

LITTLEWOOD-PALEY CHARACTERIZATION OF
HÖLDER-ZYGMUND SPACES ON STRATIFIED LIE GROUPS

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Abstract. We give a characterization of the Hölder-Zygmund spaces $\mathcal{C}^\sigma(G)$ ($0 < \sigma < \infty$) on a stratified Lie group G in terms of Littlewood-Paley type decompositions, in analogy to the well-known characterization of the Euclidean case. Such decompositions are defined via the spectral measure of a sub-Laplacian on G , in place of the Fourier transform in the classical setting. Our approach mainly relies on almost orthogonality estimates and can be used to study other function spaces such as Besov and Triebel-Lizorkin spaces on stratified Lie groups.

Keywords: stratified Lie group; Hölder-Zygmund space; Littlewood-Paley decomposition

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1. INTRODUCTION

The classical Hölder-Zygmund spaces $\mathcal{C}^\sigma(\mathbb{R}^d)$ ($0 < \sigma < \infty$) on the Euclidean space \mathbb{R}^d play an important role in harmonic analysis and partial differential equations. Let us first recall the definition of these spaces. For $0 < \sigma \leq 1$, $\mathcal{C}^\sigma(\mathbb{R}^d)$ is defined to be the space of all bounded continuous functions $f: \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$\|f\|_{\mathcal{C}^\sigma(\mathbb{R}^d)} := \begin{cases} \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x \in \mathbb{R}^d} \sup_{y \in \mathbb{R}^d \setminus \{0\}} \frac{|f(x+y) - f(x)|}{|y|^\sigma}, & 0 < \sigma < 1, \\ \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x \in \mathbb{R}^d} \sup_{y \in \mathbb{R}^d \setminus \{0\}} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|}, & \sigma = 1, \end{cases}$$

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is finite. For $\sigma = k + \sigma'$ where $k = 1, 2, \dots$ and $0 < \sigma' \leq 1$, $\mathcal{C}^\sigma(\mathbb{R}^d)$ is defined to be the space of all C^k functions $f: \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$\|f\|_{\mathcal{C}^\sigma(\mathbb{R}^d)} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{\mathcal{C}^{\sigma'}(\mathbb{R}^d)} < \infty.$$

It is well-known that the spaces $\mathcal{C}^\sigma(\mathbb{R}^d)$ ($0 < \sigma < \infty$) can be characterized in terms of the Littlewood-Paley decomposition. To recall such a characterization, choose $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\text{supp } \mathcal{F}\psi_0 \subset \{\xi \in \mathbb{R}^d: |\xi| \leq 2\} \quad \text{and} \quad |\mathcal{F}\psi_0(\xi)| \geq c \quad \text{on} \quad \{|\xi| \leq 5/3\},$$

and

$$\text{supp } \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^d: 1/2 \leq |\xi| \leq 2\} \quad \text{and} \quad |\mathcal{F}\psi(\xi)| \geq c \quad \text{on} \quad \{3/5 \leq |\xi| \leq 5/3\},$$

where c is a positive constant, and \mathcal{F} is the Fourier transform operator. For $j = 1, 2, \dots$, we set

$$\psi_j(x) := 2^{jd} \psi(2^j x), \quad x \in \mathbb{R}^d.$$

Then the Littlewood-Paley characterization of $\mathcal{C}^\sigma(\mathbb{R}^d)$ ($0 < \sigma < \infty$) can be stated as follows. For every $f \in \mathcal{C}^\sigma(\mathbb{R}^d)$, one has the estimate

$$\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|f * \psi_j\|_{L^\infty(\mathbb{R}^d)} \leq C \|f\|_{\mathcal{C}^\sigma(\mathbb{R}^d)},$$

where C is a positive constant independent of f . Conversely, every distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ that satisfies $\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|f * \psi_j\|_{L^\infty(\mathbb{R}^d)} < \infty$ can be identified with an element of $\mathcal{C}^\sigma(\mathbb{R}^d)$, and for such f one has the estimate

$$\|f\|_{\mathcal{C}^\sigma(\mathbb{R}^d)} \leq C' \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|f * \psi_j\|_{L^\infty(\mathbb{R}^d)},$$

where C' is also a positive constant independent of f . See, e.g., [9] and [15].

In the 1970s, Folland in [3] generalized the classical Hölder-Zygmund spaces to the setting of stratified Lie groups. To recall the definition of these spaces, we need first to recall some basic notions concerning stratified Lie groups. A Lie group G is called a stratified Lie group if it is connected and simply connected, and its Lie algebra \mathfrak{g} can be decomposed as a direct sum $\mathfrak{g} = V_1 \oplus \dots \oplus V_m$, with $[V_1, V_k] = V_{k+1}$ for $1 \leq k \leq m-1$ and $[V_1, V_m] = 0$. Such a group G is necessarily nilpotent, and thus the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a diffeomorphism which takes the Lebesgue measure on \mathfrak{g} to a bi-invariant Haar measure dx on G . The group identity of G will

be referred to as the origin and denoted by 0. A typical example of stratified Lie groups is the Heisenberg group \mathbb{H}^n .

The algebra \mathfrak{g} is equipped with a natural family of dilations $\{\delta_t\}_{t>0}$ which are the algebra automorphisms defined by

$$\delta_t\left(\sum_{j=1}^m Z_j\right) = \sum_{j=1}^m t^j Z_j \quad (Z_j \in V_j).$$

Under the identification of G with \mathfrak{g} (via the exponential map), δ_t may also be viewed as a map from G to G . We generally write tx instead of $\delta_t(x)$, for $x \in G$. The number $Q := \sum_{j=1}^m j(\dim V_j)$ is called the homogeneous dimension of G .

A homogeneous norm on G is a continuous function $x \mapsto |x|$ from G to $[0, \infty)$ which vanishes only at 0 and satisfies that $|x^{-1}| = |x|$ and $|\delta_t(x)| = t|x|$ for all $x \in G$ and $t > 0$. It is shown in [5] that there exists at least one homogeneous norm on G and any two homogeneous norms on G are equivalent. Henceforth we fix a homogeneous norm on G . It satisfies a triangle inequality: there exists a constant $\gamma \geq 1$ such that

$$(1.1) \quad |xy| \leq \gamma(|x| + |y|)$$

for all $x, y \in G$.

The elements of \mathfrak{g} will be considered as left-invariant vector fields on G . We fix once and for all a basis $\{X_1, \dots, X_{n_1}\}$ for $V_1 \subset \mathfrak{g}$. Then the operator

$$\mathcal{L} = - \sum_{j=1}^{n_1} X_j^2$$

is called the sub-Laplacian on G . Let

$$\mathcal{I}(n_1) = \bigcup_{k \in \mathbb{N} \cup \{0\}} \{1, \dots, n_1\}^k$$

be the set of multi-indices I with values in $\{1, \dots, n_1\}$, of arbitrary length. For $I = (i_1, \dots, i_k) \in \{1, \dots, n_1\}^k \subset \mathcal{I}(n_1)$, we set $|I| = k$ and

$$X_I = X_{i_1} \dots X_{i_k},$$

with the convention $X_I = \text{id}$ if $I \in \{1, \dots, n_1\}^0 = \emptyset$.

Now let us recall from [3] the definition of the Hölder-Zygmund spaces $\mathcal{C}^\sigma(G)$ ($0 < \sigma < \infty$) on the stratified Lie group G . See also [4]. For $0 < \sigma \leq 1$, $\mathcal{C}^\sigma(G)$ is defined to be the space of all bounded continuous functions $f: G \rightarrow \mathbb{C}$ such that

$$\|f\|_{\mathcal{C}^\sigma(G)} := \begin{cases} \|f\|_{L^\infty(G)} + \sup_{x \in G} \sup_{y \in G \setminus \{0\}} \frac{|f(xy) - f(x)|}{|y|^\sigma}, & 0 < \sigma < 1, \\ \|f\|_{L^\infty(G)} + \sup_{x \in G} \sup_{y \in G \setminus \{0\}} \frac{|f(xy) + f(xy^{-1}) - 2f(x)|}{|y|}, & \sigma = 1, \end{cases}$$

is finite. For $\sigma = k + \sigma'$ where $k = 1, 2, \dots$ and $0 < \sigma' \leq 1$, $\mathcal{C}^\sigma(G)$ is defined to be the space of all C^k functions $f: G \rightarrow \mathbb{C}$ such that

$$(1.2) \quad \|f\|_{\mathcal{C}^\sigma(G)} := \sum_{I \in \mathcal{I}(n_1), |I| \leq k} \|X_I f\|_{\mathcal{C}^{\sigma'}(G)} < \infty.$$

Note that in [3] the spaces $\mathcal{C}^\sigma(G)$ defined above are called Lipschitz spaces, and are denoted by $\Gamma_\alpha(G)$ there.

