TRACEABILITY IN $\{K_{1,4}, K_{1,4}+e\}$-FREE GRAPHS

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Abstract. A graph G is called $\{H_1, H_2, \ldots, H_k\}$-free if G contains no induced subgraph isomorphic to any graph H_i, $1 \leqslant i \leqslant k$. We define

$$\sigma_k = \min\Bigl\{\sum_{i=1}^{k} d(v_i)\colon \{v_1, \ldots, v_k\} \text{ is an independent set of vertices in } G\Bigr\}.$$

In this paper, we prove that (1) if G is a connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order n and $\sigma_3(G) \geqslant n-1$, then G is traceable, (2) if G is a 2-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order n and $|N(x_1) \cup N(x_2)| + |N(y_1) \cup N(y_2)| \geqslant n-1$ for any two distinct pairs of non-adjacent vertices $\{x_1, x_2\}$, $\{y_1, y_2\}$ of G, then G is traceable, i.e., G has a Hamilton path, where $K_{1,4}+e$ is a graph obtained by joining a pair of non-adjacent vertices in a $K_{1,4}$.

Keywords: $\{K_{1,4}, K_{1,4}+e\}$-free graph; neighborhood union; traceable

MSC 2010: 05C45, 05C38, 05C07

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. For terminology, notation and concepts not defined here, see [2]. Suppose that G is a graph with vertex set $V(G)$ and edge set $E(G)$. For $a \in V(G)$ and subgraphs H and R of G, let $N_R(a)$ and $N_R(H)$ denote the set of neighbors of the vertex a and the subgraph H in R respectively, that is

$$N_R(a) = \{v \in V(R)\colon va \in E(G)\},$$
$$N_R(H) = \Bigl(\bigcup_{u \in V(H)} N_R(u)\Bigr) \setminus V(H).$$

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The numbers $|N_R(a)|$ and $|N_R(H)|$ are called the degrees of the vertex a and the subgraph H in R, denoted as $d_R(a)$ and $d_R(H)$, respectively. If $R = G$, then $N_R(a)$ and $N_R(H)$ are written as $N(a)$ and $N(H)$, and $|N_R(a)|$ and $|N_R(H)|$ are written as $d(a)$ and $d(H)$, respectively. Let $\delta(G)$ denote the minimum degree of G, and let

$$\sigma_k = \min\left\{\sum_{i=1}^{k} d(v_i)\colon\ \{v_1,\ldots,v_k\} \text{ is an independent set of vertices in } G\right\}.$$

If G is a complete graph, we set $NC(G) = |V(G)|-1$, otherwise $NC(G)$ is denoted as

$$NC(G) = \min\{|N(x)\cup N(y)|\colon\ x,z\in V(G) \text{ and } xy\notin E(G)\}.$$

The subgraph induced by S will be denoted by $G[S]$. If $S=\{x_1,x_2,\ldots,x_{|S|}\}$, then $G[S]=G[\{x_1,x_2,\ldots,x_{|S|}\}]$ is also written as $G[x_1,x_2,\ldots,x_{|S|}]$.

Let $P = x_1x_2\ldots x_t$ be a path in G with a given orientation. For $x_i, x_j \in V(P)$, $1 \leqslant i < j \leqslant t$, let x_i^{-l}, x_i^{+l}, $1 \leqslant i-l < i+l \leqslant t$ denote the vertices x_{i-l} and x_{i+l} on P, respectively. We denote by x_iPx_j and $x_i\overline{P}x_j$ the paths $x_ix_{i+1}\ldots x_{j-1}x_j$ and $x_jx_{j-1}\ldots x_{i+1}x_i$, respectively. For convenience, we also denote x_i^{-1} and x_i^{+1} as x_i^- and x_i^+, respectively. Sometimes we denote x_i as x_i^{-0} or x_i^{+0}.

A Hamilton cycle (path) of G is a cycle (path) that contains every vertex of G. A graph is called traceable if it has a Hamilton path. A graph containing a Hamilton cycle is said to be hamiltonian.

A graph G is called $\{H_1,H_2,\ldots,H_k\}$-free if G contains no induced subgraph isomorphic to any graph H_i, $1\leqslant i\leqslant k$. The graph $K_{1,4}$ is a star with 5 vertices, and $K_{1,4}+e$ is obtained from $K_{1,4}$ by adding an edge connecting two non-adjacent vertices. In this paper, we investigate the traceability of $\{K_{1,4},K_{1,4}+e\}$-free graphs.

Li et al. in [3], [4], [5] obtained some results on the hamiltonicity of $\{K_{1,4},K_{1,4}+e\}$-free graphs.

