

ON σ -PERMUTABLY EMBEDDED SUBGROUPS
OF FINITE GROUPS

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Abstract. Let $\sigma = \{\sigma_i: i \in I\}$ be some partition of the set of all primes \mathbb{P} , G be a finite group and $\sigma(G) = \{\sigma_i: \sigma_i \cap \pi(G) \neq \emptyset\}$. A set \mathcal{H} of subgroups of G is said to be a complete Hall σ -set of G if every non-identity member of \mathcal{H} is a Hall σ_i -subgroup of G and \mathcal{H} contains exactly one Hall σ_i -subgroup of G for every $\sigma_i \in \sigma(G)$. G is said to be σ -full if G possesses a complete Hall σ -set. A subgroup H of G is σ -permutable in G if G possesses a complete Hall σ -set \mathcal{H} such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. A subgroup H of G is σ -permutably embedded in G if H is σ -full and for every $\sigma_i \in \sigma(H)$, every Hall σ_i -subgroup of H is also a Hall σ_i -subgroup of some σ -permutable subgroup of G .

By using the σ -permutably embedded subgroups, we establish some new criteria for a group G to be soluble and supersoluble, and also give the conditions under which a normal subgroup of G is hypercyclically embedded. Some known results are generalized.

Keywords: finite group; σ -subnormal subgroup; σ -permutably embedded subgroup; σ -soluble group; supersoluble group

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1. INTRODUCTION

Throughout this paper, all groups are finite and G always denotes a finite group. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing n ; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G .

In what follows, $\sigma = \{\sigma_i: i \in I\}$ is some partition of all primes \mathbb{P} , that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write $\sigma(G) = \{\sigma_i: \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [20], [29], [30], G is said to be σ -primary if $|\sigma(G)| \leq 1$; σ -soluble if every chief factor of G is σ -primary. A set \mathcal{H} of subgroups of G is said to be a *complete*

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Hall σ -set of G if every non-identity member of \mathcal{H} is a Hall σ_i -subgroup of G for some σ_i and \mathcal{H} contains exactly one Hall σ_i -subgroup for every $\sigma_i \in \sigma(G)$. G is said to be *σ -full* if G possesses a complete Hall σ -set; *σ -nilpotent* if G has a complete Hall σ -set $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ such that $G = H_1 \times H_2 \times \dots \times H_t$. Clearly, a σ -nilpotent group is σ -soluble. G is said to be a *σ -full group of Sylow type* if every subgroup of G is a D_{σ_i} -group for all $\sigma_i \in \sigma(G)$. A subgroup H of G is said to be *σ -subnormal* in G if there exists a subgroup chain $H = H_0 \leq H_1 \leq \dots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ -primary for all $i = 1, \dots, n$.

Let \mathcal{L} be some nonempty set of subgroups of G and $K \leq G$. Following [29], a subgroup H of G is called *\mathcal{L} -permutable* if $HA = AH$ for all $A \in \mathcal{L}$; *\mathcal{L}^K -permutable* if $HA^x = A^xH$ for all $A \in \mathcal{L}$ and all $x \in K$. In particular, a subgroup H of G is *σ -permutable* in G if G has a complete Hall σ -set \mathcal{H} such that H is \mathcal{H}^G -permutable.

It is well known that embedded subgroups play an important role in the theory of finite groups. For example, a subgroup H of G is said to be *normally embedded* in G (see [12], page 250) if for each prime p dividing the order of H , a Sylow p -subgroup of H is also a Sylow p -subgroup of some normal subgroup of G . A subgroup H of G is said to be *permutably embedded* in G (see [5]) if for each prime p dividing the order of H , a Sylow p -subgroup of H is also a Sylow p -subgroup of some permutable subgroup of G . (Note that a subgroup H of G is said to be *permutable* in G if $HS = SH$ for any subgroup S of G .) A subgroup H of G is said to be *s -permutably embedded* in G (see [8]) if for each prime p dividing the order of H , a Sylow p -subgroup of H is also a Sylow p -subgroup of some s -permutable subgroup of G . (Note that a subgroup H of G is said to be *s -permutable* in G if $HP = PH$ for any Sylow subgroup P of G .) A subgroup H of G is *σ -permutably embedded* in G (see [19]) if H is σ -full and for every $\sigma_i \in \sigma(H)$, every Hall σ_i -subgroup of H is also a Hall σ_i -subgroup of some σ -permutable subgroup of G . By using the above embedded subgroups, the researchers have obtained a series of interesting results (see, for example, [3], [5], [8], [19], [24], [31]).

Some properties of σ -permutably embedded subgroups were analysed in [19]. In this paper, we continue the research of σ -permutably embedded subgroups.

We first obtain the following result.