The purpose of this paper is to give a Littlewood-Paley characterization for the spaces $\mathcal{C}^\sigma(G)$ which is analogous to that of the classical Hölder-Zygmund spaces $\mathcal{C}^\sigma(\mathbb{R}^d)$. The Littlewood-Paley operators in our setting will be defined via the spectral measure of the sub-Laplacian \mathcal{L} . Note that when restricted to $C_0^\infty(G)$, \mathcal{L} is essentially self-adjoint. Its closure has the domain $\mathcal{D} = \{u \in L^2(G) : \mathcal{L}u \in L^2(G)\}$, where $\mathcal{L}u$ is a derivative in the sense of distributions. This closure is the unique self-adjoint extension of $\mathcal{L}|_{C_0^\infty(G)}$. We denote this extension also by the symbol \mathcal{L} . It admits a spectral resolution

$$\mathcal{L} = \int_0^\infty \lambda \, dE_\lambda,$$

where dE_λ is the spectral measure. Any bounded, Borel measurable function \widehat{K} on $[0, \infty)$ defines a bounded operator

$$\widehat{K}(\mathcal{L}) = \int_0^\infty \widehat{K}(\lambda) \, dE_\lambda$$

on $L^2(G)$. As shown in [2], page 76, the spectral measure of $\{0\}$ vanishes, so the point $\lambda = 0$ may be neglected in the spectral resolution, and we should regard \widehat{K} as a function on $(0, \infty)$ rather than on $[0, \infty)$. Since the operator $\widehat{K}(\mathcal{L})$ is bounded on $L^2(G)$ and commutes with left translations, it follows from the Schwartz kernel theorem that there exists a convolution distribution kernel $K \in \mathcal{S}'(G)$ such that

$$\widehat{K}(\mathcal{L})f = f * K \quad \forall f \in \mathcal{S}(G),$$

where $\mathcal{S}(G)$ and $\mathcal{S}'(G)$ are respectively the Schwartz space on G and the distribution space on G , whose definitions will be recalled in Section 2 below.

Let $\mathbb{R}^+ := (0, \infty)$. Denote by $\mathcal{S}(\mathbb{R}^+)$ the space of all smooth functions $\widehat{\Phi}$ on \mathbb{R}^+ be such that for every nonnegative integer k , $\widehat{\Phi}^{(k)}(\lambda)$ decays rapidly as $\lambda \rightarrow +\infty$ and converges to some finite number as $\lambda \rightarrow 0^+$, where $\widehat{\Phi}^{(k)}$ is the k th order derivative of $\widehat{\Phi}$. An important fact, which was originally given in [11], says that if $\widehat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$ then the convolution kernel Φ associated to $\widehat{\Phi}(\mathcal{L})$ is in $\mathcal{S}(G)$. Due to this fact, if $\widehat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$ then one naturally enlarges the domain of $\widehat{\Phi}(\mathcal{L})$ from $L^2(G)$ to $\mathcal{S}'(G)$:

$$\widehat{\Phi}(\mathcal{L})f := f * \Phi \quad \forall f \in \mathcal{S}'(G).$$

The main result of the present paper is the following:

Theorem 1.1. *Let $\widehat{\Psi}_0, \widehat{\Psi} \in \mathcal{S}(\mathbb{R}^+)$ be such that $\text{supp } \widehat{\Psi}_0 \in [0, 4]$, $\text{supp } \widehat{\Psi} \subset [1/4, 4]$, and*

$$(1.3) \quad [\widehat{\Psi}_0(\lambda)]^2 + \sum_{j=1}^{\infty} [\widehat{\Psi}(2^{-2j}\lambda)]^2 = 1 \quad \forall \lambda \in \mathbb{R}^+.$$

For $j = 1, 2, \dots$, we set

$$(1.4) \quad \widehat{\Psi}_j(\lambda) := \widehat{\Psi}(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}^+.$$

Then for every $f \in \mathcal{C}^\sigma(G)$ we have the estimate

$$(1.5) \quad \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \leq C \|f\|_{\mathcal{C}^\sigma(G)},$$

where C is a positive constant independent of f . Conversely, every distribution $f \in \mathcal{S}'(G)$ that satisfies $\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty$ can be identified with an element of $\mathcal{C}^\sigma(G)$, and for such f we have the estimate

$$(1.6) \quad \|f\|_{\mathcal{C}^\sigma(G)} \leq C' \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)},$$

where C' is also a positive constant independent of f .

Folland in [4] established a characterization of $\mathcal{C}^\sigma(G)$ in terms of Poisson integrals, which may be thought of as an earlier version of the Littlewood-Paley characterization of $\mathcal{C}^\sigma(G)$. Different from [4], the Littlewood-Paley operators used in the present paper are built via the spectral measure associated to the sub-Laplacian, and the

convolution kernels associated to our Littlewood-Paley operators are Schwartz functions on G . It should be mentioned that Saka [14] first introduced Besov spaces $\Lambda(\alpha, p, q)$ ($\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$) on stratified groups, by using the heat semigroup. Giulini in [8] defines Besov spaces $B_{p,q}^\alpha$ ($\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$) using a Littlewood-Paley type decomposition on stratified groups, and proves that $B_{p,q}^s$ coincide with $\Lambda(\alpha, p, q)$ introduced by Saka. The main result of [8] is a characterization of $B_{p,q}^\alpha$ by their approximation theoretic properties. But the result of [8] seems not to yield the equivalence between $B_{\infty,\infty}^\sigma$ and $\mathcal{C}^\sigma(G)$ directly. Recently, Führ and Mayeli in [6] studied homogeneous Besov spaces $\dot{B}_{p,q}^\alpha$ on stratified groups. Among other things, they established wavelet characterizations of $\dot{B}_{p,q}^\alpha$. However, the relation between $\dot{B}_{\infty,\infty}^\alpha$ and the homogeneous Hölder-Zygmund spaces is not investigated in [6]. It is also worth pointing out that Furioli et al. [7] recently introduced Besov spaces on Lie groups of polynomial growth, by means of the Littlewood-Paley decomposition related to the sub-Laplacian. So it is an interesting problem whether the spaces $B_{\infty,\infty}^\alpha$ in such a more general setting coincide with (appropriately defined) Hölder-Zygmund spaces.

The rest of the paper is organized as follows. In Section 2, we recall some known results on stratified Lie groups. In Section 3, we give an almost orthogonality estimate and use it to derive a Calderón type reproducing formula. In Section 4, we give the proof of our main result, Theorem 1.1.

Convention: Throughout the paper, C denotes a positive constant which is independent of the main variable quantities involved but whose value may vary from one occurrence to the next. For two variable quantities a and b , if $a \leq Cb$, then we write $a \lesssim b$ or $b \gtrsim a$. If both $a \lesssim b$ and $b \lesssim a$ are valid, then we write $a \sim b$. The set of all strictly positive integers is denoted by \mathbb{N} , and the set of all strictly positive real numbers will be denoted by \mathbb{R}^+ . For any $\sigma > 0$, $\lfloor \sigma \rfloor$ denotes the largest integer less than or equal to σ . If $\alpha, \beta \in \mathbb{R}$, we use $\alpha \wedge \beta$ to denote the number $\min\{\alpha, \beta\}$.

2. SOME KNOWN RESULTS ON STRATIFIED LIE GROUPS

Recall that we have fixed a basis $\{X_1, \dots, X_{n_1}\}$ for $V_1 \subset \mathfrak{g}$. We now let $\{X_{n_1+1}, \dots, X_{n_2}\}$ be a basis for V_2 , $\{X_{n_2+1}, \dots, X_{n_3}\}$ be a basis for V_3 , and so on, so that we obtain a basis $\{X_1, \dots, X_n\}$ for \mathfrak{g} adapted to the stratification. A complex-valued function P on G is called a polynomial on G if $P \circ \exp$ is a polynomial on the vector space $\mathfrak{g} \equiv \mathbb{R}^n$. Let ξ_1, \dots, ξ_n be the basis for the linear forms on \mathfrak{g} dual to the basis X_1, \dots, X_n for \mathfrak{g} , and set $\eta_j = \xi_j \circ \exp^{-1}$, $j = 1, \dots, n$. Then η_1, \dots, η_n are generators of the algebra of polynomials on G . Thus, every polynomial

on G can be written uniquely as

$$(2.1) \quad P = \sum_{l_1, \dots, l_n \in \mathbb{N} \cup \{0\}} a_{l_1, \dots, l_n} \eta_1^{l_1} \dots \eta_n^{l_n}, \quad a_{l_1, \dots, l_n} \in \mathbb{C},$$

where all but finitely many of the coefficients a_{l_1, \dots, l_n} vanish. A polynomial of the type (2.1) is said to be of homogeneous degree M , where $M \in \mathbb{N} \cup \{0\}$, if the inequality

$$\sum_{k=1}^n d_k l_k \leq M$$

holds for all those multi-indices (l_1, \dots, l_n) for which $a_{l_1, \dots, l_n} \neq 0$, where each d_k is a positive integer given by

$$(2.2) \quad d_k := j \quad \text{if } X_k \in V_j.$$

For $M \in \mathbb{N} \cup \{0\}$, we let \mathcal{P}_M denote the space of polynomials on G of homogeneous degree M . A function $f: G \rightarrow \mathbb{C}$ is said to have vanishing moments of order M , where $M \in \mathbb{N}$, if

$$\int_G f(x) P(x) dx = 0 \quad \forall P \in \mathcal{P}_{M-1},$$

with the absolute convergence of the integral.

The convolution of two functions f, g on G is defined by

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy = \int_G f(xy^{-1}) g(y) dy,$$

provided that the integrals converge absolutely. For $j = 1, \dots, n_1$, we let Y_j denote the right-invariant vector field which coincides with X_j at the origin. For $I = \{i_1, \dots, i_k\} \in \{1, \dots, n_1\}^k \subset \mathcal{I}(n_1)$, we set $\tilde{I} = \{i_k, \dots, i_1\}$, $X_I = X_{i_1} \dots X_{i_k}$ and $Y_I = Y_{i_1} \dots Y_{i_k}$. The operators X_I and Y_I interact the convolution in the following way:

$$(2.3) \quad X_I(f * g) = f * (X_I g), \quad Y_I(f * g) = (Y_I f) * g, \quad (X_I f) * g = f * (Y_{\tilde{I}} g).$$

If f is a function on G , we define the reflection of f by $\tilde{f}(x) = f(x^{-1})$, $x \in G$. Then we have

$$(2.4) \quad X_I \tilde{f} = (-1)^{|I|} \widetilde{Y_I f}.$$

We now recall the definition of Taylor polynomials of a function on G . Let $M \in \mathbb{N} \cup \{0\}$, $f \in C^M(G)$ and $x \in G$. The (left) Taylor polynomial of f at x of

homogeneous degree M is defined to be the unique polynomial $P_{x,M}^{(f)}(\cdot) \in \mathcal{P}_M$ such that $X_I f(0) = X_I P_{x,M}^{(f)}(0)$ for all multi-indices $I \in \mathcal{I}(n_1)$ with $|I| \leq M$. Note that $P_{x,0}^{(f)}(\cdot) \equiv f(x)$. The following stratified Taylor inequality will be frequently used.

