Czechoslovak Mathematical Journal, 69 (144) (2019), 173–181

FINITE p-GROUPS WITH EXACTLY TWO NONLINEAR NON-FAITHFUL IRREDUCIBLE CHARACTERS

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Received May 17, 2017. Published online July 23, 2018.

Abstract. Let G be a finite group with exactly two nonlinear non-faithful irreducible characters. We discuss the properties of G and classify finite p-groups with exactly two nonlinear non-faithful irreducible characters.

Keywords: p-group; nonlinear irreducible character; non-faithful character

MSC 2010: 20C15

1. INTRODUCTION

Iranmanesh and Saeidi [4] studied finite groups with exactly one nonlinear non-faithful irreducible character. And Saeidi [6] classified solvable groups with a unique nonlinear non-faithful irreducible character. We consider the following case in this note.

Hypothesis $(*)$:
A finite group has exactly two nonlinear non-faithful irreducible characters.

Let G be a finite group with exactly two nonlinear non-faithful irreducible characters χ_1, χ_2. Let $K_1 = \ker \chi_1$, $K_2 = \ker \chi_2$, and write $L = K_1 \cap K_2$.

In this note, we will show some properties of groups which satisfy Hypothesis $(*)$. Our main conclusion is the classification of finite p-groups with exactly two nonlinear non-faithful irreducible characters. In fact, we have the following result.

The authors acknowledge the support of the NSFC (11701503, 11571129, 31360277). The second author was supported of China Scholarship Council, Funds of Henan University of Technology (2014JCYJ14, 2016JJSB074), Project of Department of Education of Henan Province (17A110004), Projects of Zhengzhou Municipal Bureau of Science and Technology (20150249, 20140970), Fund of Henan Province (162300410066).

DOI: 10.21136/CMJ.2018.0230-17

Theorem 1.1. *A p-group G has exactly two nonlinear non-faithful irreducible characters if and only if one of the following assertions holds.*

(1) *G is a 2-group with nilpotence class 2, $G/\mathbf{Z}(G)$ is elementary abelian, $\mathbf{Z}(G) \cong C_2 \times C_2$ and $|G'| = 2$.*

(2) *G is a group of order 32 of nilpotence class 3 and $\mathbf{Z}(G) \cong C_2$ or $\mathbf{Z}(G) \cong C_4$.*

(3) *G is a group of order 81 of nilpotence class 3.*

All groups considered in this note are finite. The notation and terminology are standard, and one can refer to [5] and [3].

2. Preliminaries

We first discuss the intersection of the kernels of irreducible nonlinear characters of a finite group. There is a modular form of the intersection of the kernels of irreducible Brauer characters in [8]. Moreover, in this section, we give some properties of groups which satisfy Hypothesis $(*)$, and state facts which are important to prove the main theorem of this note.

Proposition 2.1. *Let $\mathrm{Irr}(G)$ and $\mathrm{Lin}(G)$ be the sets of irreducible characters and linear characters of a group G, respectively. If $\mathrm{Irr}(G) \supsetneq \mathrm{Lin}(G)$, then*

(i) $\displaystyle\bigcap_{\varphi\in \mathrm{Irr}(G)-\mathrm{Lin}(G)} \ker\varphi = 1$. *In particular, if G has a unique nonlinear irreducible character χ, then χ is faithful; and if G has exactly two nonlinear irreducible characters χ, φ, then $\ker\chi \cap \ker\varphi = 1$.*

(ii) *if G satisfies Hypothesis $(*)$ and $L > 1$, then $\mathbf{Z}(G)$ is cyclic, where $\mathbf{Z}(G)$ is the center of G;*

(iii) *if G satisfies Hypothesis $(*)$, then every normal subgroup of G not containing G' is among K_1, K_2 and L;*

(iv) *if $\mathbf{Z}(G) \neq 1$ and G satisfies Hypothesis $(*)$ and $L > 1$, then L is a minimal normal subgroup of G of a prime order. Moreover, if $G' \cap L = 1$, then $G' \cap K_i = 1$ for $i = 1, 2$.*