Theorem 1.1. *Let G be a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ -set of G such that H_i is a supersoluble σ_i -subgroup for all $i = 1, \dots, t$. If every maximal subgroup of non-cyclic H_i is σ -permutably embedded in G , then G is supersoluble.*

Recall that a normal subgroup E of G is called *hypercyclically embedded* in G ([25], page 217) if every chief factor of G below E is cyclic. Hypercyclically embedded subgroups play an important role in the theory of soluble groups (see [6], [9], [15], [25])

and the conditions under which a normal subgroup is hypercyclically embedded in G were found by many authors (see books [6], [9], [15], [25] and, for example, the recent papers [16], [18], [23], [27], [28], [32]).

In this paper, we also get the following results in this line researches.

Theorem 1.2. *Let G be a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ -set of G such that H_i is a nilpotent σ_i -subgroup for all $i = 1, \dots, t$. Let E be a normal subgroup of G . If every maximal subgroup of any non-cyclic $H_i \cap E$ is σ -permutably embedded in G , then E is hypercyclically embedded in G .*

Theorem 1.3. *Let G be a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ -set of G such that H_i is a supersoluble σ_i -subgroup for all $i = 1, \dots, t$. Let E be a normal subgroup of G . If every cyclic subgroup H of any non-cyclic $H_i \cap E$ of prime order and order 4 (if the Sylow 2-subgroup of E is non-abelian and $H \not\leq Z_\infty(G)$) is σ -permutably embedded in G , then E is hypercyclically embedded in G .*

In Section 3 and Section 4 we give the proofs of Theorems 1.1, 1.2 and 1.3. In Section 5 we will give some applications of our results.

All unexplained terminologies and notations are standard. The reader is referred to [12], [15] if necessary.

2. PRELIMINARIES

We use \mathfrak{S}_σ to denote the class of all σ -soluble groups and $F_\sigma(G)$ to denote the product of all normal σ -nilpotent subgroups of G .

Lemma 2.1 ([29], Lemma 2.1). *The class \mathfrak{S}_σ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a σ -soluble group by a σ -soluble group is a σ -soluble group as well.*

Lemma 2.2 ([29], Lemma 2.6 (11)). *Let G be a σ -full group and A be a σ -subnormal subgroup of G . If A is σ -nilpotent, then A is contained in $F_\sigma(G)$.*

Lemma 2.3 ([19], Lemma 2.6 (i)). *$F_\sigma(G)$ is σ -nilpotent.*

Lemma 2.4 ([29], Lemma 2.8). *Let H , K and N be subgroups of G . Let $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ -set of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is \mathcal{L} -permutable and N is normal in G .*

- (1) The subgroup HN/N is \mathcal{L}^* -permutable, where

$$\mathcal{L}^* = \{H_1N/N, \dots, H_tN/N\}^{KN/N}.$$

In particular, if H is σ -permutable in G , then HN/N is σ -permutable in G/N .

- (2) If G is a σ -full group of Sylow type and E/N is a σ -permutable subgroup of G/N , then E is σ -permutable in G .

Lemma 2.5 ([29], Theorem B). *Let H be a subgroup of a σ -full group G . If H is σ -permutable in G , then H is σ -subnormal in G and H^G/H_G is σ -nilpotent.*

Lemma 2.6. *Let $H \leq K$ and N be subgroups of G . Suppose that N is normal in G .*

- (1) *If G is a σ -full group of Sylow type and H is σ -permutably embedded in G , then H is σ -permutably embedded in K .*
- (2) *If H is σ -permutably embedded in G , then HN/N is σ -permutably embedded in G/N .*
- (3) *If G is a σ -full group of Sylow type and H/N is σ -permutably embedded in G/N , then H is σ -permutably embedded in G .*

Proof. (1)–(2) can be found in [19], Lemma 2.2.

(3) Let H_i be a Hall σ_i -subgroup of H , where $\sigma_i \in \sigma(H)$. Then H_iN/N is a Hall σ_i -subgroup of H/N . By the hypothesis, there exists a σ -permutable subgroup T/N of G/N such that H_iN/N is a Hall σ_i -subgroup of T/N . Then T is σ -permutable in G by Lemma 2.4 (2). Since $N \leq H$, $H_iN \leq H$ and so H_i is a Hall σ_i -subgroup of H_iN . It follows that $|T : H_i| = |T : H_iN| |H_iN : H_i|$ is a σ'_i -number. Hence, H_i is a Hall σ_i -subgroup of T . This shows that H is σ -permutably embedded in G . \square

Following [29], [20], we use $O^{\sigma_i}(G)$ to denote the subgroup of G generated by all σ'_i -subgroups of G , and $O_{\sigma_i}(G)$ and $O_{\sigma'_i}(G)$ to denote the subgroup of G generated by all normal σ_i -subgroups and normal σ'_i -subgroups of G , respectively.

Lemma 2.7 ([29], Lemma 3.1). *Let H be a σ_1 -subgroup of a σ -full group G . Then H is σ -permutable in G if and only if $O^{\sigma_1}(G) \leq N_G(H)$.*

Let P be a p -group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.8 ([17], Lemma 4.3). *Let C be a Thompson critical subgroup of a p -group P (see [13], page 185).*

- (1) *If p is odd, then the exponent of $\Omega(C)$ is p .*
- (2) *If P is a non-abelian 2-group, then the exponent of $\Omega(C)$ is 4.*

The following lemma is a corollary of [17], Lemma 4.4 and [11], Lemma 2.12.