Proposition 2.1 ([5], Corollary 1.44). *For every $M \in \mathbb{N}$, there is a constant C_M (depending on M) such that for all $f \in C^M(G)$ and $x, y \in G$,*

$$|f(xy) - P_{x,M-1}^{(f)}(y)| \leq C_M |y|^M \sup_{\substack{I \in \mathcal{I}(n_1), |I|=M \\ |z| \leq b^M |y|}} |(X_I f)(xz)|,$$

where b is a positive constant independent of M, f, x and y . In particular,

$$|f(xy) - f(x)| \leq C |y| \sup_{\substack{1 \leq k \leq n_1 \\ |z| \leq b |y|}} |(X_k f)(xz)|.$$

We denote by $\mathcal{S}(G)$ the space of all functions f on G such that $f \circ \exp^{-1} \in \mathcal{S}(\mathfrak{g}) \equiv \mathcal{S}(\mathbb{R}^n)$. As pointed out in [5], page 35, $\mathcal{S}(G)$ is a Fréchet space and several different choices of families of norms induce the same topology on $\mathcal{S}(G)$. In this paper, for our purpose it will be convenient to use the following family: for any $\Phi \in \mathcal{S}(G)$ and $M \in \mathbb{N} \cup \{0\}$, we define

$$\|\Phi\|_{\mathcal{S}_M} := \sup_{\substack{I \in \mathcal{I}(n_1), |I| \leq M \\ x \in G}} (|X_I \Phi(x)| + |Y_I \Phi(x)|)(1 + |x|)^{Q+M+|I|}.$$

It follows immediately from (2.4) that $\|\tilde{\Phi}\|_{\mathcal{S}_M} = \|\Phi\|_{\mathcal{S}_M}$. The dual space $\mathcal{S}'(G)$ of $\mathcal{S}(G)$ is called the distribution space on G . For $f \in \mathcal{S}'(G)$ and $\Phi \in \mathcal{S}(G)$, we shall denote the evaluation of f on Φ by $\langle f, \Phi \rangle$.

For any function f on G and $t > 0$, the L^1 -normalized dilation of f is defined by

$$D_t f(x) = t^Q f(tx), \quad x \in G.$$

The 2-homogeneity of \mathcal{L} implies the following fact: if $\widehat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$ and if Φ denotes the convolution kernel associated to $\widehat{\Phi}(\mathcal{L})$, then the convolution kernel associated to $\widehat{\Phi}(t^{-2}\mathcal{L})$ coincides with $D_t \Phi$.

For any $x \in G$ and $r > 0$, we define the ball centered at x of radius r by $B(x, r) = \{y \in G: |x^{-1}y| < r\}$. Denote by $|E|$ the Haar measure of any measurable $E \subset G$. Since $d(rx) = r^Q dx$, we have $|B(x, r)| = c_0 r^Q$ for all $x \in G$ and $r > 0$, where c_0 is a positive constant. Consequently, G satisfies the volume doubling condition, namely, there is a constant C such that $|B(x, 2r)| \leq C|B(x, r)|$ for all $x \in G$ and $r > 0$.

The heat kernel h_t on G is, by definition, the convolution kernel associated to the heat semigroup $e^{-t\mathcal{L}}$, i.e.,

$$e^{-t\mathcal{L}}f(x) = f * h_t(x), \quad x \in G, \quad t > 0.$$

By [16], Theorem IV.4.2, h_t and its derivatives satisfy the following Gaussian upper bound estimate: for any multi-index $I \in \mathcal{I}(n_1)$, there exist constants C, c such that

$$(2.5) \quad |X_I h_t(x)| \leq C t^{-(|I|+Q)/2} \exp\left(-\frac{|x|^2}{ct}\right), \quad x \in G, \quad t > 0.$$

This estimate together with Proposition 2.1 yields that h_t also satisfies the following Hölder continuity estimate: there exist constants C', c' such that for all $t > 0$ and all $x, x' \in G$ with $|x^{-1}x'| \leq (2b\gamma)^{-1}\sqrt{t}$, where b is the constant from Proposition 2.1 and γ is the constant from (1.1), we have

$$(2.6) \quad |h_t(x) - h_t(x')| \leq C' \left(\frac{|x^{-1}x'|}{\sqrt{t}} \right) t^{-Q/2} \exp\left(-\frac{|x|^2}{c't}\right).$$

We have seen that G satisfies the volume doubling condition, and the heat kernel h_t associated to the sub-Laplacian \mathcal{L} satisfies the Gaussian upper bound estimate and the Hölder continuity estimate. Hence the general theory developed by Kerkycharian and Petrushev in [12] can be applied to our setting. In particular, the following smooth functional calculus induced by the heat kernel is valid. (See also the remarks in [10], page 292.)

Proposition 2.2 ([12], Theorem 3.4). *For any $N \in \mathbb{N}$ with $N \geq Q+1$ there exists a constant C_N (depending on N) such that for all $\widehat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$, $t > 0$ and $x \in G$, we have*

$$|K_{\widehat{\Phi}(t^2\mathcal{L})}(x)| \leq C \|\widehat{\Phi}\|_{(N)} t^{-Q} (1 + t^{-1}|x|)^{-N},$$

where $K_{\widehat{\Phi}(t^2\mathcal{L})}(\cdot)$ denotes the convolution kernel associated to $\Phi(t^2\mathcal{L})$, and $\|\widehat{\Phi}\|_{(N)}$ is defined by

$$\|\widehat{\Phi}\|_{(N)} := \sup_{\lambda \in \mathbb{R}^+, 0 \leq k \leq N} (1 + \lambda)^{N+Q+1} |\widehat{\Phi}^{(k)}(\lambda)|.$$

Finally, we recall a result from [3]: if $\alpha \in \mathbb{C}$, $\alpha \neq 0$, and $0 < r < R < \infty$, then there exists a constant C such that

$$(2.7) \quad \int_{r \leq |x| \leq R} |x|^{-Q+\alpha} dx = C \alpha^{-1} (R^\alpha - r^\alpha).$$

From this we immediately see that $(1 + |\cdot|)^{-N} \in L^1(G)$ if and only if $N > Q$.

3. AN ALMOST ORTHOGONALITY ESTIMATE AND A CALDERÓN TYPE REPRODUCING FORMULA

The following almost orthogonality estimate will be frequently used.

Lemma 3.1. *Suppose $\Phi, \Psi \in \mathcal{S}(G)$ and both of them have vanishing moments of order M , where $M \in \mathbb{N}$. Then for any $0 < \varepsilon < 1$ there is a constant $C > 0$ such that for all $j, k \in \mathbb{Z}$,*

$$(3.1) \quad |\Phi_j * \Psi_k(x)| \leq C \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-|j-k|(M-\varepsilon)} \frac{2^{-(j \wedge k)M}}{(2^{-(j \wedge k)} + |x|)^{Q+M}},$$

where $\Phi_j(x) := D_{2^j} \Phi(x)$, $\Psi_k(x) := D_{2^k} \Psi(x)$, and $j \wedge k := \min\{j, k\}$.

Proof. We first consider the case $j \leq k$. Let $P_{x, M-1}^{(\Phi_j)}(\cdot) \in \mathcal{P}_{M-1}$ be the (left) Taylor polynomial of Φ_j at x of homogeneous degree $M-1$. By the vanishing moment condition on Ψ_k , we has

$$\begin{aligned} |\Phi_j * \Psi_k(x)| &= \left| \int_G [\Phi_j(xy^{-1}) - P_{x, M-1}^{(\Phi_j)}(y^{-1})] \Psi_k(y) dy \right| \\ &\leq \int_{|y| \leq (2^{-j} + |x|)/(2\gamma b^M)} |\Phi_j(xy^{-1}) - P_{x, M-1}^{(\Phi_j)}(y^{-1})| |\Psi_k(y)| dy \\ &\quad + \int_{|y| \geq (2^{-j} + |x|)/(2\gamma b^M)} |\Phi_j(xy^{-1})| |\Psi_k(y)| dy \\ &\quad + \int_{|y| \geq (2^{-j} + |x|)/(2\gamma b^M)} |P_{x, M-1}^{(\Phi_j)}(y^{-1})| |\Psi_k(y)| dy \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