Proof. Write $U = \displaystyle\bigcap_{\varphi\in \mathrm{Irr}(G)-\mathrm{Lin}(G)} \ker\varphi$ and $h \in U$. Then $\varphi(h) = \varphi(1)$ for $\varphi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$. For any $\lambda \in \mathrm{Lin}(G)$, λ can act on $\mathrm{Irr}(G)$ by multiplication. Then $\exists\, \theta \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ such that $\varphi = \lambda\theta$. Thus

$$\varphi(1) = \varphi(h) = \lambda(h)\theta(h) = \lambda(h)\theta(1) = \lambda(h)\varphi(1), \quad \lambda(h) = 1.$$

Since λ is arbitrary, it follows that $h \in \bigcap\limits_{\lambda \in \mathrm{Lin}(G)} \ker \lambda = G'$, where G' is the derived subgroup of G, and then

$$h \in U \cap G' = \bigcap_{\mu \in \mathrm{Irr}(G)} \ker \mu = 1.$$

Therefore, we have that $\bigcap\limits_{\varphi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)} \ker \varphi = 1$. Immediately, if G has a unique nonlinear irreducible character χ, then $\ker \chi = 1$.

If G satisfies Hypothesis $(*)$ and $L > 1$, then G has at least one nonlinear faithful irreducible character and so $\mathbf{Z}(G)$ is cyclic.

Let G satisfy Hypothesis $(*)$ and let N be a normal subgroup of G not containing the derived subgroup G'. Then the number of nonlinear irreducible characters of G/N is no more than 2. Let $\mathrm{Irr}_1(G)$ denote the set of all the nonlinear irreducible characters of G. So we have the following two cases.

Case (a): When $|\mathrm{Irr}_1(G/N)| = 1$, it follows that $\mathrm{Irr}_1(G/N) = \{\widehat{\chi}_1\}$ or $\{\widehat{\chi}_2\}$, where $\widehat{\chi}_i(Ng) = \chi_i(g)$ for $g \in G$, $i = 1, 2$. If $\mathrm{Irr}_1(G/N)| = \{\widehat{\chi}_1\}$, we deduce that $\ker \widehat{\chi}_1 = \overline{1}$ by (i). Therefore $N = \ker \chi_1$ since $\ker \chi_1/N = \ker \widehat{\chi}_1$. If $\mathrm{Irr}_1(G/N)| = \{\widehat{\chi}_2\}$, for the same reason as above, we have $N = \ker \chi_2$.

Case (b): When $|\mathrm{Irr}_1(G/N)| = 2$, it follows that $\mathrm{Irr}_1(G/N) = \{\widehat{\chi}_1, \widehat{\chi}_2\}$. By (i), we have that $\ker \widehat{\chi}_1 \cap \ker \widehat{\chi}_2 = \overline{1}$. Since $(\ker \chi_1/N) \cap (\ker \chi_2/N) = \ker \widehat{\chi}_1 \cap \ker \widehat{\chi}_2$, it follows that $N = \ker \chi_1 \cap \ker \chi_2 = L$.

By the above proof, we have that N is among K_1, K_2 and L.

By (iii), it follows that $G' \cap L = 1$ or $G' \cap L = L$. In both cases, we have that L is a minimal normal subgroup of G. If $G' \subseteq \mathbf{Z}(G)$, then G' is cyclic and so $L \subseteq \mathbf{Z}(G)$ and then L is of a prime order. If $\mathbf{Z}(G)$ is not containing G', then it follows by (iii) that $\mathbf{Z}(G)$ is among K_1, K_2 and L. Thus $L \subseteq \mathbf{Z}(G)$ and so L is of a prime order. Also, if $G' \cap L = 1$, then $G' \cap K_i = 1$ for $i = 1, 2$. Otherwise, $G' \cap K_i = L$ by (iii) and we have that

$$L = G' \cap K_i = G' \cap K_i \cap L = (G' \cap L) \cap K_i = 1,$$

a contradiction. $\square$

Seitz proved in [7] the following lemma which will be used many times in the next section.

Lemma 2.1. *Let G be a finite group. Then G has exactly one nonlinear irreducible character if and only if one of the following conditions holds.*

(i) *G is an extraspecial 2-group.*

(ii) *G is a Frobenius group with elementary abelian Frobenius kernel G' and a cyclic Frobenius complement H, where $|G'| - 1 = |H|$.*

Zhang classified in [9] the groups with exactly two nonlinear irreducible characters.