Lemma 2.9. *Let P be a normal p -subgroup of G and C be a Thompson critical subgroup of P . If either $P/\Phi(P)$ is hypercyclically embedded in $G/\Phi(P)$ or $\Omega(C)$ is hypercyclically embedded in G , then P is hypercyclically embedded in G .*

Lemma 2.10 ([28], Theorem C). *Let E be a normal subgroup of G . If $F^*(E)$ is hypercyclically embedded in G , then E is hypercyclically embedded in G .*

In this lemma, $F^*(E)$ is the generalized Fitting subgroup of E , that is, the largest normal quasinilpotent subgroup of E (see [22], Chapter X).

Recall that a class of groups \mathfrak{F} is said to be a *formation* provided that (i) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$, and (ii) if $G/M \in \mathfrak{F}$ and $G/N \in \mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$ for any normal subgroup M, N of G . A formation \mathfrak{F} is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

Lemma 2.11 ([26], Lemma 2.16 or [15], Theorem 1.2.7 (b)). *Let \mathfrak{F} be a saturated formation containing all supersoluble groups and E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. PROOFS OF THEOREMS 1.1 AND 1.2

The following fact is one of the main steps in the proofs of Theorems 1.1 and 1.2.

Proposition 3.1. *Let G be a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ -set of G such that H_i is a soluble σ_i -subgroup for all $i = 1, \dots, t$, and let the smallest prime p of $\pi(G)$ belong to σ_1 . If every maximal subgroup of H_1 is σ -permutably embedded in G , then G is soluble.*

Proof. Assume that this is false and let (G, H_1) be a counterexample with minimal $|G| + |H_1|$. Then $p = 2 \in \pi(H_1)$ by the Feit-Thompson theorem.

(1) G is not σ -soluble, and so $|\sigma(G)| > 1$. Assume that G is σ -soluble. Then for every chief factor, H/K of G is σ -primary, that is, H/K is a σ_i -group for some i . But since H_i is soluble, H/K is an elementary abelian group. It follows that G is soluble. This contradiction shows that (1) holds.

(2) $O_{\sigma_1}(G) = 1$. Assume that $O_{\sigma_1}(G) \neq 1$. Let $N = O_{\sigma_1}(G)$. If $N = H_1$, then G/N is soluble by Feit-Thompson theorem, and so G is σ -soluble, contrary to claim (1). Hence $N \neq H_1$, and so H_1/N is a non-identity Hall σ_1 -subgroup of G/N . Let M/N be a maximal subgroup of H_1/N . Then M is a maximal subgroup of H_1 . By the hypothesis and Lemma 2.6 (2), M/N is σ -permutably embedded in G/N .

Then, clearly, the hypothesis holds for $(G/N, H_1/N)$. Hence G/N is soluble by the choice of (G, H_1) . Consequently, G is σ -soluble, which contradicts claim (1). Hence we have (2).

(3) $O_{\sigma'_1}(G) = 1$. Assume that $K = O_{\sigma'_1}(G) \neq 1$. Then H_1K/K is a Hall σ_1 -subgroup of G/K . Let W/K be a maximal subgroup of H_1K/K . Then $W = (H_1 \cap W)K$ is a maximal subgroup of H_1K . If $H_1 \cap W$ is not a maximal subgroup of H_1 , then there exists a subgroup E of H_1 such that $H_1 \cap W < E < H_1$. Since $(|H_1|, |K|) = 1$, $W < EK < H_1K$. This contradiction shows that $H_1 \cap W$ is a maximal subgroup of H_1 . By the hypothesis and Lemma 2.6 (2), W/K is σ -permutably embedded in G/K . This shows that $(G/K, H_1K/K)$ satisfies the hypothesis, so G/K is soluble by the choice of (G, H_1) . But since K is soluble by Feit-Thompson theorem, it follows that G is soluble. This contradiction shows that (3) holds.

(4) Let R be a minimal normal subgroup of G . Then R is not σ -soluble and G/R is soluble. Assume that R is σ -soluble. Then R is a σ_i -group for some i . Hence, $R \leq O_{\sigma_1}(G)$ or $R \leq O_{\sigma'_1}(G)$, which contradicts claim (2) or (3). Hence, R is not σ -soluble. By the hypothesis and Lemma 2.6 (1), it is easy to see that (RH_1, H_1) satisfies the hypothesis. If $RH_1 < G$, then RH_1 is soluble by the choice of G . It follows that R is soluble, a contradiction. Hence, $G = RH_1$, and so $G/R = H_1R/R \cong H_1/(H_1 \cap R)$ is soluble since H_1 is soluble.

(5) R is the unique minimal normal subgroup of G and $F_\sigma(G) = 1$. This directly follows from claim (4).

