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A REMARK ON WEAK MCSHANE INTEGRAL

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Abstract. We characterize the weak McShane integrability of a vector-valued function on a finite Radon measure space by means of only finite McShane partitions. We also obtain a similar characterization for the Fremlin generalized McShane integral.

Keywords: weak McShane integral; finite McShane partition; Radon measure space

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1. Introduction

The purpose of this note is to develop the theory of weak McShane integral, which was initiated by Saadoune and Sayyad [6]. Our work here is motivated by two things. The weak McShane integrability of a vector-valued function on a quasi-Radon measure space is defined in terms of *infinite* McShane partitions. However, it is possible to characterize the integrability by means of *finite* McShane partitions if the measure space is compact and finite; see Proposition 2.2.

In [2] Faure and Mawhin defined the Henstock-Kurzweil integral of a vector-valued function defined on an unbounded interval of $\mathbb{R}^m$ by using finite partitions; see also [7], Section 3.7. These two things drew our interest in seeking a possibility to characterize the weak McShane integrability of a function on a non-compact space in terms of only finite McShane partitions. Our main result, Theorem 3.1, shows that this is possible in the case where the space is a finite Radon measure space. A similar assertion is also valid for the Fremlin generalized McShane integral [4]; see Theorem 3.3.

Let us see the meaning of our main result from a different point of view. Proposition 2.2 indicates that the weak McShane integral of a function on a compact finite quasi-Radon measure space is a variant of the Riemann integral in the sense that (2.1) involves only finite McShane sums. This, combined with Proposition 2.3

and our Theorem 3.1, indicates that the weak McShane integral of a function on a finite Radon measure space can be viewed as a variant of the improper integral.

2. Preliminaries

In this section several notions and terminologies are recalled from [6] for the reader's sake. Let $(S, \mathcal{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space which is outer regular (the definition of a quasi-Radon measure space is supplied in Appendix). By a *generalized McShane partition* of S we mean a sequence $\{(E_i, t_i)\}_{i=1}^{\infty}$ such that $\{E_i\}_{i=1}^{\infty}$ is a disjoint family of measurable sets of finite measure, $t_i \in S$ for each i, and $\mu\Bigl(S \setminus \bigcup\limits_{i=1}^{\infty} E_i\Bigr) = 0$. A function $\Delta\colon S \to \mathcal{T}$ is called a *gauge* if $s \in \Delta(s)$ for every $s \in S$. We say that a generalized McShane partition $\{(E_i, t_i)\}_{i=1}^{\infty}$ is *subordinate* to a gauge $\Delta\colon S \to \mathcal{T}$ if $E_i \subset \Delta(t_i)$ for every $i \in \mathbb{N}$. Given a gauge Δ, we denote by $\Pi_\infty(\Delta)$ the set of all generalized McShane partitions of S subordinate to Δ. A sequence $\{\mathcal{P}_\infty^m\}_{m=1}^{\infty}$ of generalized McShane partitions of S is said to be *adapted* to a sequence of gauges $\{\Delta_m\}_{m=1}^{\infty}$ if $\mathcal{P}_\infty^m$ is subordinate to Δ_m for each m. Let X be a Banach space. For a function $f\colon S \to X$ and a generalized McShane partition $\mathcal{P} = \{(E_i, t_i)\}_{i=1}^{\infty}$ of S, we set

$$\sigma_n(f, \mathcal{P}) = \sum_{i=1}^{n} \mu(E_i) f(t_i).$$

Definition 2.1 ([6], Definition 3.2)**.** A function $f\colon S \to X$ is said to be *weakly McShane integrable* (WM-integrable for short) on S, with weak McShane integral w, if there is a sequence of gauges $\Delta_m\colon S \to \mathcal{T}$, $m = 1, 2, \ldots$, such that, for every $x^* \in X^*$ and for every sequence $\{\mathcal{P}_\infty^m\}_{m=1}^{\infty}$ of generalized McShane partitions of S adapted to $\{\Delta_m\}_{m=1}^{\infty}$,

$$\lim_{m\to\infty} \limsup_{l\to\infty} |\langle x^*, \sigma_l(f, \mathcal{P}_\infty^m) - w\rangle| = 0.$$

We set $w = (\mathrm{WM}) \int_S f \,\mathrm{d}\mu$ in this case.