Lemma 2.2. *Let G be a finite group with exactly two nonlinear irreducible characters. Then one of the following assertions holds.*

(1) *G is an extraspecial 3-group.*

(2) *G is a Frobenius group with abelian Frobenius complement H and elementary abelian Frobenius kernel N, and $2|H| = |N| - 1$.*

(3) *$G = (C_3 \times C_3) \rtimes Q_8$ is a Frobenius group with Frobenius complement Q_8.*

(4) *G is a 2-group with nilpotence class 3 with a normal series $G \rhd G' \rhd \mathbf{Z}(G) \rhd 1$, and $G/\mathbf{Z}(G)$ is an extraspecial 2-group, $|G'| = 4$, $|\mathbf{Z}(G)| = 2$.*

(5) *G is a 2-group with nilpotence class 2 with a normal series $G \rhd \mathbf{Z}(G) \rhd G' \rhd 1$, and $G/\mathbf{Z}(G)$ is elementary abelian, $|\mathbf{Z}(G)| = 4$, $|G'| = 2$. Furthermore, $|G| = 2^{2m}$, $m \in \mathbb{Z}$. (See [1], Theorem 6, page 281).*

3. p-groups

Suppose G satisfies Hypothesis $(*)$, and $L < K_1$, $L < K_2$. The following lemma indicates that we only need to consider 2-groups if we study p-groups satisfying those conditions.

Lemma 3.1. *Let a p-group G satisfy Hypothesis $(*)$. Let $L = K_1 \cap K_2$ and $L \lneqq K_1$, $L \lneqq K_2$. Then $p = 2$ and $|K_1| = |K_2| = 4$ if $L > 1$, $|K_1| = |K_2| = 2$ if $L = 1$.*

P r o o f. Notice that G/K_1 has only one nonlinear irreducible character. Also, since G is a finite p-group, we have that $p = 2$ by Lemma 2.1.

Assume that K_1/M is a chief factor of G. Then $G' \nleqslant M$. Otherwise, $G' \leqslant K_1$ and $\chi_1 \in \mathrm{Irr}(G/K_1)$, a contradiction. Also, since $L \lneqq K_2$, we have $M \neq K_2$. Therefore, $M = L$ by Proposition 2.1 (iii) and then K_1/L is a chief factor of G. Similarly, K_2/L is also a chief factor of G.

Since every chief factor of a p-group has order p, it follows that $|K_1| = |K_2| = 2$ if $L = 1$. If $L > 1$, then it follows by Proposition 2.1 (iv) that $|L| = 2$ and so $|K_1| = |K_2| = 2^2$. □

A p-group G is said to satisfy the *strong condition* on normal subgroups provided that for any $N \trianglelefteq G$ either $G' \leqslant N$ or $N \leqslant \mathbf{Z}(G)$. If for any $N \trianglelefteq G$, either $G' \leqslant N$ or $|N\mathbf{Z}(G) : \mathbf{Z}(G)| \leqslant p$, then we say G satisfies the *weak condition* on normal subgroups. Fernández-Alcober and Moretó in [2] gave some results about finite groups satisfying the strong condition or the weak condition on normal subgroups.

Lemma 3.2. *Let G be a p-group.*

(1) *If G satisfies the strong condition on normal subgroups, then it has nilpotence class $c(G) \leqslant 3$. Furthermore,*

 (i) *if $c(G) = 2$, then $\exp G/\mathbf{Z}(G) = \exp G' = p$;*

 (ii) *if $c(G) = 3$, then $|G : \mathbf{Z}(G)| = p^3$ and $|G| \leqslant p^5$. Moreover, $|G| = 2^4$ for $p = 2$.*

(2) *If G satisfies the weak condition on normal subgroups, then it has nilpotence class $c(G) \leqslant 4$. Furthermore,*

 (i) *if $c(G) = 2$, then $\exp G/\mathbf{Z}(G) = \exp G' = p$ or p^2. Moreover, in the latter case $G/\mathbf{Z}(G) \cong C_{p^2} \times C_{p^2}$ and $G' \cong C_{p^2}$;*