(6) *Final contradiction.* Let L be any maximal subgroup of H_1 . By the hypothesis, there exists a σ -permutable subgroup T of G such that L is a Hall σ_1 -subgroup of T . If $T_G = 1$, then T is σ -nilpotent and σ -subnormal in G by Lemma 2.5. Then by Lemma 2.2 and claim (5), we get that $T \leq F_\sigma(G) = 1$. This implies that $L = 1$, and so $|G| = 2n$, where n is an odd number. It follows that G is soluble, a contradiction. Hence, $T_G \neq 1$, and so $R \leq T$ by claim (5). Then $T \geq RL$. Hence, $L = T \cap H_1 \geq RL \cap H_1 = (R \cap H_1)L$ for any maximal subgroup L of H_1 . This implies that $R \cap H_1 \leq \Phi(H_1)$. Then by [21], Lemma IV.4.6, there exists a normal subgroup M of R such that R/M is a σ_1 -group and $|R \cap H_1| \mid |R/M|$. It follows that $O^{\sigma_1}(R) \leq M$. Since $O^{\sigma_1}(R) \text{ char } R \trianglelefteq G$, we have $O^{\sigma_1}(R) \trianglelefteq G$, so $O^{\sigma_1}(R) = 1$ or R by claim (5). If $O^{\sigma_1}(R) = 1$, then $R \leq H_1$, which contradicts claim (2). Hence $O^{\sigma_1}(R) = R$, and therefore $M = R$. Moreover, since $|R \cap H_1| \mid |R/M|$, we obtain that $R \cap H_1 = 1$. But clearly, $R \cap H_1$ is a Hall σ_1 -subgroup of R . Thus, R is a σ'_1 -subgroup, so $R \leq O_{\sigma'_1}(G) = 1$, contrary to claim (4). This completes the proof. \square

Proof of Theorem 1.1. Assume that this is false and let G be a counterexample of minimal order. Then: (1) G is soluble. By Feit-Thompson theorem, we may assume that $2 \mid |G|$. Without loss of generality, we may assume that $2 \in \pi(H_1)$.

If H_1 is cyclic, then G has a cyclic Sylow 2-subgroup. Hence G is 2-nilpotent by [21], Theorem IV.2.8, and so G is soluble. If H_1 is non-cyclic, then G is soluble by Proposition 3.1.

(2) Let R be a minimal normal subgroup of G . Then G/R is supersoluble. It is clear that $\overline{\mathcal{H}} = \{H_1R/R, H_2R/R, \dots, H_tR/R\}$ is a complete Hall σ -set of G/R and $H_iR/R \cong H_i/H_i \cap R$ is supersoluble. By claim (1), R is an elementary abelian p -group for some prime p . Without loss of generality, we may assume that $R \leq H_1$. Assume that H_1/R is non-cyclic. Then H_1 is non-cyclic. Let M/R be a maximal subgroup of H_1/R . Then M is a maximal subgroup of H_1 . By the hypothesis and Lemma 2.6 (2), M/R is σ -permutably embedded in G/R . Now let M_i/R be a maximal subgroup of H_iR/R , where $i \neq 1$, and suppose that H_iR/R is non-cyclic. Then $M_i = (H_i \cap M_i)R$ is a maximal subgroup of H_iR . With the same discussion as for claim (3) in the proof of Proposition 3.1, we have that $H_i \cap M_i$ is a maximal subgroup of H_i . Then by the hypothesis and Lemma 2.6 (2), M_i/R is σ -permutably embedded in G/R . This shows that the hypothesis holds for G/R . The choice of G implies that G/R is supersoluble.

(3) R is the unique minimal normal subgroup of G , $\Phi(G) = 1$, $C_G(R) = R$, R is an elementary abelian p -group for some prime p and $|R| > p$. This directly follows from claims (1), (2) and [12], Theorem A.15.2.

Without loss of generality, we may assume that $p \in \pi(H_1)$. Then $R \leq H_1$.

(4) *Final contradiction.* Since $\Phi(G) = 1$, $R \not\leq \Phi(H_1)$ by [21], Lemma III.3.3. Hence, there exists a maximal subgroup K of H_1 such that $H_1 = RK$. Let $E = R \cap K$. By claim (3), we have that $E \leq H_1$. Since H_1 is supersoluble, $|R : E| = |RK : K| = |H_1 : K|$ is a prime. Hence, E is a maximal subgroup of R , and so $E \neq 1$ by claim (3). Since R is not cyclic by claim (3) and $R \leq H_1$, H_1 is non-cyclic. Then by the hypothesis, there exists a σ -permutable subgroup T of G such that K is a Hall σ_1 -subgroup of T . If $T_G = 1$, then T is σ -nilpotent and σ -subnormal in G by Lemma 2.5. It follows from Lemma 2.2 that $T \leq F_\sigma(G)$. But since $R \leq F_\sigma(G)$ and $C_G(R) = R$, we have that T is a σ_1 -group by Lemma 2.3, so $T = K$. It follows from Lemma 2.7 that $O^{\sigma_1}(G) \leq N_G(K)$. Hence $O^{\sigma_1}(G) \leq N_G(K \cap R) = N_G(E)$. This implies that $E \trianglelefteq G$, which contradicts the minimality of R . Hence $T_G \neq 1$. Then by claim (3), $R \leq T_G \leq T$. Consequently, $K < H_1 \leq T$. But K is a Hall σ_1 -subgroup of T , a contradiction. This completes the proof. \square

