm{e}^{(\tau-t)\kappa} C_g(\beta)\,\frac{\mathrm{e}^{-t\beta}}{g(\beta)} \leqslant (\varrho_g + \eta_g)\mathrm{e}^{(\tau-t)\kappa}\,\frac{\mathrm{e}^{-t\beta}}{g(\beta)} \tag{5.2}$$

for all $\beta > 0$, where

$$C_g(\beta) := \Big\|\int_\beta^\infty g(\lambda)\mathrm{e}^{\tau\lambda}\,\mathrm{d}E_\lambda\phi\Big\| + \int_0^\tau\Big\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\Big\|\,\mathrm{d}s \to 0 \quad \text{as } \beta\to\infty.$$

In particular,

$$\lim_{\beta\to\infty} u_\beta(t) = u(t), \quad 0 \leqslant t < \tau.$$

P r o o f. For $0 \leqslant t < \tau$, let

$$w_\beta(t) = f(t,u(t)) - f(t,u_\beta(t)).$$

From (4.1) and (4.2), we have

$$\begin{aligned}u(t) - u_\beta(t) &= \int_\beta^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi - \int_t^\tau\int_0^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\,\mathrm{d}s \\ &\quad + \int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u_\beta(s))\,\mathrm{d}s \\ &= \int_\beta^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi - \int_t^\tau\int_\beta^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\,\mathrm{d}s \\ &\quad - \int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda w_\beta(s)\,\mathrm{d}s.\end{aligned}$$

Now, by using Lemma 5.2, we have

$$\left\|\int_\beta^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi\right\| \leqslant \frac{\mathrm{e}^{-t\beta}}{g(\beta)}\left\|\int_\beta^\infty g(\lambda)\mathrm{e}^{\tau\lambda}\,\mathrm{d}E_\lambda\phi\right\| \tag{5.3}$$

and

$$\int_t^\tau\left\|\int_\beta^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\right\|\mathrm{d}s \leqslant \frac{\mathrm{e}^{-t\beta}}{g(\beta)}\int_0^\tau\left\|\int_\beta^\infty g(\lambda)\mathrm{e}^{s\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\right\|\mathrm{d}s. \tag{5.4}$$

Also,by using Lemma 4.1 , we have

$$\int_t^\tau\left\|\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda w_\beta(s)\right\|\mathrm{d}s \leqslant \int_t^\tau \mathrm{e}^{(s-t)\beta}\|w_\beta(s)\|\,\mathrm{d}s.$$

By using condition (ii) in Theorem 4.2, we get

$$\int_t^\tau\left\|\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda w_\beta(s)\right\|\mathrm{d}s \leqslant \mathrm{e}^{-t\beta}\int_t^\tau \kappa\mathrm{e}^{s\beta}\|u(s)-u_\beta(s)\|\,\mathrm{d}s. \tag{5.5}$$

Now, by using (5.3), (5.4) and (5.5), we have

$$\begin{aligned}\|u(t)-u_\beta(t)\| &\leqslant \left\|\int_\beta^\infty \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi\right\| + \int_t^\tau\left\|\int_\beta^\infty \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u(s))\right\|\mathrm{d}s\\ &\quad+\int_t^\tau\left\|\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda w_\beta(s)\right\|\mathrm{d}s\\ &\leqslant \frac{\mathrm{e}^{-t\beta}}{g(\beta)}C_g(\beta)+\mathrm{e}^{-t\beta}\int_t^\tau \kappa\mathrm{e}^{s\beta}\|u(s)-u_\beta(s)\|\,\mathrm{d}s.\end{aligned}$$

The above inequality can be written as

$$\mathrm{e}^{t\beta}\|u(t)-u_\beta(t)\| \leqslant \frac{C_g(\beta)}{g(\beta)}+\int_t^\tau \kappa\mathrm{e}^{s\beta}\|u(s)-u_\beta(s)\|\,\mathrm{d}s.$$