By Proposition 2.1 and (2.7), we have

$$\begin{aligned} I_1 &\lesssim \int_{|y| \leq (2^{-j} + |x|)/(2\gamma b^M)} |\Psi_k(y)| |y|^M \sup_{\substack{I \in \mathcal{I}(n_1), |I|=M \\ |z| \leq b^M |y|}} |(X_I \Phi_j)(xz)| dy \\ &\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \int_{|y| \leq (2^{-j} + |x|)/(2\gamma b^M)} \frac{2^{-kM}}{(2^{-k} + |y|)^{Q+M}} \\ &\quad \times \sup_{\substack{I \in \mathcal{I}(n_1), |I|=M \\ |z| \leq b^M |y|}} \frac{2^{-jM} |y|^M}{(2^{-j} + |xz|)^{Q+M+M}} dy \\ &\sim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM} 2^{-jM}}{(2^{-j} + |x|)^{Q+M+M}} \int_{|y| \leq (2^{-j} + |x|)/(2\gamma b^M)} \frac{|y|^M}{(2^{-k} + |y|)^{Q+M}} dy \end{aligned}$$

$$\begin{aligned}
&\leq \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM} 2^{-jM} 2^{k\varepsilon}}{(2^{-j} + |x|)^{Q+M+M}} \int_{|y| \leq (2^{-j} + |x|)/(2\gamma b^M)} \frac{|y|^M}{(2^{-k} + |y|)^{Q+M-\varepsilon}} dy \\
&\leq \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM} 2^{-jM} 2^{k\varepsilon}}{(2^{-j} + |x|)^{Q+M+M}} \int_{|y| \leq (2^{-j} + |x|)/(2\gamma b^M)} \frac{1}{|y|^{Q-\varepsilon}} dy \\
&\sim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-j(M-\varepsilon)} 2^{-j\varepsilon} 2^{-kM} 2^{k\varepsilon}}{(2^{-j} + |x|)^{Q+M+M-\varepsilon}} \leq \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-j\varepsilon} 2^{-kM} 2^{k\varepsilon}}{(2^{-j} + |x|)^{Q+M}} \\
&= \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(k-j)(M-\varepsilon)} \frac{2^{-jM}}{(2^{-j} + |x|)^{Q+M}}.
\end{aligned}$$

Here, in the third line we used that if $|y| \leq (2^{-j} + |x|)/(2\gamma b^M)$ and $|z| \leq b^M |y|$ then $2^{-j} + |xz| \sim 2^{-j} + |x|$, which follows from (1.1). Similarly, we have

$$\begin{aligned}
I_2 &\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \int_{|y| > (2^{-j} + |x|)/(2\gamma b^M)} \frac{2^{-jM}}{(2^{-j} + |xy^{-1}|)^{Q+M}} \frac{2^{-kM}}{(2^{-k} + |y|)^{Q+M}} dy \\
&\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM}}{(2^{-j} + |x|)^{Q+M}} \int_G \frac{2^{-jM}}{(2^{-j} + |xy^{-1}|)^{Q+M}} dy \\
&\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM}}{(2^{-j} + |x|)^{Q+M}} = \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(k-j)M} \frac{2^{-jM}}{(2^{-j} + |x|)^{Q+M}} \\
&\leq \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(k-j)(M-\varepsilon)} \frac{2^{-jM}}{(2^{-j} + |x|)^{Q+M}}.
\end{aligned}$$

To estimate I_3 , we first note that by [1], Proposition 20.3.11, the Taylor polynomial $P_{x, M-1}^{(\Phi_j)}(\cdot)$ is of the form

$$P_{x, M-1}^{(\Phi_j)}(y) = \Phi_j(x) + \sum_{l=1}^{M-1} \sum_{\nu=1}^l \sum_{\substack{1 \leq i_1, \dots, i_\nu \leq n \\ d_{i_1} + \dots + d_{i_\nu} = l}} \frac{\eta_{i_1}(y) \dots \eta_{i_\nu}(y)}{\nu!} (X_{i_1} \dots X_{i_\nu} \Phi_j)(x),$$

where each d_{i_ν} is a positive integer defined by (2.2), i.e., $d_{i_\nu} := \mu$ if $X_{i_\nu} \in V_\mu$. Hence

$$\begin{aligned}
I_3 &\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \int_{|y| > (2^{-j} + |x|)/(2\gamma b^M)} \frac{2^{-kM}}{(2^{-k} + |y|)^{Q+M}} \sum_{l=0}^{M-1} \frac{2^{-jM} |y|^l}{(2^{-j} + |x|)^{Q+M+l}} dy \\
&\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM}}{(2^{-j} + |x|)^{Q+M}} \sum_{l=0}^{M-1} \int_{|y| > (2^{-j} + |x|)/2\gamma b} \frac{2^{-jM} |y|^l}{(2^{-k} + |y|)^{Q+M+l}} dy \\
&\leq \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM}}{(2^{-j} + |x|)^{Q+M}} \sum_{l=0}^{M-1} \int_{|y| > (2^{-j} + |x|)/2\gamma b} \frac{2^{-j\varepsilon} |y|^l}{(2^{-k} + |y|)^{Q+\varepsilon+l}} dy \\
&\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} \frac{2^{-kM} 2^{k\varepsilon} 2^{-j\varepsilon}}{(2^{-j} + |x|)^{Q+M}} \int_G \frac{2^{-k\varepsilon}}{(2^{-k} + |y|)^{Q+\varepsilon}} dy \\
&\lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(k-j)(M-\varepsilon)} \frac{2^{-jM}}{(2^{-j} + |x|)^{Q+M}}.
\end{aligned}$$

Therefore, for $j \leq k$ we have

$$(3.2) \quad |\Phi_j * \Psi_k(x)| \lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(k-j)(M-\varepsilon)} \frac{2^{-kM}}{(2^{-j} + |x|)^{Q+M}}.$$

Next we consider the case $j > k$. Since $\Phi_j * \Psi_k(x) = \tilde{\Psi}_k * \tilde{\Phi}_j(x^{-1})$ and since \tilde{f} has vanishing moments of the same order as f , it follows from (3.2) that

$$(3.3) \quad |\Phi_j * \Psi_k(x)| = |\tilde{\Psi}_k * \tilde{\Phi}_j(x^{-1})| \lesssim \|\Psi\|_{S_M} \|\Phi\|_{S_M} 2^{-(j-k)(M-\varepsilon)} \frac{2^{-jM}}{(2^{-j} + |x|)^{Q+M}},$$

where we also used the fact that $\|\tilde{\Phi}\|_{S_M} = \|\Phi\|_{S_M}$ and $\|\tilde{\Psi}\|_{S_M} = \|\Psi\|_{S_M}$. Combining (3.2) and (3.3) gives the desired estimate (3.1). \square

Remark 3.2. If we only assume Φ has vanishing moment of order M , then for $j \geq k$ we have

$$(3.4) \quad |\Phi_j * \Psi_k(x)| \lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(j-k)(M-\varepsilon)} \frac{2^{-kM}}{(2^{-k} + |x|)^{Q+M}}.$$

Similarly, if we only assume Ψ has vanishing moment of order M , then for $j \leq k$ we have

$$(3.5) \quad |\Phi_j * \Psi_k(x)| \lesssim \|\Phi\|_{S_M} \|\Psi\|_{S_M} 2^{-(k-j)(M-\varepsilon)} \frac{2^{-jM}}{(2^{-j} + |x|)^{Q+M}}.$$

If $\hat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$ and k is a nonnegative integer, we let $\hat{\Phi}^{(k)}(0) := \lim_{\lambda \rightarrow 0^+} \hat{\Phi}^{(k)}(\lambda)$. Then we have the following lemma.

Lemma 3.3. Suppose $M \in \mathbb{N}$, $\hat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$ and

$$(3.6) \quad \hat{\Phi}^{(k)}(0) = 0 \quad \text{for } k = 0, 1, \dots, M-1.$$

Then the convolution kernel Φ associated to $\hat{\Phi}(\mathcal{L})$ has vanishing moments of order $2M$. In particular, if $\hat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$ vanishes identically near the origin, then Φ has vanishing moments of arbitrary order.

Proof. First we note that, for any polynomial $P \in \mathcal{P}_{2M-1}$, we have $\mathcal{L}^M P \equiv 0$. Indeed, every $P \in \mathcal{P}_{2M-1}$ can be decomposed as a sum $P = a_0 + \sum_{j=1}^{2M-1} a_j P_j$, where $a_0, a_1, a_2, \dots \in \mathbb{C}$ and $P_j \in \mathcal{P}_j \setminus \mathcal{P}_{j-1}$, $j = 1, \dots, 2M-1$. Since every P_j is smooth on G and homogeneous of degree j , $\mathcal{L}^M P_j$ is smooth on G and homogeneous of

degree $j - 2M < 0$. So $\mathcal{L}^M P_j$ must be identically zero on G , and hence $\mathcal{L}^M P \equiv 0$ on G .

Define $\widehat{\Theta}(\lambda) := \lambda^{-M} \widehat{\Phi}(\lambda)$, $\lambda \in \mathbb{R}^+$. Then (3.6) implies that $\Theta \in \mathcal{S}(\mathbb{R}^+)$. Let Φ and Θ denote the convolution kernels associated to $\widehat{\Phi}(\mathcal{L})$ and $\widehat{\Theta}(\mathcal{L})$, respectively. Then we have $\Phi = \mathcal{L}^M \Theta$. For all $P \in \mathcal{P}_{2M-1}$, by integration by parts, we have

$$\int_G \Phi(x) P(x) dx = \int_G (\mathcal{L}^M \Theta)(x) P(x) dx = \int_G \Phi(x) (\mathcal{L}^M P)(x) dx = 0.$$

This shows that Ψ has vanishing moments of order $2M$. □

We now give a Calderón type reproducing formula on G .