Proof of Theorem 1.2. Assume that this is false and let (G, E) be a counterexample with minimal $|G| + |E|$. Then: (1) E is supersoluble. It is clear that $\mathcal{H}^* = \{H_1 \cap E, H_2 \cap E, \dots, H_t \cap E\}$ is a complete Hall σ -set of E , $H_i \cap E$ is nilpotent and E is a σ -full group of Sylow type. By Lemma 2.6 (1) and Theorem 1.1, we get that E is supersoluble.

(2) Let R be a minimal normal subgroup of G contained in E . Then R is an elementary abelian p -group for some prime p , E/R is hypercyclically embedded in G/R and R is non-cyclic. By claim (1), R is an elementary abelian p -group for some prime p . Without loss of generality, we may assume that $R \leq H_1$. Clearly, $\overline{\mathcal{H}} = \{H_1/R, H_2R/R, \dots, H_tR/R\}$ is a complete Hall σ -set of G/R and $H_iR/R \cong H_i/(H_i \cap R)$ is nilpotent. Assume that $(H_1/R) \cap (E/R)$ is non-cyclic. Then $H_1 \cap E$ is non-cyclic. Let M/R be a maximal subgroup of $(H_1/R) \cap (E/R)$. Then M is a maximal subgroup of $H_1 \cap E$. Hence, M/R is σ -permutably embedded in G/R by the hypothesis and Lemma 2.6 (2). Now assume that M_i/R is a maximal subgroup of some non-cyclic $(H_iR/R) \cap (E/R)$, where $i \neq 1$. Then $H_iR \cap E$ is non-cyclic and $M_i = (H_i \cap M_i)R$ is a maximal subgroup of $H_iR \cap E$. With the same discussion as for claim (3) in the proof of Proposition 3.1, we have that $H_i \cap M_i$ is a maximal subgroup of $H_i \cap E$. Then by the hypothesis and Lemma 2.6 (2), M_i/R is σ -permutably embedded in G/R . This shows that $(G/R, E/R)$ satisfies the hypothesis. Hence E/R is hypercyclically embedded in G/R by the choice of (G, E) .

(3) R is the unique minimal normal subgroup of G contained in E . Let L be a minimal normal subgroup of G contained in E such that $L \neq R$. Then E/L is also hypercyclically embedded in G/L by claim (2). It follows that RL/L is hypercyclically embedded in G/L . Then $|R| = p$ for $RL/L \cong R$, contrary to claim (2). Hence we have (3).

Without loss of generality, we may assume that $p \in \pi(H_1)$.

(4) E is a p -group, and so $E \leq H_1$. Let Q be a Sylow q -subgroup of E , where q is the largest prime belong to $\pi(E)$. Since E is supersoluble by claim (1), we obtain that $Q \text{ char } E \trianglelefteq G$ and so $Q \trianglelefteq G$. Hence, $R \leq Q$, $p = q$ and $F(E) = Q$ is a Sylow p -subgroup of E by claim (3). It follows from [14], Theorem 1.8.18, that $C_E(Q) \leq Q$. Moreover, since $Q \leq H_1 \cap E$ and H_1 is nilpotent, we obtain that $Q = H_1 \cap E$. Hence, $H_1 \cap Q = Q = H_1 \cap E$ and $H_i \cap Q = 1$ for all $i = 2, \dots, t$. This implies the hypothesis holds for (G, Q) . Assume that $Q < E$. Then Q is hypercyclically embedded in G by the choice of (G, E) . It follows that R is hypercyclically embedded in G , so R is cyclic, contrary to claim (2). Hence $E = Q$ is a p -group, and so $E \leq H_1$.

(5) $\Phi(E) = 1$, so E is an elementary abelian p -group. Assume that $\Phi(E) \neq 1$. Then $R \leq \Phi(E)$ by claim (3). Hence $E/\Phi(E)$ is hypercyclically embedded in $G/\Phi(E)$ by claim (2) and [15], Theorem 1.2.6 (d). It follows from claim (4) and Lemma 2.9 that E is hypercyclically embedded in G . This contradiction shows that (5) holds.