 (ii) *if $c(G) = 4$, then $|G : \mathbf{Z}(G)| = p^4$, whereas for $c(G) = 3$ we have $|G : \mathbf{Z}(G)| = p^3, p^4$ or p^6 for odd p, and $|G : \mathbf{Z}(G)| = 2^3, 2^4$ when $p = 2$;*

 (iii) *if $c(G) = 4$ and $p = 2$, then $|G| = 2^5$.*

P r o o f. See Theorem D, Theorem F and Theorem G of [2]. $\square$

Let a p-group G satisfy Hypothesis $(*)$. Obviously, there are three cases for $K_1 \cap K_2$: case (i) $L = 1$; case (ii) $1 < L < K_i$, $i = 1, 2$; and case (iii) $K_1 \leqslant K_2$ (or $K_2 \leqslant K_1$, but without loss of generality, we may assume $K_1 \leqslant K_2$). Next, we respectively discuss the structure of G according to the above cases. First, we have the following theorem.

Theorem 3.1. *A p-group G satisfies Hypothesis $(*)$ with $L = 1$ if and only if G is a 2-group of nilpotence class 2, $G/\mathbf{Z}(G)$ is elementary abelian, $\mathbf{Z}(G) \cong C_2 \times C_2$ and $|G'| = 2$.*

P r o o f. When G is a 2-group of nilpotence class 2, $G/\mathbf{Z}(G)$ is elementary abelian, $\mathbf{Z}(G) \cong C_2 \times C_2$ and $|G'| = 2$, by Lemma 2.2 (5) and since $\mathbf{Z}(G)$ is not cyclic, we know that G has exactly two nonlinear irreducible characters and both of them are non-faithful. It follows that $L = 1$ from Proposition 2.1.

Now, we assume that a p-group G satisfies Hypothesis $(*)$ and $L = 1$. First we have $p = 2$ and $|K_1| = |K_2| = 2$ by Lemma 3.1. Hence both K_1, K_2 are minimal normal subgroups of G and $K_1, K_2 \leqslant \mathbf{Z}(G)$. Using the fact that the nontrivial normal subgroups of G not containing G' are K_1 and K_2, we have that G satisfies the strong condition and so $c(G) \leqslant 3$. If $c(G) = 3$, then $G' \nleqslant \mathbf{Z}(G)$ and hence $\mathbf{Z}(G) = K_1$ or $\mathbf{Z}(G) = K_2$. When $\mathbf{Z}(G) = K_1$, we get that $K_2 \leqslant K_1$, contradicting that $L = 1$. Thus $\mathbf{Z}(G) \neq K_1$. For the same reason, $\mathbf{Z}(G) \neq K_2$. Therefore, we obtain that $c(G) = 2$. Also we have that $G/\mathbf{Z}(G)$ is elementary abelian and $\exp G' = 2$ by Lemma 3.2. Furthermore, G/K_i is extra-special, we can deduce that $\mathbf{Z}(G)/K_i = \mathbf{Z}(G/K_i) \cong C_2$, and hence $|\mathbf{Z}(G)| = 4$. We claim that $|G'| = 2$, otherwise, we must have that $\mathbf{Z}(G) = G' \cong C_2 \times C_2$. Thus we can obtain three normal subgroups of G which are different from one another and do not contain G'. That is impossible as the nontrivial normal subgroups of G not containing G' are K_1 and K_2. Then the claim follows. Therefore by Lemma 2.2 (5) we have that G has exactly two nonlinear irreducible characters. Thus G has no faithful irreducible characters, and so $\mathbf{Z}(G)$ is not cyclic, which implies that $\mathbf{Z}(G) \cong C_2 \times C_2$. $\square$

In the next Lemma, we give some properties of p-groups which satisfy Hypothesis $(*)$ and $L > 1$.

Lemma 3.3. *Let a finite p-group G satisfy Hypothesis $(*)$ and let $L > 1$. Then $L < G'$ and G satisfies the weak condition on normal subgroups.*

Proof. By Proposition 2.1 (iv), we have that $G' \cap L = 1$ or $L < G'$. If $G' \cap L = 1$, then $G' \cap K_i = 1$, $i = 1, 2$ by Proposition 2.1. Thus G' and L are all minimal normal subgroups of G and since G is a p-group, we have $L, G' \subseteq \mathbf{Z}(G)$. Since $\mathbf{Z}(G)$ is cyclic, we have that $L = G'$, a contradiction. So $L < G'$.