Lemma 3.4. *Suppose $\widehat{\Phi}_0, \widehat{\Phi} \in \mathcal{S}(\mathbb{R}^+)$, $\widehat{\Phi}$ vanishes identically near the origin, and*

$$(3.7) \quad \widehat{\Phi}_0(\lambda) + \sum_{j=1}^{\infty} \widehat{\Phi}(2^{-2j} \lambda) = 1 \quad \forall \lambda \in \mathbb{R}^+.$$

Then for all $f \in \mathcal{S}'(G)$ we have

$$(3.8) \quad f = \widehat{\Phi}_0(\mathcal{L})f + \sum_{j=1}^{\infty} \widehat{\Phi}(2^{-2j} \mathcal{L})f \quad \text{with convergence in } \mathcal{S}'(G).$$

Proof. Let Φ_0 and Φ denote the convolution kernels associated to $\widehat{\Phi}_0(\mathcal{L})$ and $\widehat{\Phi}(\mathcal{L})$, respectively. For $j = 1, 2, \dots$, we define $\Phi_j := D_{2^j} \Phi$. Then, as we noted before, Φ_j coincides with the convolution kernel associated to $\widehat{\Phi}(2^{-2j} \mathcal{L})$ for $j = 1, 2, \dots$

To prove (3.8), by duality it suffices to show that for all $g \in \mathcal{S}(G)$, the partial sum $S_k(g) := \widehat{\Phi}_0(\mathcal{L})g + \sum_{j=1}^k \widehat{\Phi}(2^{-2j} \mathcal{L})g$ converges in $\mathcal{S}(G)$ to g as $k \rightarrow \infty$. To see the latter, note that by (2.3) we have that for all $g \in \mathcal{S}(G)$, $M \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$

$$(3.9) \quad \begin{aligned} \|S_{k+l}(g) - S_k(g)\|_{\mathcal{S}_M} &= \left\| \sum_{j=k+1}^{k+l} \widehat{\Phi}(2^{-2j} \mathcal{L})g \right\|_{\mathcal{S}_M} \leq \sum_{j=k+1}^{k+l} \|g * \Phi_j\|_{\mathcal{S}_M} \\ &= \sum_{j=k+1}^{k+l} \sup_{\substack{I \in \mathcal{I}(n_1), |I| \leq M \\ x \in G}} [|X_I(g * \Phi_j)(x)| + |Y_I(g * \Phi_j)(x)|] (1 + |x|)^{Q+M+|I|} \\ &= \sum_{j=k+1}^{k+l} \sup_{\substack{I \in \mathcal{I}(n_1), |I| \leq M \\ x \in G}} [2^{j|I|} |g * (X_I \Phi)_j(x)| + |(Y_I g) * \Phi_j(x)|] (1 + |x|)^{Q+M+|I|}, \end{aligned}$$

where $(X_I\Phi)_j := D_{2^j}(X_I\Phi)$. Let \widetilde{M} be a positive integer such that $\widetilde{M} \geq M + 1$, and let $0 < \varepsilon < 1$. Since $\widehat{\Phi}$ vanishes identically near the origin, it follows from Lemma 3.3 that Φ has vanishing moments of arbitrary order. In particular, Φ has vanishing moments of order \widetilde{M} . Hence, by (3.5) in Remark 3.2, we have that for $j = 1, 2, \dots$

$$(3.10) \quad |g * (X_I\Phi)_j(x)| \leq C \|g\|_{S_M} \|X_I\Phi\|_{S_M} 2^{-j(\widetilde{M}-\varepsilon)} (1 + |x|)^{-Q+\widetilde{M}}$$

and

$$(3.11) \quad |(Y_I g) * \Phi_j(x)| \leq C \|Y_I g\|_{S_M} \|\Phi\|_{S_M} 2^{-j(\widetilde{M}-\varepsilon)} (1 + |x|)^{-Q+\widetilde{M}}.$$

Put

$$C_{g,\Phi,M} := \max \left\{ \|g\|_{S_M} \left(\sup_{I \in \mathcal{I}(n_1), |I| \leq M} \|X_I\Phi\|_{S_M} \right), \left(\sup_{I \in \mathcal{I}(n_1), |I| \leq M} \|Y_I g\|_{S_M} \right) \|\Phi\|_{S_M} \right\}.$$

Since $g, \Phi \in \mathcal{S}(G)$, we have $C_{g,\Phi,M} < \infty$. Inserting (3.10) and (3.11) in (3.9), and taking into account that $\widetilde{M} - M - \varepsilon > 0$, we get

$$\|S_{k+l}(g) - S_k(g)\|_{S_M} \leq C C_{g,\Phi,M} \sum_{j=k+1}^{k+l} 2^{-j\varepsilon} \leq C C_{g,\Phi,M} \sum_{j=k+1}^{\infty} 2^{-j\varepsilon} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This shows that $\{S_k(g)\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{S}(G)$. By virtue of the completeness of $\mathcal{S}(G)$, there exists $h \in \mathcal{S}(G)$ such that $S_k(g) \rightarrow h$ in $\mathcal{S}(G)$ as $k \rightarrow \infty$. On the other hand, it follows from (3.7) and the spectral theory that $S_k(g) \rightarrow g$ in $L^2(G)$ as $k \rightarrow \infty$. Therefore, since $g, h \in C^\infty(G)$, we must have $h(x) = g(x)$ for all $x \in G$. Hence $S_k(g) \rightarrow g$ in $\mathcal{S}(G)$ as $k \rightarrow \infty$. The proof of the lemma is completed. \square

4. PROOF OF THEOREM 1.1

We need somme lemmas. First, we have the following estimate for derivatives of the Bessel potentials associated to \mathcal{L} .

Lemma 4.1. *Suppose $\alpha \in \mathbb{R}$, $\alpha > Q/2$, $k \in \mathbb{N} \cup \{0\}$ and $k < 2\alpha - Q$. Let J_α denote the convolution kernel of the operator $(\text{id} + \mathcal{L})^{-\alpha}$. Then for any $I \in \mathcal{I}(n_1)$ with $|I| \leq k$ and any $N > 0$, there exists a constant $C_{\alpha,I,N}$ (depending on α , I and N) such that for all $x \in G$,*

$$|X_I J_\alpha(x)| \leq C_{\alpha,I,N} (1 + |x|)^{-N}.$$

Proof. Note that J_σ can be expressed as

$$(4.1) \quad J_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} h_t(x) dt, \quad x \in G.$$

Using (2.5) it is easy to see that the integral $\int_0^\infty t^{\alpha-1} e^{-t} X_I h_t(x) dt$ converges uniformly in $x \in G$. So we may differentiate (4.1) under the integral to get

$$X_I J_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} X_I h_t(x) dt, \quad x \in G.$$

Hence by (2.5) we have

$$\begin{aligned} |X_I J_\alpha(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} |X_I h_t(x)| dt \\ &\leq C_{\alpha,I} \int_0^\infty t^{\alpha-1} e^{-t} t^{-(|I|+Q)/2} e^{-|x|^2/(ct)} dt \\ &= C_{\alpha,I} \left(\int_0^{\sqrt{2/c}|x|} + \int_{\sqrt{2/c}|x|}^\infty \right) t^{\alpha-1} e^{-t} t^{-(|I|+Q)/2} e^{-|x|^2/(ct)} dt. \end{aligned}$$

For $0 < t \leq \sqrt{2/c}|x|$ we have the estimate $e^{-|x|^2/(ct)} \leq e^{-|x|/\sqrt{2c}}$, while for $\sqrt{2/c}|x| \leq t < \infty$ we have the estimates $e^{-t} = e^{-t/2} e^{-t/2} \leq e^{-t/2} e^{-|x|/\sqrt{2c}}$ and $e^{-|x|^2/(ct)} \leq 1$. Therefore we have

$$\begin{aligned} |X_I J_\alpha(x)| &\leq C_{\alpha,I} e^{-|x|/\sqrt{2c}} \left(\int_0^{\sqrt{2/c}|x|} t^{\alpha-1-(|I|+Q)/2} e^{-t} dt + \int_{\sqrt{2/c}|x|}^\infty t^{\alpha-1-(|I|+Q)/2} e^{-t/2} dt \right) \\ &\leq C_{\alpha,I} e^{-|x|/\sqrt{2c}} \leq C_{\alpha,I,N} (1 + |x|)^{-N}, \end{aligned}$$

as desired. \square

Lemma 4.2. Suppose $\alpha \in \mathbb{R}$. Then $\mathcal{S}(G) \subset \mathcal{D}((\text{id} + \mathcal{L})^\alpha)$. Furthermore, $(\text{id} + \mathcal{L})^\alpha$ maps $\mathcal{S}(G)$ into $\mathcal{S}(G)$ continuously.

Proof. We first show $\mathcal{S}(G) \subset \mathcal{D}((\text{id} + \mathcal{L})^\alpha)$. Note that this is trivial if $\alpha \leq 0$, since for all $\alpha \leq 0$ we have $\mathcal{D}((\text{id} + \mathcal{L})^\alpha) = L^2(X)$. Assume now $\alpha > 0$. Let $j = [\alpha] + 1$, and set $\Phi(\lambda) = (1 + \lambda)^\alpha (1 + \lambda^j)^{-1}$, $\lambda \in \mathbb{R}^+$. Then $(1 + \lambda)^\alpha = \Phi(\lambda)(1 + \lambda^j)$. Hence by [13], Theorem 13.24 (b), we have

$$(4.2) \quad \mathcal{D}(\Phi(\mathcal{L}) \circ (\text{id} + \mathcal{L}^j)) \subset \mathcal{D}((\text{id} + \mathcal{L})^\alpha).$$

On the other hand, since $\Phi \in L^\infty(\mathbb{R}^+)$, we have $\mathcal{D}(\Phi(\mathcal{L})) = L^2(G)$, and hence

$$\begin{aligned}\mathcal{D}(\Phi(\mathcal{L}) \circ (\text{id} + \mathcal{L}^j)) &= \{f \in L^2(X) \mid f \in \mathcal{D}(\text{id} + \mathcal{L}^j), (\text{id} + \mathcal{L}^j)f \in \mathcal{D}(\Phi(\mathcal{L}))\} \\ &= \mathcal{D}(\text{id} + \mathcal{L}^j).\end{aligned}$$

Combining this with (4.2) we see that $\mathcal{D}(\text{id} + \mathcal{L}^j) \subset \mathcal{D}((\text{id} + \mathcal{L})^\alpha)$, which along with the obvious fact that $\mathcal{S}(G) \subset \mathcal{D}(\text{id} + \mathcal{L}^j)$ implies $\mathcal{S}(G) \subset \mathcal{D}((\text{id} + \mathcal{L})^\alpha)$.