(6) *Final contradiction.* Let R_1 be a maximal subgroup of R such that $R_1 \trianglelefteq H_1$. Then $|R_1| > 1$ by claim (3). By claim (5), there exists a complement S of R in E (maybe $S = 1$). Let $V = R_1S$. Then clearly $R_1 = R \cap V$ and V is a maximal subgroup of E . By the hypothesis and claims (2)–(5), there exists a σ -permutable subgroup T of G such that V is a Hall σ_1 -subgroup of T . We show that V is also

σ -permutable in G . Let L be a Hall σ_i -subgroup of G . If $i = 1$, then $V \leq L$ by claim (4). This implies that $VL = LV$. If $i \neq 1$, then V is a Hall σ_1 -subgroup of $TL = LT$. (Note that since T is σ -permutable in G , $TL = LT$.) Since E is normal in G , V is subnormal in G by claim (4), so V is also subnormal in TL . But as V is a Hall σ_1 -subgroup of T , V is normal in TL . Hence $VL = LV$. This implies that V is σ -permutable in G . Then by Lemma 2.7, $O^{\sigma_1}(G) \leq N_G(V)$, and so $O^{\sigma_1}(G) \leq N_G(V \cap R) = N_G(R_1)$. Moreover, since $R_1 \trianglelefteq H_1$, we obtain that $R_1 \trianglelefteq G$. This contradiction completes the proof. \square

4. PROOF OF THEOREM 1.3

In order to prove Theorem 1.3, we first prove the following:

Lemma 4.1. *Let G be a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ -set of G such that H_i is a supersoluble σ_i -subgroup for all $i = 1, \dots, t$. Let P be a normal p -group of G and $P \leq H_j$ for some j . If every cyclic subgroup H of P of prime order and order 4 (if P is a non-abelian 2-group and $H \not\leq Z_\infty(G)$) is σ -permutably embedded in G , then P is hypercyclically embedded in G .*

Proof. Assume that this is false and let (G, P) be a counterexample with minimal $|G| + |P|$. Without loss of generality, we may assume that $j = 1$.

(1) *Let P/N be a chief factor of G . Then N is hypercyclically embedded in G . Hence, N is a unique normal subgroup of G such that P/N is a chief factor of G and $|P/N| > p$.*

It is clear that (G, N) satisfies the hypothesis. Hence, N is hypercyclically embedded in G by the choice of (G, P) . Assume that G has another normal subgroup $R \neq N$ such that P/R is a chief factor of G . Then R is also hypercyclically embedded in G . It follows that $P/N = RN/N$ is hypercyclically embedded in G/N . Hence, P is hypercyclically embedded in G . This contradiction shows that N is a unique normal subgroup such that P/N is a chief factor of G . It is also clear that $|P/N| > p$.

(2) *The exponent of P is p or 4 (if P is a non-abelian 2-group).* Let C be a Thompson critical subgroup of P (see [13], page 185). If $\Omega(C) < P$, then $\Omega(C) \leq N$ is hypercyclically embedded in G by claim (1). Hence, by Lemma 2.9, P is hypercyclically embedded in G , a contradiction. Hence, $\Omega(C) = P$, so by Lemma 2.8, the exponent of P is p or 4 (if P is a non-abelian 2-group).

(3) *Final contradiction.* Since H_1/N is supersoluble and $|P/N| > p$, H_1/N has a minimal normal subgroup L/N such that $1 \neq L/N < P/N$ and L/N is cyclic. Let $x \in L \setminus N$ and $H = \langle x \rangle$. Then $L = HN$ and $|H| = p$ or 4 (if P is a non-abelian 2-group)

by claim (2). If $H \leq Z_\infty(G)$, then $L/N = HN/N \leq Z_\infty(G)N/N \leq Z_\infty(G/N)$ by [15], Theorem 1.2.6 (d). So $Z_\infty(G/N) \cap P/N \neq 1$. Hence, $P/N \leq Z_\infty(G/N)$ since P/N is a chief factor of G . It follows from claim (1) that P is hypercyclically embedded in G . This contradiction shows that $H \not\leq Z_\infty(G)$. Then by the hypothesis, there exists a σ -permutable subgroup T of G such that H is a Hall σ_1 -subgroup of T . With a similar argument as for claim (6) in the proof of Theorem 1.2, we have that H is σ -permutable in G . Then HN/N is σ -permutable in G/N by Lemma 2.4 (1). Hence, $O^{\sigma_1}(G/N) \leq N_{G/N}(HN/N)$ by Lemma 2.7. Moreover, since $L/N \leq H_1/N$, we obtain that $HN/N = L/N \leq G/N$, and so $L \leq G$. This contradiction completes the proof. \square

Proof of Theorem 1.3. Assume that this is false and let (G, E) be a counterexample with minimal $|G| + |E|$. Let P be a Sylow p -subgroup of E , where p is the smallest prime contained in $\pi(E)$. Without loss of generality, we may assume that $P \leq H_1 \cap E$.