Next, we prove that G satisfies the weak condition. First, $L \leqslant \mathbf{Z}(G)$. And by Proposition 2.1 (iii), we only need to prove that $|K_i\mathbf{Z}(G) : \mathbf{Z}(G)| = |K_i/(K_i \cap \mathbf{Z}(G))| \leqslant p$, $i = 1, 2$. Since $L \leqslant K_i \cap \mathbf{Z}(G) \leqslant K_i$, and K_i/L are chief factors of G or $K_i/L = 1$, $i = 1, 2$, the proof follows. $\square$

In the rest of this paper, we consider the cases (ii) and (iii) stated above.

Theorem 3.2. *A p-group G satisfies Hypothesis $(*)$ with $1 < L < K_1$ and $1 < L < K_2$ if and only if G is a group of nilpotence class 3, order 32, and $\mathbf{Z}(G) \cong C_2$ or C_4.*

Proof. A computation in GAP using the GAP libraries shows that the groups G of order 32 and nilpotence class 3 are $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$, $(C_8 \rtimes C_2) \rtimes C_2$, $C_2((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2)(C_4 \times C_2)$, $(C_2 \times D_8) \rtimes C_2$, $(C_2 \times Q_8) \rtimes C_2$ with $\mathbf{Z}(G) \cong C_2$ and $(C_4 \times C_4) \rtimes C_2$, $C_4D_8 = C_4(C_4 \times C_2)$, $(C_8 \times C_2) \rtimes C_2$ with $\mathbf{Z}(G) \cong C_4$. Using GAP's program to compute character tables we verified that in each case Hypothesis $(*)$ holds for G with $1 < L < K_1$ and $1 < L < K_2$.

Now we assume that G satisfies Hypothesis $(*)$, and $1 < L < K_1$, $1 < L < K_2$. First, $p = 2$ by Lemma 3.1. Notice that $L < G'$ by Lemma 3.3. Then L is the unique minimal normal subgroup of G. And we have that the nilpotence class satisfies $c(G) \leqslant 4$ by Lemma 3.2.

If $c(G) = 2$, then $G' \leqslant \mathbf{Z}(G)$ and so G' is cyclic. By Lemma 3.2, it follows that $\exp G' = 2$ or 2^2. We must have $\exp G' = 2^2$, otherwise, $|L| = |G'| = 2$ and since $L, G' \subseteq \mathbf{Z}(G)$, we get that $L = G'$, a contradiction. Thus $G' \cong C_4$ and $G/\mathbf{Z}(G) \cong C_4 \times C_4$. Note that G/L has exactly two nonlinear irreducible characters.

By Lemma 2.2 and $c(G/L) = 2$, it follows that $|\mathbf{Z}(G/L)| = 4$ and then $|\mathbf{Z}(G)/L| \leqslant 4$, $|\mathbf{Z}(G)| \leqslant 8$. Also, by [1], Theorem 6, page 281 we have that $|G/L| = 2^{2m}$, $m \in \mathbb{Z}$. Since $G' \leqslant \mathbf{Z}(G)$, it follows that $|\mathbf{Z}(G)| = 8$ or 4. If $|\mathbf{Z}(G)| = 4$, then $|G| = 2^6$ and $|G/L| = 2^5$, a contradiction. Thus $\mathbf{Z}(G) \cong C_8$. Also, note that G/K_1 has only one nonlinear irreducible character. By Lemma 2.1 (Seitz's Theorem), it follows that

$$2 = |\mathbf{Z}(G/K_1)| = |\mathbf{Z}(\chi_1) : K_1|.$$

Since $|K_1| = 4$, it follows that $|\mathbf{Z}(\chi_1)| = 8$. Therefore $\mathbf{Z}(G) = \mathbf{Z}(\chi_1) \supset K_1$. Since $K_1 \subseteq \mathbf{Z}(G)$ and $G' \subseteq \mathbf{Z}(G)$, we have that $K_1 = G'$ as they have the same order, a contradiction.