Hence, by Lemma 4.3, we get

$$\mathrm{e}^{t\beta}\|u(t)-u_\beta(t)\| \leqslant \frac{C_g(\beta)}{g(\beta)}\mathrm{e}^{(\tau-t)\kappa},$$

i.e.,

$$\|u(t)-u_\beta(t)\| \leqslant \mathrm{e}^{(\tau-t)\kappa}C_g(\beta)\frac{\mathrm{e}^{-t\beta}}{g(\beta)} \tag{5.6}$$

for all $t\in[0,\tau)$. Note that $C_g(\beta)\leqslant \varrho_g+\eta_g$ for all $\beta>0$. Hence, from (5.6), we obtain inequality (5.2). Since $\int_0^\infty g^2(\lambda)\mathrm{e}^{2\tau\lambda}\,\mathrm{d}\|E_\lambda\phi\|^2<\infty$, we have

$$\left\|\int_\beta^\infty g(\lambda)\mathrm{e}^{\tau\lambda}\,\mathrm{d}E_\lambda\phi\right\|^2 = \int_\beta^\infty g^2(\lambda)\mathrm{e}^{2\tau\lambda}\,\mathrm{d}\|E_\lambda\phi\|^2 \to 0 \quad \text{as } \beta\to\infty. \tag{5.7}$$

By using (5.7) and Lemma 5.1, we get $C_g(\beta)\to 0$ as $\beta\to\infty$. Hence, from (5.6), $\|u(t)-u_\beta(t)\|\to 0$ as $\beta\to\infty$. □

Remark 5.4. We observe that the estimate obtained in Theorem 5.3 includes the estimate in (Jana and Nair [8], Theorem 3.2) as a particular case for the choice $f(t)$ in place of $f(t,u(t))$ under same conditions on ϕ and f.

Remark 5.5. When (i) $g(\lambda)=\mathrm{e}^{p\lambda}$, $p\geqslant 0$ and (ii) $g(\lambda)=\lambda^q$, $q>0$, by Theorem 5.3 we have

(i) $\|u(t)-u_\beta(t)\|\leqslant(\varrho_g+\eta_g)\mathrm{e}^{(\tau-t)\kappa}\mathrm{e}^{-(t+p)\beta}$ and

(ii) $\|u(t)-u_\beta(t)\|\leqslant(\varrho_g+\eta_g)\mathrm{e}^{(\tau-t)\kappa}\mathrm{e}^{-t\beta}/\beta^q$,

respectively, for $0\leqslant t<\tau$.

5.2. Error analysis with noisy data. Suppose that the data ϕ is noisy, that is, we have ϕ_ε in place of ϕ with

$$\|\phi-\phi_\varepsilon\|\leqslant\varepsilon$$

for some $\varepsilon>0$. Let $u(\cdot)$ be the *mild solution* of the nonlinear FVP given by (1.1)–(1.2) and let $u_{\beta,\varepsilon}(\cdot):=u_\beta(\phi_\varepsilon,\cdot)$ be the solution of the integral equation (4.2) with ϕ replaced by ϕ_ε, that is

$$u_{\beta,\varepsilon}(t)=\int_0^\beta \mathrm{e}^{(\tau-t)\lambda}\,\mathrm{d}E_\lambda\phi_\varepsilon-\int_t^\tau\int_0^\beta \mathrm{e}^{(s-t)\lambda}\,\mathrm{d}E_\lambda f(s,u_{\beta,\varepsilon}(s))\,\mathrm{d}s$$

for each $\beta>0$.

Theorem 5.6. *Let $\phi\in H$ and let $f\colon[0,\tau]\times H\to H$ be a Borel measurable function satisfying the conditions* (i)–(ii) *in Theorem* 4.2. *Let g, ϕ and $f(\cdot,\cdot)$ satisfy Assumption* (A). *Then*

$$\|u(t)-u_{\beta,\varepsilon}(t)\|\leqslant c_g(t)\Big(\mathrm{e}^{(\tau-t)\beta}\mathrm{e}+\frac{\mathrm{e}^{-t\beta}}{g(\beta)}\Big),\quad 0\leqslant t<\tau,$$

where $c_g(t)=\mathrm{e}^{(\tau-t)\kappa}\max\{1,\varrho_g+\eta_g\}$.