(1) $H_1 \cap E$ is non-cyclic. Assume that $H_1 \cap E$ is cyclic. Then P is cyclic. By [21], Theorem IV.2.8, E is p -nilpotent. Let $E_{p'}$ be a normal Hall p' -subgroup of E . Then $E_{p'} \leq G$. If $E_{p'} = 1$, then E is cyclic, so E is hypercyclically embedded in G , a contradiction. Hence $E_{p'} \neq 1$. Clearly, $H_i \cap E_{p'} = H_i \cap E$ for $i = 2, \dots, t$. This shows the hypothesis holds for $(G, E_{p'})$, so $E_{p'}$ is hypercyclically embedded in G by the choice of (G, E) . But as $E/E_{p'} \cong P$ is cyclic, it follows that E is hypercyclically embedded in G . This contradiction shows that (1) holds.

(2) If $E = P$, then E is hypercyclically embedded in G . This directly follows from Lemma 4.1 and claim (1).

(3) E is not p -nilpotent. Assume that E is p -nilpotent. Let $E_{p'}$ be a normal Hall p' -subgroup of E . Then $E_{p'} \leq G$. By claim (2), $E_{p'} \neq 1$. Clearly, $\overline{\mathcal{H}} = \{H_1 E_{p'}/E_{p'}, H_2 E_{p'}/E_{p'}, \dots, H_t E_{p'}/E_{p'}\}$ is a complete Hall σ -set of $G/E_{p'}$ and $H_i E_{p'}/E_{p'} \cong H_i/H_i \cap E_{p'}$ is supersoluble.

We claim that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. In fact, $H_i E_{p'}/E_{p'} \cap E/E_{p'} = 1$ for $i = 2, \dots, t$ and $H_1 E_{p'}/E_{p'} \cap E/E_{p'} = E/E_{p'}$. It is trivial when $E/E_{p'}$ is cyclic. We may therefore assume that $E/E_{p'}$ is non-cyclic. Let $H/E_{p'}$ be a cyclic subgroup of $E/E_{p'}$ of order p or 4 (if the Sylow 2-subgroup of $E/E_{p'}$ is non-cyclic and $H/E_{p'} \not\leq Z_\infty(G/E_{p'})$). Then by Schur-Zassenhaus theorem, $H = E_{p'} \rtimes L$ and without loss of generality, we may assume that $L \leq E \cap H_1$. Note that if $L \leq Z_\infty(G)$, then $H/E_{p'} = L E_{p'}/E_{p'} \leq Z_\infty(G) E_{p'}/E_{p'} \leq Z_\infty(G/E_{p'})$ by [15], Theorem 1.2.6 (d). Hence, L is of order p or 4 (if the Sylow 2-subgroup of E is non-cyclic and $L \not\leq Z_\infty(G)$). Then by Lemma 2.6 (2), we see that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. Hence, $E/E_{p'}$ is hypercyclically embedded in $G/E_{p'}$ by the choice of (G, E) . On the other hand, it is clear that the hypothesis holds for

$(G, E_{p'})$, so $E_{p'}$ is hypercyclically embedded in G by the choice of (G, E) . Therefore E is hypercyclically embedded in G , a contradiction. Hence we have (3).

(4) *Final contradiction.* By claim (3), [21], Theorem IV.5.4, and [14], Theorem 3.4.11, E has a p -closed Schmit subgroup $S = P_1 \rtimes Q$, where P_1 is a Sylow p -subgroup of S of exponent p or 4 (if P_1 is non-abelian 2-group), Q is a Sylow q -subgroup of S for some prime $q \neq p$, $P_1/\Phi(P_1)$ is an S -chief factor, $Z_\infty(S) = \Phi(S)$ and $\Phi(S) \cap P_1 = \Phi(P_1)$.

We claim that $|P_1 : \Phi(P_1)| = p$. If $q \in \pi(H_1)$, then S is a σ_1 -group, and so $S \leq H_1^g$ for some $g \in G$ since G is a σ -full group of Sylow type. Since H_1 is supersoluble and $P_1/\Phi(P_1)$ is an S -chief factor, $|P_1 : \Phi(P_1)| = p$. Now we consider that $q \notin \pi(H_1)$. Assume that there exists a minimal subgroup $D/\Phi(P_1)$ of $P_1/\Phi(P_1)$ such that $D/\Phi(P_1)$ is not σ -permutable in $S/\Phi(P_1)$. Let $x \in D \setminus \Phi(P_1)$ and $U = \langle x \rangle$. Then $D = U\Phi(P_1)$ and $|U| = p$ or 4 (if P_1 is non-abelian 2-group). If $U \leq Z_\infty(G)$, then $U \leq Z_\infty(S) \cap P_1 = \Phi(S) \cap P_1 = \Phi(P_1)$, a contradiction. Hence $U \not\leq Z_\infty(G)$. Then by the hypothesis and Lemma 2.6 (1), there exists a σ -permutable subgroup T of S such that U is a Hall σ_1 -subgroup of T . Let K be a Hall σ_i -subgroup of S , where $\sigma_i \cap \pi(S) \neq \emptyset$. If $i = 1$, then $K = P_1$, and so $UK = KU = P_1$. If $i \neq 1$, then U is a Hall σ_1 -subgroup of $TK = KT$. But as $D < P_1 \leq S$ and $p \in \sigma_1$, we have that $TK < S$. Hence, TK is nilpotent, and so $U \leq TK$. Thus $UK = KU$. This implies that U is σ -permutable in S . It follows from Lemma 2.4 (1) that $D/\Phi(P_1) = U\Phi(P_1)/\Phi(P_1)$ is σ -permutable in $S/\Phi(P_1)$. This contradiction shows that every minimal subgroup of $P_1/\Phi(P_1)$ is σ -permutable in $S/\Phi(P_1)$. Consequently, every minimal subgroup of $P_1/\Phi(P_1)$ is s -permutable in $S/\Phi(P_1)$ since $\pi(S) = \{p, q\}$ and $q \notin \pi(H_1)$. Then by [26], Lemma 2.12, we also have that $|P_1 : \Phi(P_1)| = p$. Hence, P_1 is cyclic of exponent p . This implies that P_1 is a group of order p . Since $N_S(P_1)/C_S(P_1) \lesssim \text{Aut}(P_1)$ is a group of order $p - 1$ and p is the smallest prime contained in $\pi(E)$, it follows that $C_S(P_1) = N_S(P_1) = S$. Thus $Q \leq S$. This contradiction completes the proof. \square