A function $f\colon S \to X$ is said to be WM-integrable on a measurable subset E of S if $\chi_E f$ is WM-integrable on S. Here χ_E stands for the characteristic function of E. We say that f is WM-integrable on Σ if it is WM-integrable on every measurable subset of S. It is proved in [6], Proposition 3.2 that a function $f\colon S \to X$ is WM-integrable on S, with weak McShane integral w, if and only if there exists a sequence of gauges $\Delta_m\colon S \to \mathcal{T}$, $m = 1, 2, \ldots$, such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{l\to\infty} |\langle x^*, \sigma_l(f, \mathcal{P}_\infty) - w\rangle| = 0 \quad \forall\, x^* \in X^*.$$

By a *finite partial McShane partition* of S we mean a finite sequence $\{(E_i, t_i)\}_{i=1}^{p}$ such that $E_1, E_2, \ldots, E_p$ is a disjoint family of measurable sets of finite measure and $t_i \in S$ for each $i \leqslant p$; it is said to be *subordinate* to a gauge $\Delta\colon S \to \mathcal{T}$ if $E_i \subset \Delta(t_i)$ for each $i \leqslant p$. A finite partial McShane partition $\{(E_i, t_i)\}_{i=1}^{p}$ of S is called a *finite strict generalized McShane partition* of S if $\bigcup\limits_{i=1}^{p} E_i = S$. It is also useful to recall the following result:

Proposition 2.2 ([6], Proposition 3.3). *Let $(S, \mathcal{T}, \Sigma, \mu)$ be a compact finite quasi-Radon measure space, X a Banach space. Let $f : S \to X$ be a function. Then f is* WM*-integrable, with weak McShane integral w, if and only if there is a sequence of gauges $\Delta_m\colon S \to \mathcal{T}$, $m = 1, 2, \ldots$, such that*

$$\lim_{m\to\infty} \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in\Pi_f(\Delta_m)} \left|\left\langle x^*, \sum_{i=1}^{p} \mu(E_i) f(t_i) - w \right\rangle\right| = 0 \quad \forall\, x^* \in X^*. \tag{2.1}$$

Here, $\Pi_f(\Delta_m)$ stands for the set of all finite strict generalized McShane partitions of S subordinate to Δ_m.

It is worth mentioning that this assertion is an analogy to [4], 1E Proposition. We further recall the following implication.

Proposition 2.3 ([6], Proposition 3.4). *Let $(S, \mathcal{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and let $E \in \Sigma$. Let X be a Banach space, $f\colon S \to X$ a function. Let $\mathcal{T}_E$ stand for the relativization of $\mathcal{T}$ to E. Set $\Sigma_E = \{F \in \Sigma\colon F \subset E\}$ and $\mu_E = \mu|_{\Sigma_E}$. Then, f is* WM*-integrable on E if and only if $f|_E$ is* WM*-integrable on E with respect to the quasi-Radon measure space $(E, \mathcal{T}_E, \Sigma_E, \mu_E)$, and the two integrals are equal.*

We also need additional terminologies. Let $(S, \mathcal{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space which is outer regular. We call a generalized McShane partition $\{(E_i, t_i)\}_{i=1}^{\infty}$ of S a *strict generalized McShane partition* of S if $\bigcup\limits_{i=1}^{\infty} E_i = S$. Let X be a Banach space. For a Pettis integrable function $f\colon S \to X$ and for $E \in \Sigma$, there is a unique $w_E \in X$ such that $\int_E \langle x^*, f\rangle \,\mathrm{d}\mu = \langle x^*, w_E\rangle$ for every $x^* \in X^*$; we write $w_E = (\mathrm{Pe}) \int_E f \,\mathrm{d}\mu$. The map $\Sigma \ni E \mapsto w_E \in X$ is called the indefinite Pettis integral of f. We consult [1] or [5] for the notions and terminologies of the standard measure theory. We refer to [8] for the theory of Pettis integral.

3. Weak McShane integrability in terms of finite McShane partitions

By a *Radon measure space* we mean a quasi-Radon measure space which is inner regular for the compact sets. The following theorem characterizes the WM-integrability of a function on a Radon measure space in terms of finite McShane partitions.