Theorem 1.1 ([5]). *Let G be a 3-connected $\{K_{1,4},K_{1,4}+e\}$-free graph of order $n\geqslant 30$. If $\delta(G)\geqslant (n+5)/5$, then G is hamiltonian.*

Theorem 1.2 ([4]). *Let G be a 2-connected $\{K_{1,4},K_{1,4}+e\}$-free graph of order $n\geqslant 13$. If $\delta(G)\geqslant n/4$, then G is hamiltonian or $G\in\mathcal{F}$, where $\mathcal{F}$ is a family of non-hamiltonian graphs of connectivity 2.*

Theorem 1.3 ([3]). *Suppose that G is a connected $\{K_{1,4},K_{1,4}+e\}$-free graph of order n that is isomorphic to none of graphs G_1 and G_2 shown in Figure 1. If $\delta(G)\geqslant (n-2)/3$, then G is traceable.*

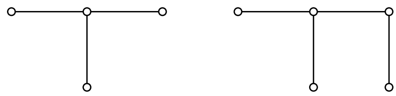

Figure 1. G_1 and G_2

We first get the following result by considering $\sigma_3(G)$ as follows:

Theorem 1.4. *Let G be a connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order n. If $\sigma_3(G) \geqslant n-1$, then G is traceable.*

Remark 1.5. The degree condition of Theorem 1.4 is sharp. The infinite class of graphs $\mathcal{G}_1$ depicted in Figure 2 is not traceable with $\sigma_3(G) = n-2$. Figure 3 gives an infinite class of graphs $\mathcal{G}_2$. Each graph G in $\mathcal{G}_2$ is a connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order $2m$ with $\delta(G) = 2$ and $\sigma_3(G) = n-1$. It is easy to see that G has a Hamilton path. So there is an infinite class of traceable graphs satisfying the condition of Theorem 1.4 but not satisfying the condition of Theorem 1.3.

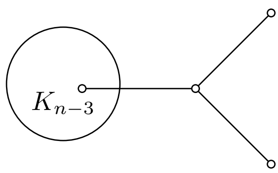

Figure 2. Graphs $\mathcal{G}_1$

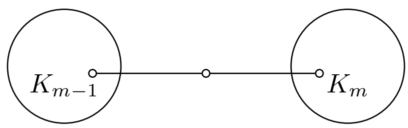

Figure 3. Graphs $\mathcal{G}_2$

On the other hand, the neighborhood union of vertices is another factor that can impact the traceability of a graph. A combination of Theorem 1.4 and the following lemma yields a corollary that can ensure graph's traceability by its neighborhood union.

Lemma 1.6 ([1])**.** *Let G be a graph of order $n \geqslant 3$. Then $\sigma_3(G) \geqslant 3NC(G) - n + 3$.*

Corollary 1.7. *Let G be a connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order n. If $NC(G) \geqslant (2n-4)/3$, then G is traceable.*

For 2-connected graphs, the neighborhood union also can help to judge whether a graph is traceable.

Theorem 1.8 ([6])**.** *If G is a 2-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order n such that $NC(G) \geqslant (n-2)/2$, then G is traceable.*

Our second main result further extends Theorem 1.8 as follows:

Theorem 1.9. *Let G be a 2-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order n. If $|N(x_1) \cup N(x_2)| + |N(y_1) \cup N(y_2)| \geqslant n-1$ for any two distinct pairs of non-adjacent vertices $\{x_1, x_2\}$, $\{y_1, y_2\}$ of G, then G is traceable.*

Remark 1.10. In the graphs of Figure 4, the three vertices of the upper triangle dominate the vertices of the three complete graphs indicated by K_m, K_m and K_{2m-2}, and $\min\{|N(x_1) \cup N(x_2)| + |N(y_1) \cup N(y_2)|\} = n-1$. Obviously, every graph of Figure 4 is connected $\{K_{1,4}, K_{1,4}+e\}$-free, but not traceable. Hence, the infinite class of graphs $\mathcal{G}_3$ depicted in Figure 4 is an evidence showing that the connectivity of Theorem 1.9 cannot be relaxed to 1.