5. SOME APPLICATIONS OF THE RESULTS

It is clear that every σ -permutable subgroup of G is σ -permutable embedded in G . In the case when $\sigma = \{\{2\}, \{3\}, \dots\}$, every normal subgroup, every normally embedded subgroup, every permutable subgroup, every permutable embedded subgroup, every s -permutable subgroup and every s -permutable embedded subgroup of G are all σ -permutable embedded in G . However, the converse is not true in general (see [19], Example 1.2). Hence, the following results directly follow from Theorem 1.1.

Corollary 5.1 ([31], Theorem 1). *If all maximal subgroups of every Sylow subgroup of G are normal in G , then G is supersoluble.*

Corollary 5.2 ([5], Theorem 4). *If all maximal subgroups of every Sylow subgroup of G are permutably embedded in G , then G is supersoluble.*

Corollary 5.3 ([31], Theorem 2). *If all maximal subgroups of every Sylow subgroup of G are s -permutable in G , then G is supersoluble.*

Corollary 5.4 ([8], Theorem 1). *If all maximal subgroups of every Sylow subgroup of G are s -permutable embedded in G , then G is supersoluble.*

By Theorems 1.2 and 1.3, we may obtain the following results.

Corollary 5.5. *Let \mathfrak{F} be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ -set of G such that H_i is a nilpotent σ_i -subgroup for all $i = 1, \dots, t$. If every maximal subgroup of any non-cyclic $H_i \cap E$ is σ -permutable embedded in G , then $G \in \mathfrak{F}$.*

Corollary 5.6. *Let \mathfrak{F} be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ -set of G such that H_i is a supersoluble σ_i -subgroup for all $i = 1, \dots, t$. If every cyclic subgroup H of any non-cyclic $H_i \cap E$ of prime order and order 4 (if the Sylow 2-subgroup of E is non-abelian and $H \not\leq Z_\infty(G)$) is σ -permutable embedded in G , then $G \in \mathfrak{F}$.*

Theorems 1.2–1.3 and Corollaries 5.5–5.6 cover a lot of known results, in particular, [4], Theorem 4.1, [8], Corollary, [2], Theorem 1.3, [3], Theorem 3.3, [10], Theorem 3, [1], Theorem 3.1, [24], Theorem 3.3 and [7], Theorem 2 and Theorem 5.

Corollary 5.7. *Let \mathfrak{F} be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ -set of G such that H_i is a nilpotent σ_i -subgroup for all $i = 1, \dots, t$. If every maximal subgroup of any non-cyclic $H_i \cap F^*(E)$ is σ -permutable embedded in G , then $G \in \mathfrak{F}$.*

Proof. By the hypothesis and Theorem 1.2, we have that $F^*(E)$ is hypercyclically embedded in G . Then E is hypercyclically embedded in G by Lemma 2.10. Therefore $G \in \mathfrak{F}$ by Lemma 2.11. \square

By using a similar argument as in the proof of Corollary 5.7, we deduce the following corollary from Theorem 1.3.

Corollary 5.8. *Let \mathfrak{F} be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ -full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ -set of G such that H_i is a supersoluble σ_i -subgroup for all $i = 1, \dots, t$. If every cyclic subgroup H of any non-cyclic $H_i \cap F^*(E)$ of prime order and order 4 (if the Sylow 2-subgroup of $F^*(E)$ is non-abelian and $H \not\leq Z_\infty(G)$) is σ -permutably embedded in G , then $G \in \mathfrak{F}$.*

Corollaries 5.7 and 5.8 also cover many known results, in particular, [24], Theorem 3.1, Theorem 3.4 and Corollary 3.5, [2], Theorem 1.4, [3], Corollary 3.4, [4], Theorem 3.2 and [8], Theorem 2.

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