If $c(G) = 4$, by Lemma 3.2 it follows that $|G| = 2^5$, that is, G has maximal class. And then G' has index 4 in G and so $|G'| = 8$. Since G/L has exactly two nonlinear irreducible characters $\widehat{\chi}_1$, $\widehat{\chi}_2$ and $\ker\widehat{\chi}_i = K_i/L \neq \overline{1}$ for $i = 1, 2$, it follows that $\mathbf{Z}(G/L)$ is not cyclic and hence $|(G/L)'| = |G'/L| = 2$ by Lemma 2.2. So $|G'| = 4$. We arrive at a contradiction.

The remaining case is $c(G) = 3$. By Lemma 3.2 it follows that $|G : \mathbf{Z}(G)| = 2^3$ or 2^4. Note that $G' \nleq \mathbf{Z}(G)$. So $\mathbf{Z}(G) \in \{L, K_1, K_2\}$ by Proposition 2.1 (iii). Thus $|\mathbf{Z}(G)| = 2$ or 2^2. Again by $|G/L| = 2^{2m}$, we have that $|G| = 2^5$. $\square$

Theorem 3.3. *A p-group G satisfies Hypothesis $(*)$ and $K_1 \leqslant K_2$ if and only if G is a group of nilpotence class 3 and order 3^4.*

P r o o f. First assume that G satisfies Hypothesis $(*)$ and $K_1 \leqslant K_2$. Then $L = K_1$ and so by Proposition 2.1 we have that $\mathbf{Z}(G)$ cyclic and K_1 is a minimal normal subgroup of G with $K_1 \leqslant \mathbf{Z}(G)$. Hence $|K_1| = p$. Moreover, Lemma 3.3 and Lemma 3.2 show that $K_1 < G'$ and $c(G) \leqslant 4$.

Assume that $K_1 \lneq K_2$. Then $\mathrm{Irr}_1(G/K_2) = \{\widehat{\chi}_2\}$, where $\mathrm{Irr}_1(G/K_2)$ denotes the set of nonlinear irreducible characters of G/K_2. So $p = 2$ by Lemma 2.1. Moreover, we have $\mathrm{Irr}_1(G/K_1) = \{\widehat{\chi}_1, \widehat{\chi}_2\}$. It follows that $c(G/K_1) = 2$ or 3 by Lemma 2.2.

Case (1), when $c(G/K_1) = 3$.

If $c(G) = 4$, then $|G| = 2^5$ by Lemma 3.2, and so $|G/K_1| = 2^4$. But groups with order 2^4 and nilpotence class 3 have three nonlinear irreducible characters, which is a contradiction.

If $c(G) = 3$, then $|G/\mathbf{Z}(G)| = 2^3$ or 2^4 by Lemma 3.2. Note that $G' \nleq \mathbf{Z}(G)$, hence $\mathbf{Z}(G) = K_1$ or K_2. If $\mathbf{Z}(G) = K_1$, then $|\mathbf{Z}(G)| = 2$ and so $|G| = 2^4$ or 2^5. When $|G| = 2^4$, since $c(G) = 3$, it follows that G has just one nonlinear non-faithful irreducible character. This is a contradiction. When $|G| = 2^5$, we would obtain a contradiction in the same way as in the situation of $c(G) = 4$. If $\mathbf{Z}(G) = K_2$, then G satisfies the strong condition on normal subgroups. By Lemma 3.2, we have

$|G| = 2^4$ which is not possible as above. Since $c(G/K_1) = 3$, it is not possible that $c(G) = 2$.

Case (2), when $c(G/K_1) = 2$.

By Lemma 2.2, we get $|G'/K_1| = 2$ and $|\mathbf{Z}(G/K_1)| = 4$. Since G/K_1 has a faithful nonlinear irreducible character $\widehat{\chi}_1$, it follows that $\mathbf{Z}(G/K_1) = C_4$. Therefore, we deduce that G'/K_1 is the unique minimal normal subgroup of G/K_1. Thus we get that all nonlinear irreducible characters of G/K_1 are faithful, contradicting the fact that $\widehat{\chi}_2$ is non-faithful for G/K_1.

Therefore we get that $K_1 = K_2$.