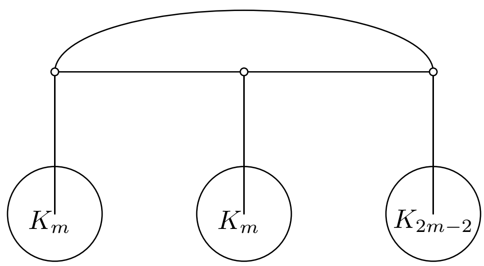

Figure 4. Graphs $\mathcal{G}_3$

Figure 5 shows an infinite class of graphs $\mathcal{G}_4$. The graph G in $\mathcal{G}_4$ is composed of two disjoint complete subgraphs G_1, G_2 of order $2m-1$ and two non-adjacent vertices x, y. The vertex x joins $m-4$ vertices of G_1 and 3 vertices of G_2, the vertex y joins 3 vertices of G_1 and $m-4$ vertices of G_2, and $N(x) \cap N(y) = \emptyset$. Then each graph G in $\mathcal{G}_4$ is a 2-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph of order $4m$, $NC(G) = 2(m-1) < (n-2)/2$, $|N(x) \cup N(y)| + |N(y) \cup N(u_4)| = n-1$, and there are no other two different pairs of vertices such that their sum of neighborhood union is less than $n-1$. It is easy to see that G has a Hamilton path. So there is an infinite class of traceable graphs satisfying the condition of Theorem 1.9 but not satisfying the condition of Theorem 1.8.

Since every claw-free graph is $\{K_{1,4}, K_{1,4}+e\}$-free, we have the following corollary of Theorem 1.9.

Corollary 1.11. *If G is a 2-connected claw-free graph of order n such that $|N(x_1) \cup N(x_2)| + |N(y_1) \cup N(y_2)| \geqslant n-1$ for any two distinct pairs of non-adjacent vertices $\{x_1, x_2\}$, $\{y_1, y_2\}$ of G, then G is traceable.*

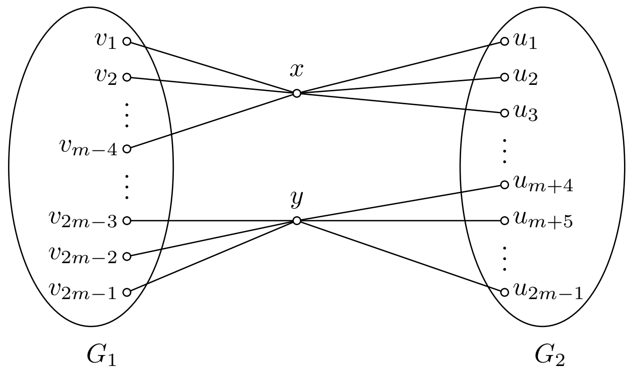

Figure 5. Graphs $\mathcal{G}_4$

2. Proof of Theorem 1.4

Suppose that a graph G satisfies the conditions of Theorem 1.4, but G has no Hamilton path. Let $P = x_1x_2\ldots x_t$ be a longest path in G where $t \leqslant n-1$. Let $R = G - P$, and let H be a component of R. Since G is connected, there is an edge $y_1x_i \in E(G)$, where $y_1 \in V(H)$. Then we have the following observation.

Observation 2.1.
(1) $2 \leqslant i \leqslant t-1$, $N(x_1), N(x_t) \subseteq V(P)$ and $x_{i-1}, x_{i+1} \notin N(y_1)$, $x_1x_t \notin E(G)$.
(2) $x_i, x_{i+1} \notin N(x_1)$, $x_i, x_{i-1} \notin N(x_t)$ for $3 \leqslant i \leqslant t-2$.

Proof. (1) Suppose the opposite. We obtain a path longer than P in all cases easily.

(2) If $x_{i+1} \in N(x_1)$, then the path $x_t\overline{P}x_{i+1}x_1Px_iy_1$ is longer than P, a contradiction. If $x_i \in N(x_1)$, since $y_1x_1, y_1x_{i-1}, y_1x_{i+1}, x_1x_{i+1} \notin E(G)$, if $x_{i-1}x_{i+1} \in E(G)$, the path $x_t\overline{P}x_{i+1}x_{i-1}\overline{P}x_1x_iy_1$ is longer than P, so $x_{i-1}x_{i+1} \notin E(G)$. Then

$$G[x_i, x_1, y_1, x_{i-1}, x_{i+1}] \cong K_{1,4} \quad \text{or} \quad G[x_i, x_1, y_1, x_{i-1}, x_{i+1}] \cong K_{1,4} + e,$$

a contradiction. In a similar way, we can show that $x_i, x_{i-1} \notin N(x_t)$. □

Claim 2.2.
(1) $N_R(x_1) \cup N_R(x_t) \cup N_R(y_1) \subseteq V(R) \setminus \{y_1\}$.
(2) $N_R(x_1) \cap N_R(x_t) = \emptyset$, $N_R(x_1) \cap N_R(y_1) = \emptyset$, $N_R(x_t) \cap N_R(y_1) = \emptyset$, $N_R(x_1) \cap N_R(x_t) \cap N_R(y_1) = \emptyset$.

Proof. (1) Since $N_R(x_1) \cup N_R(x_t) = \emptyset$, $N_R(y_1) \subseteq V(H) \setminus \{y_1\} \subseteq V(R) \setminus \{y_1\}$, so $N_R(x_1) \cup N_R(x_t) \cup N_R(y_1) \subseteq V(R) \setminus \{y_1\}$.