Theorem 3.1. *Let $(S,\mathcal{T},\Sigma,\mu)$ be a finite Radon measure space, X a Banach space. Let $f\colon S\to X$ be a function. Then the conditions* (a) *and* (b) *below are equivalent:*

(a) *f is* WM-*integrable on Σ.*

(b) *For each $E\in\Sigma$, there exist a sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets, a sequence of sequences of gauges $\{\Delta_j^n\}_{j=1}^{\infty}$ $(n=1,2,\dots)$ on S and a vector $w_E\in X$ such that the following* (i)–(iii) *are valid:*

  (i) *$K_n\subset E$ $(n=1,2,\dots)$;*

  (ii) *f is* WM-*integrable on each K_n and*

$$\lim_{n\to\infty}\left\|(\mathrm{WM})\int_{K_n} f\,\mathrm{d}\mu-w_E\right\|=0;$$

  (iii) *for each $x^*\in X^*$,*

$$\lim_{n\to\infty}\limsup_{j\to\infty}\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_j^n)}\left|\left\langle x^*,\sum_{i=1}^{p}\mu(E_i)(\chi_{E\setminus K_n}f)(t_i)\right\rangle\right|=0.$$

  Here, $P\Pi_f(\Delta_j^n)$ stands for the set of all finite partial McShane partitions of S subordinate to Δ_j^n.

Proof. First, we prove that (b) yields (a). Let E, $\{K_n\}_{n=1}^{\infty}$, $\{\Delta_j^n\}_{j=1}^{\infty}$ $(n=1,2,\dots)$ and w_E be as in (b). It follows from the first condition of (ii) that, for each n, there exists a sequence $\{\delta_j^n\}_{j=1}^{\infty}$ of gauges on S such that, for each $x^*\in X^*$,

$$\lim_{j\to\infty}\sup_{\mathcal{P}_\infty\in\Pi_\infty(\delta_j^n)}\limsup_{l\to\infty}\left|\left\langle x^*,\sigma_l(\chi_{K_n}f,\mathcal{P}_\infty)-(\mathrm{WM})\int_{K_n} f\,\mathrm{d}\mu\right\rangle\right|=0.$$

Define a sequence of gauges $\tilde{\delta}_j\colon S\to\mathcal{T}$ $(j=1,2,\dots)$ by

$$\tilde{\delta}_j(t)=\bigcap_{n=1}^{j}(\Delta_j^n(t)\cap\delta_j^n(t)).$$

We arbitrarily pick a sequence $\{(E_i^j, t_i^j)\}_{i=1}^\infty$ ($j = 1, 2, \ldots$) of generalized McShane partitions of S adapted to $\{\tilde{\delta}_j\}_{j=1}^\infty$. Pick also an $x^* \in X^*$, arbitrarily. Let

$$M(n,j) = \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_j^n)} \left| \left\langle x^*, \sum_{i=1}^p \mu(E_i)(\chi_{E\setminus K_n} f)(t_i) \right\rangle \right|$$

and

$$L(n,j) = \sup_{\mathcal{P}_\infty\in\Pi_\infty(\delta_j^n)} \limsup_{l\to\infty} \left| \left\langle x^*, \sigma_l(\chi_{K_n} f, \mathcal{P}_\infty) - (\mathrm{WM})\int_{K_n} f\,\mathrm{d}\mu \right\rangle \right|.$$

Pick an $\varepsilon > 0$, arbitrarily. It follows from (iii) that there is an $N_1 \in \mathbb{N}$ such that for any integer $n \geqslant N_1$, $\limsup_{j\to\infty} M(n,j) < \varepsilon$. By the last condition of (ii), we see that there is an $N_2 \in \mathbb{N}$ such that for any integer $n \geqslant N_2$, $\|(\mathrm{WM})\int_{K_n} f\,\mathrm{d}\mu - w_E\| < \varepsilon$. Put $N = \max\{N_1, N_2\}$. Note that

$$\begin{aligned}
&\left| \left\langle x^*, \sum_{i=1}^l \mu(E_i^j)(\chi_E f)(t_i^j) - w_E \right\rangle \right| \\
&\quad\leqslant \left| \left\langle x^*, \sum_{i=1}^l \mu(E_i^j)(\chi_{K_N} f)(t_i^j) - (\mathrm{WM})\int_{K_N} f\,\mathrm{d}\mu \right\rangle \right| \\
&\qquad + \left| \left\langle x^*, (\mathrm{WM})\int_{K_N} f\,\mathrm{d}\mu - w_E \right\rangle \right| + \left| \left\langle x^*, \sum_{i=1}^l \mu(E_i^j)(\chi_{E\setminus K_N} f)(t_i^j) \right\rangle \right|.
\end{aligned}$$