Proof. Let $0\leqslant t<\tau$ and let $u_\beta(t):=u_\beta(\phi,t)$ be the solution of the integral equation (4.2). Now, by using Theorem 4.4, we get

$$\|u_\beta(t)-u_{\beta,\varepsilon}(t)\|\leqslant\mathrm{e}^{(\tau-t)\kappa}\mathrm{e}^{(\tau-t)\beta}\|\phi-\phi_\varepsilon\|$$

so that

$$\|u(t)-u_{\beta,\varepsilon}(t)\|\leqslant\mathrm{e}^{(\tau-t)\kappa}\mathrm{e}^{(\tau-t)\beta}\|\phi-\phi_\varepsilon\|+\|u(t)-u_\beta(t)\|.$$

Since $\|\phi-\phi_\varepsilon\|\leqslant\varepsilon$,

$$\|u(t)-u_{\beta,\varepsilon}(t)\|\leqslant\mathrm{e}^{(\tau-t)\kappa}\mathrm{e}^{(\tau-t)\beta}\varepsilon+\|u(t)-u_\beta(t)\|. \tag{5.8}$$

From (5.8), using Theorem 5.3, we obtain the required estimate. □

Remark 5.7. When noise is only in the final value ϕ, the estimate obtained in [8], Theorem 3.6 is a constant multiple of the estimate in Theorem 5.6 for $f(t)$ in place of $f(t,u(t))$ under the same conditions on ϕ and f.

Remark 5.8. In particular, when (i) $g(\lambda)=\mathrm{e}^{p\lambda}$, $p\geqslant 0$ and (ii) $g(\lambda)=\lambda^q$, $q>0$, by Theorem 5.6 we have

(i) $\|u(t)-u_{\beta,\varepsilon}(t)\|\leqslant c_g(t)(\mathrm{e}^{(\tau-t)\beta}\varepsilon+\mathrm{e}^{-(p+t)\beta})$ and
(ii) $\|u(t)-u_{\beta,\varepsilon}(t)\|\leqslant c_g(t)(\mathrm{e}^{(\tau-t)\beta}\varepsilon+\mathrm{e}^{-\beta t}/\beta^q)$,

respectively, for $0\leqslant t<\tau$.

5.3. Error estimates under parameter choice strategies. From Theorem 5.6, we have

$$\|u(t)-u_{\beta,\varepsilon}(t)\|\leqslant c_g(t)\Big(\mathrm{e}^{(\tau-t)\beta}\varepsilon+\frac{\mathrm{e}^{-t\beta}}{g(\beta)}\Big),\quad 0\leqslant t<\tau,$$

where $c_g(t)=\mathrm{e}^{(\tau-t)\kappa}\max\{1,\varrho_g+\eta_g\}$. Note that, for a fixed ε and $0\leqslant t<\tau$, $\lim\limits_{\beta\to\infty}\varepsilon\mathrm{e}^{(\tau-t)\beta}=\infty$. Further,

$$\lim_{\beta\to\infty}\frac{\mathrm{e}^{-t\beta}}{g(\beta)}=0,$$

when $0<t<\tau$, and for $t=0$,

$$\lim_{\beta\to\infty}\frac{\mathrm{e}^{-t\beta}}{g(\beta)}=0,$$

when g is unbounded.