(2) Since $N_R(x_1) \cup N_R(x_t) = \emptyset$, so (2) is correct obviously. □

Set $P_1 = x_1 P x_i$, $P_2 = x_{i+1} P x_t$.

$$N^+_{P_i}(v) = \{u^+ \colon u^+ \in P,\, u \in N_{P_i}(v)\}, \quad N^-_{P_i}(v) = \{u^- \colon u^- \in P,\, u \in N_{P_i}(v)\}.$$

Claim 2.3.

(1) *If $i = 2$, then $N_{P_1}(x_1) = N_{P_1}(y_1) = \{x_2\}$, $N_{P_1}(x_t) = \emptyset$, and $|N_{P_1}(x_1)| + |N_{P_1}(y_1)| + |N_{P_1}(x_t)| = 2 = |V(P_1)|$.*

(2) *If $i \neq 2$, then*

  (a) $N^-_{P_1}(x_1) \cup N_{P_1}(x_t) \cup N_{P_1}(y_1) \subseteq V(P_1) \setminus \{x_{i-1}\}$.

  (b) $N^-_{P_1}(x_1) \cap N_{P_1}(x_t) = \emptyset$, $N^-_{P_1}(x_1) \cap N_{P_1}(y_1) = \emptyset$, $N_{P_1}(x_t) \cap N_{P_1}(y_1) = \emptyset$, $N^-_{P_1}(x_1) \cap N_{P_1}(x_t) \cap N_{P_1}(y_1) = \emptyset$.

P r o o f. The item (1) is an obvious fact, and we start the proof of item (2).

(a) From Observation 2.1 we have $N_{P_1}(x_1) \subseteq V(P_1) \setminus \{x_1, x_i\}$, so $N^-_{P_1}(x_1) \subseteq V(P_1) \setminus \{x_{i-1}, x_i\}$, $N_{P_1}(x_t) \subseteq V(P_1) \setminus \{x_1, x_{i-1}, x_i\}$, $N_{P_1}(y_1) \subseteq V(P_1) \setminus \{x_1, x_{i-1}\}$. Thus $N^-_{P_1}(x_1) \cup N_{P_1}(x_t) \cup N_{P_1}(y_1) \subseteq V(P_1) \setminus \{x_{i-1}\}$.

(b) Suppose that $x_k \in N^-_{P_1}(x_1) \cap N_{P_1}(x_t)$. From (a) we know that $k \neq 1, i-1, i$, hence the path $y_1 x_i P x_t x_k \overline{P} x_1 x_{k+1} P x_{i-1}$ is longer than P, a contradiction. Suppose that $x_k \in N^-_{P_1}(x_1) \cap N_{P_1}(y_1)$. Then it contradicts Observation 2.1, item (2). Suppose that $x_k \in N_{P_1}(x_t) \cap N_{P_1}(y_1)$. Then it contradicts Observation 2.1, item (2). □

Claim 2.4.

(1) *If $i = t-1$, then $N_{P_2}(x_1) = N_{P_2}(y_1) = N_{P_1}(x_t) = \emptyset$, and $|N_{P_2}(x_1)| + |N_{P_2}(y_1)| + |N_{P_2}(x_t)| = 0 = |V(P_2)| - 1$.*

(2) *If $i \neq t-1$, then*

  (a) $N^-_{P_2}(x_1) \cup N_{P_2}(x_t) \cup N_{P_2}(y_1) \subseteq V(P_2) \setminus \{x_t\}$.

  (b) $N^-_{P_2}(x_1) \cap N_{P_2}(x_t) = \emptyset$, $N^-_{P_2}(x_1) \cap N_{P_2}(y_1) = \emptyset$, $N_{P_2}(x_t) \cap N_{P_2}(y_1) = \emptyset$, $N^-_{P_2}(x_1) \cap N_{P_2}(x_t) \cap N_{P_2}(y_1) = \emptyset$.

P r o o f. The item (1) is an obvious fact, and we start the proof of item (2).

(a) From Observation 2.1 we have $N_{P_2}(x_1) \subseteq V(P_2) \setminus \{x_{i+1}, x_t\}$, so $N^-_{P_2}(x_1) \subseteq V(P_2) \setminus \{x_{t-1}, x_t\}$, $N_{P_2}(x_t) \subseteq V(P_2) \setminus \{x_t\}$, $N_{P_2}(y_1) \subseteq V(P_2) \setminus \{x_{i+1}, x_t\}$. Thus $N^-_{P_2}(x_1) \cup N_{P_2}(x_t) \cup N_{P_2}(y_1) \subseteq V(P_2) \setminus \{x_t\}$.