We next show that $(\text{id} + \mathcal{L})^\alpha$ maps $\mathcal{S}(G)$ into $\mathcal{S}(G)$ continuously. Let $\alpha \in \mathbb{R}$ and $\Phi \in \mathcal{S}(G)$. We must show that for any multi-index $I \in \mathcal{I}(n_1)$ and any $N > Q$, the estimate

$$(4.3) \quad |X_I(\text{id} + \mathcal{L})^\alpha \Phi(x)| \leq C(1 + |x|)^{-N}$$

holds with a constant C depending on α , I , N and some Schwartz norms of Φ . To this end, we choose l to be the smallest integer such that $l > \max\{\alpha + (Q + |I|)/2, (N - Q)/2\}$, and then write

$$\begin{aligned}(4.4) \quad X_I(\text{id} + \mathcal{L})^\alpha \Phi &= X_I(\text{id} + \mathcal{L})^{-(l-\alpha)}(\text{id} + \mathcal{L})^l \Phi \\ &= \int_G (\text{id} + \mathcal{L})^l \Phi(y) (X_I J_{l-\alpha})(y^{-1}x) \, dy.\end{aligned}$$

Since $l - \alpha > Q/2$ and $|I| < 2(l - \alpha) - Q$, it follows from Lemma 4.1 that

$$|(X_I J_{l-\alpha})(y^{-1}x)| \leq C_{\alpha, I, N}(1 + |y^{-1}x|)^{-N}.$$

On the other hand, since $l > (N - Q)/2$, we have

$$|(\text{id} + \mathcal{L})^l \Phi(y)| \leq \|\Phi\|_{\mathcal{S}_{2l}}(1 + |y|)^{-(Q+2l)} \leq \|\Phi\|_{\mathcal{S}_{2l}}(1 + |y|)^{-N}.$$

Substituting these into (4.4), and using (1.1) and (2.7), we get

$$\begin{aligned}&|X_I(\text{id} + \mathcal{L})^\alpha \Phi(x)| \\ &\leq C_{\alpha, I, N} \|\Phi\|_{\mathcal{S}_{2l}} \int_G (1 + |y|)^{-N} (1 + |y^{-1}x|)^{-N} \, dy \\ &= C_{\alpha, I, N} \|\Phi\|_{\mathcal{S}_{2l}} \left(\int_{|y| \leq |x|/2\gamma} + \int_{|y| \geq |x|/2\gamma} \right) (1 + |y|)^{-N} (1 + |y^{-1}x|)^{-N} \, dy \\ &\leq C_{\alpha, I, N} \|\Phi\|_{\mathcal{S}_{2l}} \int_{|y| \leq \frac{|x|}{2\gamma}} (1 + |y|)^{-N} (1 + |x|)^{-N} \, dy \\ &\quad + C_{\alpha, I, N} \|\Phi\|_{\mathcal{S}_{2l}} \int_{|y| \geq |x|/2\gamma} (1 + |x|)^{-N} (1 + |y^{-1}x|)^{-N} \, dy \\ &\leq C_{\alpha, I, N} \|\Phi\|_{\mathcal{S}_{2l}} (1 + |x|)^{-N}.\end{aligned}$$

Hence (4.3) is established and the proof is completed. \square

Definition 4.3. For $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(G)$, we define $(\text{id} + \mathcal{L})^\alpha f$ to be an element of $\mathcal{S}'(G)$ such that

$$\langle (\text{id} + \mathcal{L})^\alpha f, \Phi \rangle = \langle f, (\text{id} + \mathcal{L})^\alpha \Phi \rangle, \quad \Phi \in \mathcal{S}(G).$$

Lemma 4.4. Let $\widehat{\Psi}_0$ and $\widehat{\Psi}$ be as in Theorem 1.1. For $j = 1, 2, \dots$, we set

$$(4.5) \quad \widehat{\Psi}_j(\lambda) := \widehat{\Psi}(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}^+.$$

Let $\{\widehat{\Theta}_j\}_{j=0}^\infty$ be a system of functions in $\mathcal{S}(\mathbb{R}^+)$. Let $\alpha \in \mathbb{R}$ and let $L \in \mathbb{N}$ be such that $L > Q + 1 + |\alpha|/2$. Then for all $f \in \mathcal{S}'(G)$ we have

$$(4.6) \quad \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Theta}_j(\mathcal{L})f\|_{L^\infty(G)} \lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)},$$

where

$$(4.7) \quad C(\{\widehat{\Theta}_j\}_{j=0}^\infty) := \sup_{\substack{\lambda \in \mathbb{R}^+ \\ 0 \leq k \leq L}} (1+\lambda)^L \left| \frac{d^k \widehat{\Theta}_0}{d\lambda^k}(\lambda) \right| + \sup_{\substack{\lambda \in \mathbb{R}^+ \\ 0 \leq k \leq L \\ j \in \mathbb{N}}} (\lambda^L + \lambda^{-L}) \left| \frac{d^k [\widehat{\Theta}_j(2^{2j}\cdot)]}{d\lambda^k}(\lambda) \right|.$$

Proof. By (1.3), (4.5) and Lemma 3.4, for all $f \in \mathcal{S}'(G)$ we have

$$f = \sum_{l=0}^{\infty} \widehat{\Psi}_l(\mathcal{L}) \widehat{\Psi}_l(\mathcal{L}) f \quad \text{with convergence in } \mathcal{S}'(G).$$

Hence for all $j \in \mathbb{N} \cup \{0\}$ and $x \in G$ we have

$$\widehat{\Theta}_j(\mathcal{L})f(x) = \sum_{l=0}^{\infty} \widehat{\Theta}_j(\mathcal{L}) \widehat{\Psi}_l(\mathcal{L}) \widehat{\Psi}_l(\mathcal{L}) f(x).$$

It follows that

$$(4.8) \quad \begin{aligned} & 2^{j\alpha} |\widehat{\Theta}_j(\mathcal{L})f(x)| \\ & \leq \sum_{l=0}^{\infty} 2^{(j-l)\alpha} 2^{l\alpha} \int_G |K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})}(y^{-1}x)| |\widehat{\Psi}_l(\mathcal{L})f(y)| dy \\ & \leq \sup_{\substack{l \in \mathbb{N} \cup \{0\} \\ y \in G}} 2^{l\alpha} |\widehat{\Psi}_l(\mathcal{L})f(y)| \sum_{l=0}^{\infty} 2^{(j-l)\alpha} \int_G |K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})}(y^{-1}x)| dy \\ & \leq \sup_{\substack{l \in \mathbb{N} \cup \{0\} \\ y \in G}} 2^{l\alpha} |\widehat{\Psi}_l(\mathcal{L})f(y)| \sum_{l=0}^{\infty} 2^{(j-l)\alpha} I_{j,l}, \end{aligned}$$

where $K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})}(\cdot)$ denotes the convolution kernel associated to $\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})$, and for $j, l \in \mathbb{N} \cup \{0\}$ we have set

$$I_{j,l} := \int_G |K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})}(y^{-1}x)| \, dy = \int_G |K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})}(y)| \, dy.$$

Let $N \in \mathbb{N}$ be such that $N \geq Q + 1$ and $2L - 2N - |\alpha| > 0$. Such N exists since we have assumed that $L > Q + 1 + |\alpha|/2$. Now we claim that for all $j, l \in \mathbb{N} \cup \{0\}$ we have

$$(4.9) \quad I_{j,l} \lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) 2^{-2|j-l|(L-N)}.$$

Assume the claim for the moment. Then substituting (4.9) into (4.8), we see that for all $j \in \mathbb{N} \cup \{0\}$ and $x \in G$

$$\begin{aligned} 2^{j\alpha} |\widehat{\Theta}_j(\mathcal{L})f(x)| &\lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) \sup_{\substack{l \in \mathbb{N} \cup \{0\} \\ y \in G}} 2^{l\alpha} |\widehat{\Psi}_l(\mathcal{L})f(y)| \sum_{l=0}^\infty 2^{-|j-l|(2L-2N-|\alpha|)} \\ &\lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) \sup_{\substack{l \in \mathbb{N} \cup \{0\} \\ y \in G}} 2^{l\alpha} |\widehat{\Psi}_l(\mathcal{L})f(y)|. \end{aligned}$$

Taking the supremum over $j \in \mathbb{N} \cup \{0\}$ and $x \in G$ yields the desired estimate (4.6).