We see that there exists a $J \in \mathbb{N}$ such that, for any integer $j \geqslant J$, $M(N,j) < \varepsilon$ and $L(N,j) < \varepsilon$. Pick arbitrarily an integer j satisfying $j \geqslant \max\{J, N\}$. Since $j \geqslant N$, it holds that $\tilde{\delta}_j(t) \subset \Delta_j^N(t)$ and $\tilde{\delta}_j(t) \subset \delta_j^N(t)$. Thus, the generalized McShane partition $\{(E_i^j, t_i^j)\}_{i=1}^\infty$ is subordinate to both Δ_j^N and δ_j^N. Hence,

$$\begin{aligned}
&\limsup_{l\to\infty} \left| \left\langle x^*, \sum_{i=1}^l \mu(E_i^j)(\chi_E f)(t_i^j) - w_E \right\rangle \right| \\
&\qquad\qquad \leqslant L(N,j) + \|x^*\|\varepsilon + M(N,j) < (2 + \|x^*\|)\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{j\to\infty} \limsup_{l\to\infty} \left| \left\langle x^*, \sum_{i=1}^l \mu(E_i^j)(\chi_E f)(t_i^j) - w_E \right\rangle \right| = 0,$$

that is, f is WM-integrable on E and $(\mathrm{WM})\int_E f\,\mathrm{d}\mu = w_E$.

Next, we prove the converse; suppose (a). It follows from [6], Corollary 3.3 that f is Pettis integrable. So, the function $\nu\colon \Sigma \to X$ defined by

$$\nu(G) = (\mathrm{WM})\int_G f \,\mathrm{d}\mu \quad \Bigl(= (\mathrm{Pe})\int_G f \,\mathrm{d}\mu\Bigr), \quad G \in \Sigma,$$

is a countably additive vector measure that is absolutely continuous with respect to μ (see [4], Subsection 2A). Fix $E \in \Sigma$. Since $(S, \mathcal{T}, \Sigma, \nu)$ is a finite Radon measure space, there is an increasing sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets which satisfies (i) and $\mu(E \setminus K_n) \to 0$ (as $n \to \infty$). Pick an $\varepsilon > 0$, arbitrarily. There is an $m \in \mathbb{N}$ such that, for any $A \in \Sigma$ satisfying $A \subset E \setminus K_m$,

$$\Bigl\|(\mathrm{WM})\int_A f \,\mathrm{d}\mu\Bigr\| < \varepsilon. \tag{3.1}$$

Particularly, if we put $w_E = (\mathrm{WM})\int_E f \,\mathrm{d}\mu$, then (ii) holds. For each n, there exists a sequence of gauges $\Delta_j^n\colon S \to \mathcal{T}$ $(j = 1, 2, \ldots)$ such that, for any $x^* \in X^*$,

$$\lim_{j\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_j^n)} \limsup_{l\to\infty} \Bigl|\Bigl\langle x^*, \sigma_l(\chi_{E\setminus K_n} f, \mathcal{P}_\infty) - (\mathrm{WM})\int_{E\setminus K_n} f \,\mathrm{d}\mu \Bigr\rangle\Bigr| = 0,$$

because f is WM-integrable on $E\setminus K_n$. This, combined with the weak Saks-Henstock lemma [6], Lemma 3.2, implies that, for each $n \in \mathbb{N}$ and $x^* \in X^*$,

$$\begin{aligned}\lim_{j\to\infty} \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p} \in P\Pi_f(\Delta_j^n)} \Bigl|\Bigl\langle x^*, \sum_{i=1}^{p} \mu(E_i)(\chi_{E\setminus K_n} f)(t_i) \\ - (\mathrm{WM})\int_{\bigcup_{i=1}^{p} E_i} \chi_{E\setminus K_n} f \,\mathrm{d}\mu \Bigr\rangle\Bigr| = 0.\end{aligned} \tag{3.2}$$