(b) Suppose that $x_k \in N^-_{P_2}(x_1) \cap N_{P_2}(x_t)$. From (a) we know that $k \neq t-1, t$, hence the path $y_1 x_i P x_k x_t \overline{P} x_{k+1} x_1 P x_{i-1}$ is longer than P, a contradiction. Suppose that $x_k \in N^-_{P_2}(x_1) \cap N_{P_2}(y_1)$. Then it contradicts Observation 2.1, item (2). Suppose that $x_k \in N_{P_2}(x_t) \cap N_{P_2}(y_1)$. Then it contradicts Observation 2.1, item (2). □

From Claim 2.2, we have

$$\begin{aligned}(2.1)\qquad |N_R(x_1)| + |N_R(x_t)| + |N_R(y_1)| &\\ = |N_R(x_1) \cup N_R(x_t) \cup N_R(y_1)| + |N_R(x_1) \cap N_R(x_t)| &\\ + |N_R(x_1) \cap N_R(y_1)| + |N_R(x_t) \cap N_R(y_1)| &\\ - |N_R(x_1) \cap N_R(x_t) \cap N_R(y_1)| \leqslant |V(R)| - 1. &\end{aligned}$$

From Claim 2.3, we have:

If $i = 2$, then

$$(2.2)\qquad |N_{P_2}(x_1)| + |N_{P_1}(y_1)| + |N_{P_1}(x_t)| = |V(P_1)|.$$

If $i \neq 2$, then

$$\begin{aligned}(2.3)\quad |N_{P_1}(x_1)| + |N_{P_1}(x_t)| + |N_{P_1}(y_1)| &\\ = |N^-_{P_1}(x_1)| + |N_{P_1}(x_t)| + |N_{P_1}(y_1)| &\\ = |N^-_{P_1}(x_1) \cup N_{P_1}(x_t) \cup N_{P_1}(y_1)| + |N^-_{P_1}(x_1) \cap N_{P_1}(x_t)| &\\ + |N^-_{P_1}(x_1) \cap N_{P_1}(y_1)| + |N_{P_1}(x_t) \cap N_{P_1}(y_1)| &\\ - |N^-_{P_1}(x_1) \cap N_{P_1}(x_t) \cap N_{P_1}(y_1)| \leqslant |V(P_1)| - 1. &\end{aligned}$$

Similarly, from Claim 2.4, we have:

If $i = t - 1$, then

$$(2.4)\qquad |N_{P_2}(x_1)| + |N_{P_2}(y_1)| + |N_{P_2}(x_t)| = |V(P_2)| - 1.$$

If $i \neq t - 1$, then

$$\begin{aligned}(2.5)\quad |N_{P_2}(x_1)| + |N_{P_2}(x_t)| + |N_{P_2}(y_1)| &= |N^-_{P_2}(x_1)| + |N_{P_2}(x_t)| + |N_{P_2}(y_1)| \\ &\leqslant |V(P_2)| - 1.\end{aligned}$$

From inequalities (2.1)–(2.5), we have

$$|N(x_1)| + |N(x_t)| + |N(y_1)| \leqslant n - 2.$$

Since x_1, x_t, y_1 are pairwise non-adjacent, this contradicts the condition $\sigma_3(G) \geqslant n - 1$ of Theorem 1.4. This completes the proof of Theorem 1.4. □

3. Proof of Theorem 1.9

Suppose that a graph G satisfies the conditions of Theorem 1.9, but G has no Hamilton path. Let $P = x_1x_2 \ldots x_t$ be a longest path in G with $t \leqslant n-1$. Let $R = G - P$, and let H be a component of R. Since G is 2-connected, there are $x_i, x_j \in N_p(H)$, $i < j$, such that $N(H) \cap V(x_{i+1}Px_{j-1}) = \emptyset$. Choose a longest path $P' = y_1y_2 \ldots y_l$ in $G[H]$, $l \geqslant 1$, such that $x_iy_1, x_jy_l \in E(G)$. Then we have the following observation.

Observation 3.1.

(1) $i \geqslant 2$, $i+2 \leqslant j \leqslant t-1$ and $N(x_1), N(x_t) \subseteq V(P)$.

(2) For $3 \leqslant i \leqslant t-2$, $x_i, x_{i+1}, x_{j-1}, x_j, x_{j+1}, x_t \notin N(x_1)$ and $x_j, x_{j-1}, x_{i+1}, x_i, x_{i-1}, x_1 \notin N(x_t)$.

(3) $x_{i-1}x_{j-1} \notin E(G)$, $x_{i+1}x_{j+1} \notin E(G)$.

Proof. (1) Suppose the opposite. Then we obtain a path longer than P in all cases easily.