Thus, in order to obtain approximate regularized solutions for a fixed t, we need to choose the regularization parameter $\beta_t(\varepsilon)$ depending on ε such that

$$\Big(\varepsilon\mathrm{e}^{(\tau-t)\beta_t(\varepsilon)}+\frac{\mathrm{e}^{-t\beta_t(\varepsilon)}}{g(\beta_t(\varepsilon))}\Big)\to 0\quad\text{as } \varepsilon\to 0.$$

Also, it is desirable for such a $\beta_t(\varepsilon)$ to satisfy

$$\varepsilon\mathrm{e}^{(\tau-t)\beta_t(\varepsilon)}+\frac{\mathrm{e}^{-t\beta_t(\varepsilon)}}{g(\beta_t(\varepsilon))}=\inf_{\beta>0}\Big(\mathrm{e}^{(\tau-t)\beta}\varepsilon+\frac{\mathrm{e}^{-t\beta}}{g(\beta)}\Big).$$

This can be done following the method adopted in [9]. For the sake of completion, we supply the details in the following subsections by considering two cases, namely when (i) g is continuous and (ii) g is piecewise continuous.

5.3.1. Parameter choice when g is continuous.

Lemma 5.9. *Let $g\colon (0,\infty)\to(0,\infty)$ be a continuous and monotonically increasing function. For $t\in[0,\tau)$ and $\varepsilon>0$, let*

$$\Phi_t^\varepsilon(\beta)=\varepsilon\mathrm{e}^{(\tau-t)\beta}+\frac{\mathrm{e}^{-t\beta}}{g(\beta)}.$$

(i) *If $\lim\limits_{\beta\to 0} g(\beta)=0$, then there exists $\beta_t(\varepsilon)>0$ such that*

$$\inf_{\beta>0}\Phi_t^\varepsilon(\beta)=\Phi_t^\varepsilon(\beta_t(\varepsilon)).$$

(ii) *If $\lim\limits_{\beta\to 0} g(\beta)>0$, then there exists $\beta_t(\varepsilon)\geqslant 0$ such that*

$$\inf_{\beta\geqslant 0}\Phi_t^\varepsilon(\beta)=\Phi_t^\varepsilon(\beta_t(\varepsilon)).$$

(iii) *For $0<t<\tau$,*

$$\Phi_t^\varepsilon(\beta_t(\varepsilon))\to 0\quad \text{as } \varepsilon\to 0.$$

(iv) *If g is unbounded function, then*

$$\Phi_0^\varepsilon(\beta_0(\varepsilon))\to 0\quad \text{as } \varepsilon\to 0.$$

Proof. Since g is continuous, Φ_t^ε is also continuous on $(0,\infty)$.

(i) Let $\gamma>0$ be a fixed real number. Since $\lim\limits_{\beta\to 0} g(\beta)=0$, $\lim\limits_{\beta\to 0}\Phi_t^\varepsilon(\beta)=\infty$ and $\lim\limits_{\beta\to\infty}\Phi_t^\varepsilon(\beta)=\infty$, there exist $\beta_t^0>0$ and $\beta_t^\infty>0$ such that

$$\Phi_t^\varepsilon(\beta)>\Phi_t^\varepsilon(\gamma)\quad \forall\, 0<\beta<\beta_t^0 \quad\text{and}\quad \Phi_t^\varepsilon(\beta)>\Phi_t^\varepsilon(\gamma)\quad \forall\,\beta>\beta_t^\infty.$$

Since Φ_t^ε is continuous on the compact set $[\beta_t^0,\beta_t^\infty]$, there exists a $\beta_t(\varepsilon)$ in $[\beta_t^0,\beta_t^\infty]$ such that

$$\inf_{\beta>0}\Phi_t^\varepsilon(\beta)=\inf_{\beta_t^0\leqslant\beta\leqslant\beta_t^\infty}\Phi_t^\varepsilon(\beta)=\Phi_t^\varepsilon(\beta_t(\varepsilon)).$$

(ii) Define $g(0):=\lim\limits_{\beta\to 0} g(\beta)>0$. Then Φ_t^ε is well defined and continuous on $[0,\infty)$. Since $\lim\limits_{\beta\to\infty}\Phi_t^\varepsilon(\beta)=\infty$, there exists $\beta_t^\infty>0$ such that

$$\Phi_t^\varepsilon(\beta)>\Phi_t^\varepsilon(0)\quad \forall\,\beta>\beta_t^\infty.$$