It remains to prove the claim. To do this, we consider the following four cases:

Case (i) $j \in \mathbb{N}$, $l \in \mathbb{N}$; *Case (ii)* $j \in \mathbb{N}$, $l = 0$; *Case (iii)* $j = 0$, $l \in \mathbb{N}$; *Case (iv)* $j = 0$, $l = 0$. We shall only give the details here for Case (i) since the other cases can be done similarly. Let $j, l \in \mathbb{N}$. Put $\widehat{\Omega}(\lambda) := \widehat{\Theta}_j(2^{2l}\lambda)\widehat{\Psi}(\lambda)$, $\lambda \in \mathbb{R}^+$. Then by Proposition 2.2 we have

$$\begin{aligned} |K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})}(y)| &= |K_{\widehat{\Theta}_j(\mathcal{L})\widehat{\Psi}(2^{-2l}\mathcal{L})}(y)| = |K_{\widehat{\Omega}(2^{-2l}\mathcal{L})}(y)| \\ &\leq C_N \|\widehat{\Omega}\|_{(N)} 2^{lQ} (1 + 2^l|y|)^{-N}. \end{aligned}$$

From this and (2.7) it follows that

$$\begin{aligned} I_{k,l} &\lesssim \|\widehat{\Omega}\|_{(N)} = \sup_{\substack{\lambda \in \mathbb{R}^+ \\ 0 \leq k \leq N}} (1 + \lambda)^{N+Q+1} \left| \frac{d^k \widehat{\Omega}}{d\lambda^k}(\lambda) \right| \\ &\leq \sup_{\substack{\lambda \in \mathbb{R}^+ \\ 0 \leq k \leq N}} (1 + \lambda)^{N+Q+1} \sum_{k_1+k_2=k} C_k^{k_1} \left| \frac{d^{k_1} \widehat{\Psi}}{d\lambda^{k_1}}(\lambda) \right| \left| \frac{d^{k_2} [\widehat{\Theta}_j(2^{2l}\cdot)]}{d\lambda^{k_2}}(\lambda) \right| \\ &= \sup_{\substack{\lambda \in \mathbb{R}^+ \\ 0 \leq k \leq N}} (1 + \lambda)^{N+Q+1} \sum_{k_1+k_2=\nu} 2^{2(l-j)k_2} \left| \frac{d^{k_1} \widehat{\Psi}}{d\lambda^{k_1}}(\lambda) \right| \left| \frac{d^{k_2} [\widehat{\Theta}_j(2^{2j}\cdot)]}{d\lambda^{k_2}}(2^{2(l-j)}\lambda) \right| \end{aligned}$$

$$\begin{aligned}
&\leq C(\{\widehat{\Theta}_j\}_{j=0}^\infty) \sup_{\substack{\lambda \in \mathbb{R}^+ \\ 0 \leq k \leq N}} (1+\lambda)^{N+Q+1} \\
&\quad \times \sum_{k_1+k_2=k} 2^{2(l-j)k_2} \left| \frac{d^{k_1} \widehat{\Psi}}{d\lambda^{k_1}}(\lambda) \right| (|2^{2(l-j)}\lambda|^L + |2^{2(l-j)}\lambda|^{-L})^{-1} \\
&\lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) 2^{-2|j-l|(L-N)},
\end{aligned}$$

where for the last inequality we used the fact that $\text{supp } \widehat{\Psi} \subset [1/4, 4]$. Hence the claim is true and the proof is completed. \square

Lemma 4.5. *Let $\widehat{\Psi}_0$ and $\widehat{\Psi}$ be as in Theorem 1.1. For $j = 1, 2, \dots$, we set*

$$\widehat{\Psi}_j(\lambda) := \widehat{\Psi}(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}^+.$$

Then for any $\alpha, \beta \in \mathbb{R}$ there exists a constant C such that for all $f \in \mathcal{S}'(G)$,

$$(4.10) \quad \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\beta-2\alpha)} \|\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^\alpha f\|_{L^\infty(G)} \sim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\beta} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

Proof. For $j = 0, 1, 2, \dots$, we define

$$\widehat{\Theta}_j(\lambda) := 2^{-2j\alpha} \widehat{\Psi}_j(\lambda)(1+\lambda)^\alpha, \quad \lambda \in \mathbb{R}^+.$$

Obviously, each $\widehat{\Theta}_j$ belongs to $\mathcal{S}(\mathbb{R}^+)$. Note that for $j = 1, 2, \dots$ we have

$$\widehat{\Theta}_j(2^{2j}\cdot) = 2^{-2j\alpha} \widehat{\Psi}(\cdot)(1+2^{2j}\cdot)^\alpha.$$

Hence, the assumption $\text{supp } \widehat{\Psi} \subset [1/4, 4]$ implies that the number $C(\{\widehat{\Theta}_j\}_{j=0}^\infty)$ defined by (4.7) is finite. Therefore it follows from Lemma 4.4 that for all $f \in \mathcal{S}'(G)$,

$$\begin{aligned}
&\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\beta-2\alpha)} \|\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^\alpha f\|_{L^\infty(G)} = \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\beta} \|\widehat{\Theta}_j(\mathcal{L})f\|_{L^\infty(G)} \\
&\lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\beta} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.
\end{aligned}$$

This estimate also implies

$$\begin{aligned}
&\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\beta} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \\
&= \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j[(\beta-2\alpha)-(-2\alpha)]} \|\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^{-\alpha}(\text{id} + \mathcal{L})^\alpha f\|_{L^\infty(G)} \\
&\lesssim C(\{\widehat{\Theta}_j\}_{j=0}^\infty) \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\beta-2\alpha)} \|\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^\alpha f\|_{L^\infty(G)}.
\end{aligned}$$

Therefore (4.10) is true, and the proof is completed. \square

Lemma 4.6. Let $\widehat{\Psi}_0$ and $\widehat{\Psi}$ be as in Theorem 1.1. For $j = 1, 2, \dots$, we set

$$(4.11) \quad \widehat{\Psi}_j(\lambda) := \widehat{\Psi}(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}^+.$$

Then for any $I \in \mathcal{I}(n_1)$ and $\alpha \in \mathbb{R}$ there exists a constant C such that for all $f \in \mathcal{S}'(G)$,

$$(4.12) \quad \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\alpha - |I|)} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} \leq C \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

Proof. By (1.3), (4.11) and Lemma 3.4 we have

$$f = \sum_{l=0}^{\infty} \widehat{\Psi}_l(\mathcal{L}) \widehat{\Psi}_l(\mathcal{L}) f$$

with convergence in $\mathcal{S}'(G)$. Hence for any $I \in \mathcal{I}(n_1)$ we have

$$X_I f = \sum_{l=0}^{\infty} X_I \widehat{\Psi}_l(\mathcal{L}) \widehat{\Psi}_l(\mathcal{L}) f.$$

Consequently, for all $j \in \mathbb{N} \cup \{0\}$ and $x \in G$,

$$(4.13) \quad \widehat{\Psi}_j(\mathcal{L})(X_I f)(x) = \sum_{l=0}^{\infty} \widehat{\Psi}_j(\mathcal{L}) X_I \widehat{\Psi}_l(\mathcal{L}) \widehat{\Psi}_l(\mathcal{L}) f(x).$$

Let Ψ_0 and Ψ denote the convolution kernel associated to $\widehat{\Psi}_0(\mathcal{L})$ and $\widehat{\Psi}(\mathcal{L})$, respectively. For $j = 1, 2, \dots$, we set $\Psi_j := D_{2^j} \Psi$. Then for $j \in \mathbb{N} \cup \{0\}$, Ψ_j coincides with the convolution kernel associated to $\widehat{\Psi}_j(\mathcal{L})$. Hence by (2.3) we can rewrite (4.13) as

$$\begin{aligned} \widehat{\Psi}_j(\mathcal{L})(X_I f)(x) &= \sum_{l=0}^{\infty} \widehat{\Psi}_j(\mathcal{L}) X_I (f * \Psi_l * \Psi_l)(x) \\ &= \sum_{l=0}^{\infty} [X_I (f * \Psi_l * \Psi_l)] * \Psi_j(x) \\ &= \sum_{l=0}^{\infty} (f * \Psi_l * \Psi_l) * (Y_{\bar{I}} \Psi_j)(x) \\ &= 2^{j|I|} \sum_{l=0}^{\infty} f * \Psi_l * \Psi_l * (Y_{\bar{I}} \Psi)_j(x), \end{aligned}$$

where $(Y_{\bar{I}} \Psi)_j := D_{2^j}(Y_{\bar{I}} \Psi)$.

Let $\varepsilon \in (0, 1)$ and let $M \in \mathbb{N}$ be such that $M - \varepsilon - |\alpha| > 0$. Since both Ψ and $Y_{\bar{I}}\Psi$ have vanishing moments of arbitrary order, it follows from Lemma 3.1 that for any $M \in \mathbb{N}$ and any $\varepsilon \in (0, 1)$ there exists a constant C (depending on Ψ , M , I and ε) such that for all $j, l \in \mathbb{N}$,

$$|\Psi_l * (Y_{\bar{I}}\Phi)_j(x)| \leq C 2^{-|j-l|(M-\varepsilon)} \frac{2^{-(j \wedge l)M}}{(2^{-(j \wedge l)} + |x|)^{Q+M}}.$$

This along with (2.7) implies that $\|\Psi_l * (Y_{\bar{I}}\Psi)_j\|_{L^1(G)} \lesssim 2^{-|j-l|(M-\varepsilon)}$. Therefore,

$$\begin{aligned} & \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-|I|)} |\widehat{\Psi}_j(\mathcal{L})(X_I f)| \\ &= \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j\alpha} \left| \sum_{l=0}^{\infty} f * \Psi_l * \Psi_l * (Y_{\bar{I}}\Psi)_j(x) \right| \\ &\leq \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j\alpha} \sum_{l=0}^{\infty} \|f * \Psi_l\|_{L^\infty(G)} \|\Psi_l * (Y_{\bar{I}}\Psi)_j\|_{L^1(G)} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^{l\alpha} \|f * \Psi_l\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} \sum_{l=0}^{\infty} 2^{(j-l)\alpha} \|\Psi_l * (Y_{\bar{I}}\Psi)_j\|_{L^1(G)} \\ &\lesssim \sup_{l \in \mathbb{N} \cup \{0\}} 2^{l\alpha} \|f * \Psi_l\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} \sum_{l=0}^{\infty} 2^{-|j-l|(M-\varepsilon-|\alpha|)} \\ &\lesssim \sup_{l \in \mathbb{N} \cup \{0\}} 2^{l\alpha} \|f * \Psi_l\|_{L^\infty(G)} = \sup_{l \in \mathbb{N} \cup \{0\}} 2^{l\alpha} \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)}. \end{aligned}$$

The proof is thus complete. \square

Lemma 4.7. *Let $\widehat{\Psi}_0$ and $\widehat{\Psi}$ be as in Theorem 1.1. For $j = 1, 2, \dots$, we set*

$$\widehat{\Psi}_j(\lambda) := \widehat{\Psi}(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}^+.$$