(2) If $x_{i+1} \in N(x_1)$, then the path $x_t\overline{P}x_jy_l\overline{P'}y_1x_i\overline{P}x_1x_{i+1}Px_{j-1}$ is longer than P, a contradiction. If $x_i \in N(x_1)$, since $y_1x_1, y_1x_{i-1}, y_1x_{i+1}, x_1x_{i+1} \notin E(G)$, if $x_{i-1}x_{i+1} \in E(G)$, the path $x_t\overline{P}x_jy_l\overline{P'}y_1x_ix_1Px_{i-1}x_{i+1}Px_{j-1}$ is longer than P, so $x_{i-1}x_{i+1} \notin E(G)$. Then

$$G[x_i, x_1, y_1, x_{i-1}, x_{i+1}] \cong K_{1,4} \quad \text{or} \quad G[x_i, x_1, y_1, x_{i-1}, x_{i+1}] \cong K_{1,4} + e,$$

a contradiction. If $x_{j-1} \in N(x_1)$, then the path $x_t\overline{P}x_jy_l\overline{P'}y_1x_i\overline{P}x_1x_{j-1}\overline{P}x_{i+1}$ is longer than P, a contradiction. If $x_{j+1} \in N(x_1)$, then the path $x_t\overline{P}\ x_{j+1}x_1Px_iy_1 P'y_lx_j\overline{P}x_{i-1}$ is longer than P, a contradiction. If $x_j \in N(x_1)$, then

$$G[x_j, x_1, x_{j-1}, y_l, x_{j+1}] \cong K_{1,4} \quad \text{or} \quad G[x_j, x_1, x_{j-1}, y_l, x_{j+1}] \cong K_{1,4} + e,$$

a contradiction. If $x_t \in N(x_1)$, then the path $x_{i-1}\overline{P}x_1x_t\overline{P}x_iy_1P'y_l$ is longer than P, a contradiction. In a similar way, we can show that $x_j, x_{j-1}, x_{i+1}, x_i, x_{i-1}, x_1 \notin N(x_t)$.

(3) If $x_{j-1}x_{i-1} \in E(G)$, then the path $x_1Px_{i-1}x_{j-1}\overline{P}x_iy_1P'y_lx_jPx_t$ is longer than P, a contradiction. If $x_{i+1}x_{j+1} \in E(G)$, then the path $x_1Px_iy_1P'y_lx_j \overline{P}x_{i+1}x_{j+1}Px_t$ is longer than P, a contradiction. □

Claim 3.2.

(1) $[N_R(x_1) \cup N_R(x_t)] \cup [N_R(y_1) \cup N_R(x_{i+1})] \subseteq V(R) \setminus \{y_1\}$.

(2) $[N_R(x_1) \cup N_R(x_t)] \cap [N_R(y_1) \cup N_R(x_{i+1})] = \emptyset$.

P r o o f. (1) $N_R(x_1) \cup N_R(x_t) = \emptyset$, $N_R(y_1) \subseteq V(H) \setminus \{y_1\}$, $N_R(x_{i+1}) \subseteq V(R) \setminus V(H)$. So, $N_R(y_1) \cup N_R(x_{i+1}) \subseteq V(R) \setminus \{y_1\}$, $[N_R(x_1) \cup N_R(x_t)] \cup [N_R(y_1) \cup N_R(x_{i+1})] \subseteq V(R) \setminus \{y_1\}$.

(2) Since $N_R(x_1) \cup N_R(x_t) = \emptyset$, $[N_R(x_1) \cup N_R(x_t)] \cap [N_R(y_1) \cup N_R(x_{i+1})] = \emptyset$. □

Set $P_1 = x_1 P x_i$, $P_2 = x_{i+1} P x_{j-1}$, $P_3 = x_j P x_t$.

$$N^+_{P_i}(v) = \{u^+ \colon u^+ \in P,\, u \in N_{P_i}(v)\}, \quad N^-_{P_i}(v) = \{u^- \colon u^- \in P,\, u \in N_{P_i}(v)\}.$$

Claim 3.3.

(1) *If $i = 2$, then $N_{P_1}(x_1) \cup N_{P_1}(x_t) = N_{P_1}(y_1) \cup N_{P_1}(x_{i+1}) = \{x_i\}$, and $|N_{P_1}(x_1) \cup N_{P_1}(x_t)| + |N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})| = 2 = |V(P_1)|$.*

(2) *If $i \neq 2$, then*

(a) $[N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)] \cup [N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})] \subseteq V(P_1)$.

(b) $[N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)] \cap [N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})] = \emptyset$.

P r o o f. The item (1) is an obvious fact, and we start the proof of item (2).