We fix $x^* \in X^*$. Pick arbitrarily an integer n satisfying $n \geqslant m$. Since

$$\begin{aligned}&\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p} \in P\Pi_f(\Delta_j^n)} \Bigl|\Bigl\langle x^*, \sum_{i=1}^{p} \mu(E_i)(\chi_{E\setminus K_n} f)(t_i) \Bigr\rangle\Bigr| \\ &\leqslant \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p} \in P\Pi_f(\Delta_j^n)} \Bigl|\Bigl\langle x^*, \sum_{i=1}^{p} \mu(E_i)(\chi_{E\setminus K_n} f)(t_i) - (\mathrm{WM})\int_{\bigcup_{i=1}^{p} E_i} \chi_{E\setminus K_n} f \,\mathrm{d}\mu \Bigr\rangle\Bigr| \\ &\quad + \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p} \in P\Pi_f(\Delta_j^n)} \Bigl|\Bigl\langle x^*, (\mathrm{WM})\int_{(E\setminus K_n)\cap \bigcup_{i=1}^{p} E_i} f \,\mathrm{d}\mu \Bigr\rangle\Bigr|\end{aligned}$$

and since $(E\backslash K_n)\cap\bigcup_{i=1}^{p} E_i \subset E\backslash K_m$ for $\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_j^n)$, we see from (3.1) and (3.2) that

$$\limsup_{j\to\infty}\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_j^n)}\Bigl|\Bigl\langle x^*,\sum_{i=1}^{p}\mu(E_i)(\chi_{E\backslash K_n}f)(t_i)\Bigr\rangle\Bigr|\leqslant\varepsilon\|x^*\|.$$

Thereby, (iii) holds. □

We include the following proposition in this section, which belongs to remarks. Recall that a topological space S is said to have the Lindelöf property if every open cover of S admits a countable subcover.

Proposition 3.2. *Let $(S,\mathcal{T},\Sigma,\mu)$ be a σ-finite outer regular quasi-Radon measure space. Suppose that $(S,\mathcal{T})$ possesses the Lindelöf property. Let X be a Banach space, $\varphi\colon S\to X$ a function. In order that φ be WM-integrable, with WM-integral w, it is necessary and sufficient that there exists a sequence of gauges $\Delta_j\colon S\to\mathcal{T}$, $j=1,2,\ldots$, such that, for every sequence $\{(E_i^j,t_i^j)\}_{i=1}^{\infty}$, $j=1,2,\ldots$, of strict generalized McShane partitions of S adapted to $\{\Delta_j\}_{j=1}^{\infty}$ and for every $x^*\in X^*$,*

$$\lim_{j\to\infty}\limsup_{l\to\infty}\Bigl|\Bigl\langle x^*,\sum_{i=1}^{l}\mu(E_i^j)\varphi(t_i^j)-w\Bigr\rangle\Bigr|=0.$$

P r o o f. The necessity is clear. We deduce the sufficiency by the following observation. Let $\Delta\colon S\to\mathcal{T}$ be a gauge and let $\{(E_i,t_i)\}_{i=1}^{\infty}$ be a generalized McShane partition of S subordinate to Δ. Then, $N\equiv S\setminus\bigcup_{i=1}^{\infty}E_i$ is a null set. Because $(S,\mathcal{T})$ possesses the Lindelöf property and $\{\Delta(t)\}_{t\in S}$ is an open cover of S, there is a sequence $\{\xi_i\}_{i=1}^{\infty}$ in S for which $S=\bigcup_{i=1}^{\infty}\Delta(\xi_i)$. Put $N_i=N\cap\Delta(\xi_i)$. We define a sequence $\{\widetilde{N}_i\}_{i=1}^{\infty}$ of subsets of S by $\widetilde{N}_1=N_1$ and $\widetilde{N}_i=N_i\setminus\bigcup_{j=1}^{i-1}N_j$ $(i>1)$. For each i, $\widetilde{N}_i$ is a null set, and $\widetilde{N}_i\subset\Delta(\xi_i)$. We define a sequence $\{F_i\}_{i=1}^{\infty}$ of subsets of S and a sequence $\{\eta_i\}_{i=1}^{\infty}$ in S by

$$F_{2i-1}=E_i,\quad F_{2i}=\widetilde{N}_i,\quad \eta_{2i-1}=t_i,\quad \eta_{2i}=\xi_i.$$