Write $K_1 = K_2 = K$. Then K is the only nontrivial normal subgroup of G which does not contain G'. Since $K \leqslant \mathbf{Z}(G)$, it follows that G satisfies the strong condition on normal subgroups and hence $c(G) \leqslant 3$. If $c(G) = 2$, then G' is cyclic as $G' \leqslant \mathbf{Z}(G)$. Therefore $|G'| = p$ as $\exp G' = p$. It follows that $K = G'$, a contradiction. So $c(G) = 3$. If $p = 2$, then $|G| = 2^4$ by Lemma 3.2. That is impossible as above. Hence $p \neq 2$. Note that G/K has exactly two nonlinear irreducible characters. Then G/K must be an extra-special 3-group by Lemma 2.2 and so $p = 3$. Moreover, by Lemma 3.2, we have $|G : \mathbf{Z}(G)| = 3^3$. Since $G' \nleqslant \mathbf{Z}(G)$ and by Proposition 2.1 (iii), we obtain that $K = \mathbf{Z}(G)$. It follows that $|\mathbf{Z}(G)| = 3$. Hence $|G| = 3^4$ and G has maximal class.

Conversely, assume that $|G| = 3^4$ and $c(G) = 3$. Then $|\mathbf{Z}(G)| = 3$ and $G/\mathbf{Z}(G)$ is an extra-special 3-group by the properties of groups of order 3^4 with maximal class. Hence $G/\mathbf{Z}(G)$ has exactly two nonlinear irreducible characters by Lemma 2.2 and they are faithful. It indicates that there are $\chi_1, \chi_2 \in \mathrm{Irr}_1(G)$ and $\ker\chi_1 = \ker\chi_2 = \mathbf{Z}(G)$. Since $\ker\chi_1 \cap \ker\chi_2 \neq 1$, it follows that G must have $\varphi \in \mathrm{Irr}_1(G)$ and $\varphi \neq \chi_1, \chi_2$. Suppose $\ker\varphi \neq 1$. Since $\mathbf{Z}(G) \leqslant \ker\varphi$, it follows that $\varphi \in \mathrm{Irr}_1(G/\mathbf{Z}(G))$. So we find three nonlinear irreducible characters in $G/\mathbf{Z}(G)$, which is impossible. Consequently, φ must be faithful and so G has exactly two nonlinear non-faithful irreducible characters. $\square$

Finally, the proof of Theorem 1.1 in Introduction is immediately available by Theorems 3.1, 3.2 and 3.3.

Acknowledgement. The authors are very much indebted to the anonymous referee for his comments that improved significantly the presentation of the paper.

References

[1] *Ya. G. Berkovich, E. M. Zhmud'*: Characters of Finite Groups. Part 2. Translations of Mathematical Monographs 181, American Mathematical Society, Providence, 1998. zbl MR

[2] *G. A. Fernández-Alcober, A. Moretó*: Groups with two extreme character degrees and their normal subgroups. Trans. Am. Math. Soc. *353* (2001), 2171–2192. zbl MR doi

[3] The GAP Group. GAP—Groups, Algorithms, and Programming, Version 4.8.3 (2016), `http://www.gap-system.org`.

[4] *A. Iranmanesh, A. Saeidi*: Finite groups with a unique nonlinear nonfaithful irreducible character. Arch. Math., Brno *47* (2011), 91–98. zbl MR

[5] *I. M. Isaacs*: Character Theory of Finite Groups. Pure and Applied Mathematics 69, Academic Press, New York, 1976. zbl MR doi

[6] *A. Saeidi*: Classification of solvable groups possessing a unique nonlinear non-faithful irreducible character. Cent. Eur. J. Math. *12* (2014), 79–83. zbl MR doi

[7] *G. M. Seitz*: Finite groups having only one irreducible representation of degree greater than one. Proc. Am. Math. Soc. *19* (1968), 459–461. zbl MR doi

[8] *H. Wang, X. Chen, J. Zeng*: Zeros of Brauer characters. Acta Math. Sci., Ser. B, Engl. Ed. *32* (2012), 1435–1440. zbl MR doi

[9] *G. X. Zhang*: Finite groups with exactly two nonlinear irreducible characters. Chin. Ann. Math., Ser. A *17* (1996), 227–232. (In Chinese.) zbl MR

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