(a) From Observation 3.1 we have $N_{P_1}(x_1) \cup N_{P_1}(x_t) \subseteq V(P_1) \setminus \{x_1, x_i\}$, so $N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t) \subseteq V(P_1) \setminus \{x_{i-1}, x_i\}$, $N_{P_1}(y_1) \cup N_{P_1}(x_{i+1}) \subseteq V(P_1) \setminus \{x_1\}$. Thus $[N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)] \cup [N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})] \subseteq V(P_1)$.

(b) Suppose that $x_k \in [N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)] \cap [N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})]$. From (a) we know that $k \neq 1, i-1, i$.

Case 1: $x_1 x_k^+ \in E(G)$. If $y_1 x_k \in E(G)$, then the path $x_t \overline{P} x_j y_l \overline{P}' y_1 x_k \overline{P} x_1 x_k^+ P x_{j-1}$ is longer than P, a contradiction. If $x_{i+1} x_k \in E(G)$, then the path $x_t \overline{P} x_j y_l \overline{P}' y_1 x_i \overline{P} x_k^+ x_1 P x_k x_{i+1} P x_{j-1}$ is longer than P, a contradiction.

Case 2: $x_t x_k^+ \in E(G)$. If $y_1 x_k \in E(G)$, then the path $x_1 P x_k y_1 P' y_l x_j P x_t x_k^+ P x_{j-1}$ is longer than P, a contradiction. If $x_{i+1} x_k \in E(G)$, then the path $x_1 P x_k x_{i+1} P x_j y_l \overline{P}' y_1 x_i \overline{P} x_k^+ x_t \overline{P} x_{j+1}$ is longer than P, a contradiction. □

Claim 3.4.

(1) $[N^+_{P_2}(x_1) \cup N^+_{P_2}(x_t)] \cup [N_{P_2}(y_1) \cup N_{P_2}(x_{i+1})] \subseteq V(P_2) \setminus \{x_{i+1}\}$.

(2) $[N^+_{P_2}(x_1) \cup N^+_{P_2}(x_t)] \cap [N_{P_2}(y_1) \cup N_{P_2}(x_{i+1})] = \emptyset$.

P r o o f. (1) From Observation 3.1 we have

$$N_{P_2}(x_1) \cup N_{P_2}(x_t) \subseteq V(P_2) \setminus \{x_{i+1}, x_{j-1}\},$$

so $N^+_{P_2}(x_1) \cup N^+_{P_2}(x_t) \subseteq V(P_2) \setminus \{x_{i+1}, x_{i+2}\}$, $N_{P_2}(y_1) \cup N_{P_2}(x_{i+1}) \subseteq V(P_2) \setminus \{x_{i+1}\}$. Thus $[N^+_{P_2}(x_1) \cup N^+_{P_2}(x_t)] \cup [N_{P_2}(y_1) \cup N_{P_2}(x_{i+1})] \subseteq V(P_2) \setminus \{x_{i+1}\}$.

(2) Suppose that $x_k \in [N^+_{P_2}(x_1) \cup N^+_{P_2}(x_t)] \cap [N_{P_2}(y_1) \cup N_{P_2}(x_{i+1})]$. We know that $k \neq i+1, i+2, j$. From the assumption that $N(H) \cap V(x_{i+1}Px_{j-1}) = \emptyset$ we have $y_1x_k \notin E(G)$, so $x_{i+1}x_k \in E(G)$.

Case 1: $x_1x_k^- \in E(G)$. Then the path $x_t\overline{P}x_jy_l\overline{P}'y_1x_i\overline{P}x_1x_k^-\overline{P}x_{i+1}x_kPx_{j-1}$ is longer than P, a contradiction.

Case 2: $x_tx_k^- \in E(G)$. Then the path $x_1Px_iy_1P'y_lx_jPx_tx_k^-\overline{P}x_{i+1}x_kPx_{j-1}$ is longer than P, a contradiction. □

Claim 3.5.

(1) *If* $j = t-1$, *then* $N_{P_3}(x_1) \cup N_{P_3}(x_t) = \{x_{t-1}\}$, $N_{P_3}(y_1) \cup N_{P_3}(x_{i+1}) \subseteq \{x_{t-1}\}$, *and* $|N_{P_3}(x_1) \cup N_{P_3}(x_t)| + |N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})| \leqslant 2 = |V(P_3)|$.

(2) *If* $j \neq t-1$, *then*

(a) $[N^+_{P_3}(x_1) \cup N^+_{P_3}(x_t)] \cup [N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})] \subseteq V(P_3) \setminus \{x_{j+1}\}$.

(b) $[N^+_{P_3}(x_1) \cup N^+_{P_3}(x_t)] \cap [N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})] = \emptyset$.

P r o o f. The item (1) is an obvious fact, and we start the proof of item (2).