Then, $\{(F_i,\eta_i)\}_{i=1}^{\infty}$ is a strict generalized McShane partition of S subordinate to Δ, and it holds for every $l\in\mathbb{N}$ that

$$\sum_{i=1}^{l}\mu(E_i)\varphi(t_i)=\sum_{i=1}^{2l-1}\mu(F_i)\varphi(\eta_i).$$

We obtain the conclusion at once from this fact. □

We shall see that a characterization similar to Theorem 3.1 is valid also for the Fremlin generalized McShane integral (for the definition of this integral, see [4], 1A Definitions). To this end we introduce a terminology. Let $(S,\mathcal{T},\Sigma,\mu)$ be a σ-finite outer regular quasi-Radon measure space and let $E\in\Sigma$. Let X be a Banach space, $f\colon S\to X$ a function. We say that f is McShane integrable (M-integrable for short) on E if $\chi_E f$ is McShane integrable. In this case we designate by (M) $\int_E f\,\mathrm{d}\mu$ the McShane integral of $\chi_E f$. We have the following assertion.

Theorem 3.3. *Let $(S,\mathcal{T},\Sigma,\mu)$ be a finite Radon measure space and let $E\in\Sigma$. Let X be a Banach space, $f\colon S\to X$ a function. Then, the following conditions* (a) *and* (b) *are equivalent:*

(a) *f is* M-*integrable on E.*

(b) *There is a $w_E\in X$ such that, for any $\varepsilon>0$, there exist a compact set K and a gauge $\Delta\colon S\to\mathcal{T}$ for which the following* (i)–(iii) *are valid:*
  (i) *$K\subset E$;*
  (ii) *f is M-integrable on K and $\|(\mathrm{M})\int_K f\,\mathrm{d}\mu-w_E\|<\varepsilon$;*
  (iii)
$$\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta)}\left\|\sum_{i=1}^{p}\mu(E_i)(\chi_{E\setminus K}f)(t_i)\right\|<\varepsilon.$$

We omit the proof, as it is similar to that of Theorem 3.1.

4. Appendix

We recall from [3], Subsection 72 the definition of a quasi-Radon measure space for the sake of convenience; several auxiliary notions are also recalled.

Definition A.1. Let (S,Σ,μ) be a measure space.

(i) (S,Σ,μ) is said to be *finite* (or *totally finite*) if $\mu(S)<\infty$. (S,Σ,μ) is *semi-finite* if, whenever $E\in\Sigma$ and $\mu(E)=\infty$, there is an $F\subset E$ such that $F\in\Sigma$ and $0<\mu(F)<\infty$.

(ii) Set $\Sigma^f=\{F\in\Sigma\colon \mu(F)<\infty\}$. (S,Σ,μ) is said to be *locally determined* if it is semi-finite and, for any set $E\subset S$,
$$E\cap F\in\Sigma \text{ for every } F\in\Sigma^f\Rightarrow E\in\Sigma.$$

Definition A.2. Let $\mathcal{G}$ be a family of sets. We say that $\mathcal{G}$ is *directed upwards* if for every pair A, B of elements of $\mathcal{G}$ there is a $C\in\mathcal{G}$ such that $A\subset C$ and $B\subset C$.

Definition A.3. A *quasi-Radon measure space* is a quadruple $(S,\mathcal{T},\Sigma,\mu)$, where (S,Σ,μ) is a measure space and $\mathcal{T}$ is a topology on S such that:

(i) (S,Σ,μ) is complete and locally determined;

(ii) $\mathcal{T}\subset\Sigma$;

(iii) if $E\in\Sigma$ and $\mu(E)>0$, then there is a $G\in\mathcal{T}$ such that $\mu(G)<\infty$ and $\mu(E\cap G)>0$;

(iv) $\mu(E)=\sup\{\mu(F)\colon F\subset E,\ F \text{ closed}\}$ for every $E\in\Sigma$;

(v) if $\mathcal{G}$ is a nonempty subfamily of $\mathcal{T}$ which is directed upwards, then

$$\mu\Big(\bigcup\mathcal{G}\Big)=\sup\{\mu(G)\colon G\in\mathcal{G}\}.$$

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