Now, by the property of continuity of Φ_t^ε on the compact set $[0, \beta_t^\infty]$, there exists a $\beta_t(\varepsilon)$ in $[0, \beta_t^\infty]$ such that

$$\inf_{\beta \geqslant 0} \Phi_t^\varepsilon(\beta) = \inf_{0 \leqslant \beta \leqslant \beta_t^\infty} \Phi_t^\varepsilon(\beta) = \Phi_t^\varepsilon(\beta_t(\varepsilon)).$$

(iii) For sufficiently small ε and $0 \leqslant t < \tau$, we would like to find $\beta := \beta(\varepsilon) > 0$ such that

$$\varepsilon \mathrm{e}^{(\tau - t)\beta} = \frac{\mathrm{e}^{-t\beta}}{g(\beta)}, \quad \text{that is,} \quad \frac{1}{g(\beta)\mathrm{e}^{\tau\beta}} = \varepsilon.$$

Let

$$\vartheta(\beta) = \frac{1}{g(\beta)\mathrm{e}^{\tau\beta}}, \quad \beta > 0. \tag{5.9}$$

Clearly, ϑ is a continuous and strictly monotonically decreasing function. Let $0 < \varepsilon < \lim_{\beta \to 0} 1/g(\beta)$. Thus, by intermediate value theorem, there exits $\beta(\varepsilon) > 0$ such that $\vartheta(\beta(\varepsilon)) = \varepsilon$. Since ϑ is strictly monotonically decreasing, $\beta(\varepsilon)$ is unique and $\beta(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Note that

$$\Phi_t^\varepsilon(\beta(\varepsilon)) = 2\, \frac{\mathrm{e}^{-t\beta(\varepsilon)}}{g(\beta(\varepsilon))}$$

and

$$\Phi_t^\varepsilon(\beta_t(\varepsilon)) \leqslant \Phi_t^\varepsilon(\beta(\varepsilon)).$$

Hence the conclusions follow immediately.

(iv) Proof is immediate from (iii). □

Proof of the following theorem is immediate from Theorem 5.6.

Theorem 5.10. *Let $\phi \in H$ and let $f\colon [0, \tau] \times H \to H$ be a Borel measurable function satisfying the conditions* (i)–(ii) *in Theorem* 4.2. *If g, ϕ and $f(\cdot, \cdot)$ satisfy Assumption* (A), *then for Φ_t^ε and $\beta_t(\varepsilon)$ as in Lemma* 5.9, *we have the estimate*

$$\|u(t) - u_{\beta_t(\varepsilon), \varepsilon}(t)\| \leqslant c_g(t)\ \Phi_t^\varepsilon(\beta_t(\varepsilon)) \quad \textit{for } 0 \leqslant t < \tau.$$

Remark 5.11. (i) Due to Theorem 5.10, taking $g(\lambda) = \mathrm{e}^{p\lambda}$, $p \geqslant 0$, by finding the critical value

$$\beta_t(\varepsilon) = \frac{1}{\tau + p} \ln\Big(\frac{(t+p)}{(\tau - t)} \frac{1}{\varepsilon}\Big)$$

of the function Φ_t^ε in Lemma 5.9, we have an estimate for $0 \leqslant t < \tau$ and $t+p > 0$,

$$\|u(t) - u_{\beta_t(\varepsilon),\varepsilon}(t)\| \leqslant c_g(t)\frac{\tau+p}{\tau-t}\Big(\frac{\tau-t}{t+p}\Big)^{(t+p)/(\tau+p)}\varepsilon^{(t+p)/(\tau+p)}. \tag{5.10}$$