(a) From Observation 3.1 we have $N_{P_3}(x_1) \cup N_{P_3}(x_t) \subseteq V(P_3) \setminus \{x_j, x_t\}$, so $N^+_{P_3}(x_1) \cup N^+_{P_3}(x_t) \subseteq V(P_3) \setminus \{x_j, x_{j+1}\}$, $N_{P_3}(y_1) \cup N_{P_3}(x_{i+1}) \subseteq V(P_3) \setminus \{x_{j+1}, x_t\}$. Thus $[N^+_{P_3}(x_1) \cup N^+_{P_3}(x_t)] \cup [N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})] \subseteq V(P_3) \setminus \{x_{j+1}\}$.

(b) Suppose that $x_k \in [N^+_{P_3}(x_1) \cup N^+_{P_3}(x_t)] \cap [N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})]$. From (a) we know that $k \neq j, j+1, t$.

Case 1: $x_1x_k^- \in E(G)$. If $y_1x_k \in E(G)$, then the path $x_t\overline{P}x_ky_1x_i\overline{P}x_1x_k^-\overline{P}x_{i+1}$ is longer than P, a contradiction. If $x_{i+1}x_k \in E(G)$, then the path $x_t\overline{P}x_kx_{i+1}Px_jy_l$ $\overline{P}'y_1x_i\overline{P}x_1x_k^-\overline{P}x_{j+1}$ is longer than P, a contradiction.

Case 2: $x_tx_k^- \in E(G)$. If $y_1x_k \in E(G)$, then the path $x_1Px_iy_1x_kPx_tx_k^-\overline{P}x_{i+1}$ is longer than P, a contradiction. If $x_{i+1}x_k \in E(G)$, then the path $x_1Px_iy_1P'y_lx_j$ $\overline{P}x_{i+1}x_kPx_tx_k^-\overline{P}x_{j+1}$ is longer than P, a contradiction. □

From Claim 3.2, we have

$$
\begin{aligned}
(3.1)\qquad & |N_R(x_1) \cup N_R(x_t)| + |N_R(y_1) \cup N_R(x_{i+1})| \\
&= |[N_R(x_1) \cup N_R(x_t)] \cup [N_R(y_1) \cup N_R(x_{i+1})]| \\
&\quad + |[N_R(x_1) \cup N_R(x_t)] \cap [N_R(y_1) \cup N_R(x_{i+1})]| \leqslant |V(R)| - 1.
\end{aligned}
$$

From Claim 3.3, we have:

If $i = 2$, then

$$
(3.2)\qquad |N_{P_1}(x_1) \cup N_{P_1}(x_t)| + |N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})| = |V(P_1)|.
$$

If $i \neq 2$, then

$$(3.3)\qquad \begin{aligned} &|N_{P_1}(x_1) \cup N_{P_1}(x_t)| + |N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})| \\ &\quad = |N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)| + |N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})| \\ &\quad = |[N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)] \cup [N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})]| \\ &\qquad + |[N^-_{P_1}(x_1) \cup N^-_{P_1}(x_t)] \cap [N_{P_1}(y_1) \cup N_{P_1}(x_{i+1})]| \leqslant |V(P_1)|. \end{aligned}$$

From Claim 3.4 , we have

$$(3.4)\qquad \begin{aligned} &|N_{P_2}(x_1) \cup N_{P_2}(x_t)| + |N_{P_2}(y_1) \cup N_{P_2}(x_{i+1})| \\ &\quad = |N^+_{P_2}(x_1) \cup N^+_{P_2}(x_t)| + |N_{P_2}(y_1) \cup N_{P_2}(x_{i+1})| \leqslant |V(P_2)| - 1. \end{aligned}$$

From Claim 3.5 , we have:
If $j = t - 1$, then

$$(3.5)\qquad |N_{P_3}(x_1) \cup N_{P_3}(x_t)| + |N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})| \leqslant |V(P_3)|.$$

If $j \neq t - 1$, then

$$(3.6)\qquad \begin{aligned} &|N_{P_3}(x_1) \cup N_{P_3}(x_t)| + |N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})| \\ &\quad = |N^+_{P_3}(x_1) \cup N^+_{P_3}(x_t)| + |N_{P_3}(y_1) \cup N_{P_3}(x_{i+1})| \leqslant |V(P_3)| - 1. \end{aligned}$$

From inequalities (3.1)–(3.6), we have

$$|N(x_1) \cup N(x_t)| + |N(y_1) \cup N(x_{i+1})| \leqslant n - 2.$$

This contradicts the condition $|N(x_1) \cup N(x_2)| + |N(y_1) \cup N(y_2)| \geqslant n - 1$ of Theorem 1.9. The proof of Theorem 1.9 is completed.

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