Since $((\tau+p)/(\tau-t))((\tau-t)/(t+p))^{(t+p)/(\tau+p)} \leqslant 2$,

$$\|u(t) - u_{\beta_t(\varepsilon),\varepsilon}(t)\| \leqslant 2c_g(t)\varepsilon^{(t+p)/(\tau+p)}. \tag{5.11}$$

(ii) For $g(\lambda) = \lambda^q$, $q > 0$, taking

$$\beta_t(\varepsilon) := \frac{\gamma}{\tau - t}\ln\frac{1}{\varepsilon}$$

for some $\gamma \in (0,1)$, from Remark 5.8 (ii) we have an estimate

$$\|u(t) - u_{\beta_t(\varepsilon),\varepsilon}(t)\| \leqslant c_g(t)\Big(\varepsilon^{1-\gamma} + \varepsilon^{\gamma t/(\tau-t)}\Big(\frac{\tau-t}{\gamma}\Big)^q\Big(\ln\frac{1}{\varepsilon}\Big)^{-q}\Big), \quad 0 \leqslant t < \tau. \tag{5.12}$$

Moreover, taking $\beta(\varepsilon) = 1/\tau\ln(1/\varepsilon)$, we have the estimate from Remark 5.8 (ii)

$$\|u(t) - u_{\beta_t(\varepsilon),\varepsilon}(t)\| \leqslant c_g(t)\varepsilon^{t/\tau}\Big(1 + \tau^q\Big(\ln\frac{1}{\varepsilon}\Big)^{-q}\Big), \quad 0 \leqslant t < \tau. \tag{5.13}$$

Remark 5.12. When the noise is only in the final value ϕ, the orders of the estimates obtained in Theorems 3.10, 3.11, 3.12 in Jana and Nair [8] are particular cases of the estimates obtained in (5.10), (5.11) when $p = 0$; (5.10), (5.11) and (5.10), (5.11) when $p > 0$, respectively, for $f(t)$ in place of $f(t,u(t))$ under the same conditions on ϕ and f. Also, the order of the estimate in (5.10) has been obtained in [7], Theorem 4.16 for $f(t)$ in place of $f(t,u(t))$ by a different method, namely, the quasi-reversibility method when noise is only in the final value ϕ.

Remark 5.13. In [12], Nam considered the FVP (1.1)–(1.2) with the operator A having discrete spectrum, which is a special case of our consideration. We see that the estimates obtained in [12], Lemma 1 are also, consequences of the results considered in Remark 5.8. More specifically, we have the following result.

(i) Putting $\beta := \beta(\varepsilon) = l^{-1}\ln(1/\varepsilon)$ for some $l \geqslant \tau$ in Remark 5.8 (i) with $p = 0$, we have the estimate

$$\|u(t) - u_{\beta(\varepsilon),\varepsilon}(t)\| \leqslant c_g^1\varepsilon^{t/l}, \quad 0 \leqslant t < \tau, \tag{5.14}$$

where $c_g^1 = 2\mathrm{e}^{\tau\kappa}\max\{1, \varrho_g + \eta_g\}$. The above estimate is the same as in [12], Lemma 1 (i) under the condition $\|\mathrm{e}^{tA}u(t)\| \leqslant M_0$ for all $t \in [0,\tau]$ for some $M_0 > 0$, in place of our Assumption (A) with $g(\lambda) = 1$. It can be seen that our condition implies the above condition in [12].

(ii) Taking $\beta := \beta(\varepsilon) = l^{-1}\ln(1/\varepsilon)$ for some $l \geqslant \tau$ in Remark 5.8 (ii), we have the estimate,

$$\|u(t) - u_{\beta(\varepsilon),\varepsilon}(t)\| \leqslant c_g^2 \max\Big\{\Big(\lambda\Big(\frac{1}{\varepsilon}\Big)\Big)^{-q},\ \varepsilon^{(l-\tau)/l}\Big\}\varepsilon^{t/l}, \quad 0 \leqslant t < \tau, \tag{5.15}$$

where $q > 0$ and $c_g^2 = 2\max\{1, l^q\}e^{\tau\kappa}\max\{1, \varrho_g + \eta_g\}$. The above estimate is the same as in [12], Lemma 1 (ii) under the condition $\|A^q e^{tA}u(t)\| \leqslant M_1$ for all $t \in [0,\tau]$, for some $M_1 > 0$, in place of our Assumption (A) for $g(\lambda) = \lambda^q$ with $q > 0$. It can be seen that our condition implies the above condition in [12].

(iii) Putting $\beta := \beta(\varepsilon) = l^{-1}\ln(1/\varepsilon)$ for some $l \geqslant \tau$ in Remark 5.8 (i) with $p > 0$, we have the estimate

$$\|u(t) - u_{\beta(\varepsilon),\varepsilon}(t)\| \leqslant c_g^1 \max\{\varepsilon^{p/l}, \varepsilon^{(l-\tau)/l}\}\varepsilon^{t/l}, \quad 0 \leqslant t < \tau, \tag{5.16}$$

where $c_g^1 = 2e^{\tau\kappa}\max\{1, \varrho_g + \eta_g\}$. The above estimate is the same as in [12], Lemma 1 (iii) under the condition $\|e^{(\tau+p)A}u(t)\| \leqslant M_2$ for all $t \in [0,\tau]$, for some $M_2 > 0$, in place of our Assumption (A) with $g(\lambda) = e^{p\lambda}$, and this condition is not comparable with those in [12].

5.3.2. Parameter choice when g is piecewise continuous. Let $t \in [0,\tau)$ and $\varepsilon > 0$, and let Φ_t^ε be as in Lemma 5.9. That is,

$$\Phi_t^\varepsilon(\beta) = \varepsilon e^{(\tau-t)\beta} + \frac{e^{-t\beta}}{g(\beta)}, \quad \beta > 0.$$

Since $\Phi_t^\varepsilon(\beta) > 0$ for all $\beta > 0$, $\inf_{\beta>0}\Phi_t^\varepsilon(\beta)$ exists. Let $\{\beta_t^n(\varepsilon)\}$ be such that

$$E(t,\varepsilon,g) := \inf_{\beta>0}\Phi_t^\varepsilon(\beta) = \lim_{n\to\infty}\Phi_t^\varepsilon(\beta_t^n(\varepsilon)).$$

Let $k_\varepsilon^t \in \mathbb{N}$ be such that $\Phi_t^\varepsilon(\beta_t^n(\varepsilon)) \leqslant 2E(t,\varepsilon,g)$ for all $n \geqslant k_\varepsilon^t$. Let

$$n_t^\varepsilon = \min\{k_\varepsilon^t\colon\ \Phi_t^\varepsilon(\beta_t^n(\varepsilon)) \leqslant 2E(t,\varepsilon,g) \quad \forall\, n \geqslant k_\varepsilon^t\}.$$

Now, from Theorem 5.6, we get the estimate

$$\|u(t) - u_{\beta_t^{n_t^\varepsilon}(\varepsilon),\varepsilon}(t)\| \leqslant c_g(t)\ \Phi_t^\varepsilon\big(\beta_t^{n_t^\varepsilon}(\varepsilon)\big) \leqslant 2c_g(t)E(t,\varepsilon,g).$$

6. Conclusion

We defined the *mild solution* for nonlinear nonhomogeneousness FVP for the parabolic problem and considered regularized approximations for it, and carried out error estimates when ϕ is exact as well as inexact. We considered appropriate parameter choice strategies when the final value ϕ is noisy. The results obtained in [8] for linear nonhomogeneous FVP are particular results of this paper for $f(t)$ in place of $f(t, u(t))$ when noise is only in the final value. Also, it extends the work of Tuan (see [16]) for homogeneous linear FVP to nonhomogeneous nonlinear FVP. Further, the paper incldues the considerations in [12] resulting in some of the error estimates special cases consequences of out results.

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