Then for any $\alpha \in \mathbb{R}$ and $l \in \mathbb{N}$,

$$\begin{aligned} (4.14) \quad & \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \\ & \sim \sum_{I \in \mathcal{I}(n), |I| \leq l} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\alpha-l)} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)}, \quad f \in \mathcal{S}'(G). \end{aligned}$$

Proof. The inequality

$$\sum_{I \in \mathcal{I}(n), |I| \leq l} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\alpha-k)} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} \lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}$$

follows immediately from Lemma 4.6. To see the converse inequality, note that by Lemma 4.5 and Lemma 4.6

$$\begin{aligned}
& \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \\
&= \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\alpha} \|\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^{-1/2}(\text{id} + \mathcal{L})^{1/2}f\|_{L^\infty(G)} \\
&\lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\alpha-1)} \|\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^{1/2}f\|_{L^\infty(G)} \\
&= \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-1)} |\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^{-1/2}(\text{id} + \mathcal{L})f(x)| \\
&\lesssim \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-1)} |\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^{-1/2}f(x)| \\
&\quad + \sum_{k=1}^{n_1} \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-1)} |\widehat{\Psi}_j(\mathcal{L})(\text{id} + \mathcal{L})^{-1/2}X_k(X_k f)(x)| \\
&\lesssim \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-2)} |\widehat{\Psi}_j(\mathcal{L})f(x)| + \sum_{k=1}^{n_1} \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-1)} |\widehat{\Psi}_j(\mathcal{L})(X_k f)(x)| \\
&\leq \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-1)} |\widehat{\Psi}_j(\mathcal{L})f(x)| + \sum_{k=1}^{n_1} \sup_{\substack{j \in \mathbb{N} \cup \{0\} \\ x \in G}} 2^{j(\alpha-1)} |\widehat{\Psi}_j(\mathcal{L})(X_k f)(x)|.
\end{aligned}$$

By induction, we obtain the direction “ \lesssim ” in (4.14). Hence the proof of the lemma is completed. \square

Lemma 4.8 ([3], Proposition 5.8). *Suppose $0 < \beta < 1$ and f is a bounded continuous function. Then $f \in \mathcal{C}^1(G)$ if and only if there is a constant $B > 0$ such that for every $\tau > 0$ there exist $f_\tau \in \mathcal{C}^{1+\beta}(G)$, $f^\tau \in \mathcal{C}^{1-\beta}(G)$ with $\|f_\tau\|_{\mathcal{C}^{1+\beta}(G)} \leq B\tau$, $\|f^\tau\|_{\mathcal{C}^{1-\beta}(G)} \leq B\tau^{-1}$, and $f = f_\tau + f^\tau$. In this case, the smallest such B is comparable to $\|f\|_{\mathcal{C}^1(G)}$.*

Lemma 4.9. *Let $\widehat{\Psi}_0$ and $\widehat{\Psi}$ be as in Theorem 1.1. For $j = 1, 2, \dots$, we set*

$$(4.15) \quad \widehat{\Psi}_j(\lambda) := \widehat{\Psi}(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}^+.$$

Suppose $0 < \beta < 1$ and $f \in \mathcal{S}'(G)$. Then

$$\sup_{j \in \mathbb{N} \cup \{0\}} 2^j \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty$$

if and only if there is a constant $B > 0$ such that for every $\tau > 0$ there exist $f_\tau, f^\tau \in \mathcal{S}'(G)$ with

$$\begin{aligned} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1+\beta)} \|\widehat{\Psi}_j(\mathcal{L})f_\tau\|_{L^\infty(G)} &\leq B\tau, \\ \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1-\beta)} \|\widehat{\Psi}_j(\mathcal{L})f^\tau\|_{L^\infty(G)} &\leq B\tau^{-1}, \end{aligned}$$

and $f = f_\tau + f^\tau$. In this case, the smallest such B is comparable to

$$\sup_{j \in \mathbb{N} \cup \{0\}} 2^j \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

Proof. Let Ψ_0 and Ψ denote the convolution kernel associated to $\widehat{\Psi}_0(\mathcal{L})$ and $\widehat{\Psi}(\mathcal{L})$, respectively. For $j = 1, 2, \dots$ we set $\Psi_j := D_{2^j}\Psi$. Then for $j \in \mathbb{N} \cup \{0\}$, Ψ_j coincides with the convolution kernel associated to $\widehat{\Psi}_j(\mathcal{L})$.

We first prove the “if” part of the lemma. Suppose we can find B, f_τ, f^τ as above. Then, putting $\tau_j := 2^{j\beta}$, $j = 0, 1, 2, \dots$ we have $f = f_{\tau_j} + f^{\tau_j}$, and

$$\begin{aligned} \sup_{j \in \mathbb{N} \cup \{0\}} 2^j \|\widehat{\Psi}(\mathcal{L})f\|_{L^\infty(G)} &\leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{-j\beta} 2^{j(1+\beta)} \|\widehat{\Psi}_j(\mathcal{L})f_{\tau_j}\|_{L^\infty(G)} \\ &\quad + \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\beta} 2^{j(1-\beta)} \|\widehat{\Psi}_j(\mathcal{L})f^{\tau_j}\|_{L^\infty(G)} \\ &\leq \sup_{j \in \mathbb{N} \cup \{0\}} B 2^{-j\beta} \tau_j + \sup_{j \in \mathbb{N} \cup \{0\}} B 2^{j\beta} \tau_j^{-1} = 2B. \end{aligned}$$

To prove the “only if” part, note first that it suffices to consider $\tau \geq 1$, since for $\tau < 1$ we can simply take $B = \sup_{j \in \mathbb{N} \cup \{0\}} 2^j \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}$, $f_\tau = 0$ and $f^\tau = f$.

Given $f \in \mathcal{S}'(G)$ with $\sup_{j \in \mathbb{N} \cup \{0\}} 2^j \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty$, and $\tau \geq 1$, we set $f_\tau =$

$\sum_{l=0}^k \widehat{\Psi}_l(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})f$ and $f^\tau = \sum_{l=k+1}^{\infty} \widehat{\Psi}_l(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})f$, where k is the unique positive integer such that

$$(4.16) \quad 2^{k-1} \leq \tau^{1/\beta} < 2^k.$$

Then, it follows from (1.3), (4.15) and Lemma 3.4 that $f = f_\tau + f^\tau$. Since $\widehat{\Psi}$ vanishes identically near the origin, by Lemma 3.3 Ψ has vanishing moments of arbitrary order. Hence it follows from Lemma 3.1 and (2.7) that there exists a constant C_Ψ such that for all $j, l \in \mathbb{N} \cup \{0\}$

$$(4.17) \quad \|\Psi_j * \Psi_l\|_{L^1(G)} \leq C_\Psi 2^{-2|j-l|}.$$

Now let us set

$$B := 2^\beta \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} \sum_{l=0}^{\infty} 2^{-|j-l|[2-(1-\beta)]}.$$

Obviously, B is a finite positive number. By (4.16) and (4.17) we have

$$\begin{aligned} & \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1+\beta)} \|\widehat{\Psi}_j(\mathcal{L})f_\tau\|_{L^\infty(G)} \\ &= \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1+\beta)} \left\| \sum_{l=0}^k \widehat{\Psi}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})f \right\|_{L^\infty(G)} \\ &\leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1+\beta)} \sum_{l=0}^k \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \|\Psi_l * \Psi_j\|_{L^1(G)} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1+\beta)} \sum_{l=0}^k 2^{-l} \|\Psi_l * \Psi_j\|_{L^1(G)} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1+\beta)} \sum_{l=0}^k 2^{-l} 2^{-2|j-l|} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} 2^{k\beta} \sup_{j \in \mathbb{N} \cup \{0\}} \sum_{l=0}^k 2^{-|j-l|[2-(1+\beta)]} \\ &\leq B\tau. \end{aligned}$$

Similarly,

$$\begin{aligned} & \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1-\beta)} \|\widehat{\Psi}_j(\mathcal{L})f^\tau\|_{L^\infty(G)} \\ &= \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1-\beta)} \left\| \sum_{l=k+1}^{\infty} \widehat{\Psi}_j(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})\widehat{\Psi}_l(\mathcal{L})f \right\|_{L^\infty(G)} \\ &\leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1-\beta)} \sum_{l=k+1}^{\infty} \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \|\Psi_l * \Psi_j\|_{L^1(G)} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1-\beta)} \sum_{l=k+1}^{\infty} 2^{-l} \|\Psi_l * \Psi_j\|_{L^1(G)} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(1-\beta)} \sum_{l=k+1}^{\infty} 2^{-l} 2^{-2|j-l|} \\ &\leq \sup_{l \in \mathbb{N} \cup \{0\}} 2^l \|\widehat{\Psi}_l(\mathcal{L})f\|_{L^\infty(G)} 2^{-(k+1)\beta} \sup_{j \in \mathbb{N} \cup \{0\}} \sum_{l=k+1}^{\infty} 2^{-|j-l|[2-(1-\beta)]} \\ &\leq B\tau^{-1}. \end{aligned}$$

The proof of the lemma is completed. \square

Now we are ready to give the proof of the main theorem.

Proof of Theorem 1.1. Let Ψ_0 and Ψ denote the convolution kernel associated to $\widehat{\Psi}_0(\mathcal{L})$ and $\widehat{\Psi}(\mathcal{L})$, respectively. For $j = 1, 2, \dots$ we set $\Psi_j := D_{2^j}\Psi$. Then for $j \in \mathbb{N} \cup \{0\}$, Ψ_j coincides with the convolution kernel associated to $\widehat{\Psi}_j(\mathcal{L})$.

Case I: $0 < \sigma < 1$. For any $f \in \mathcal{C}^\sigma(G)$ we have

$$(4.18) \quad \|\widehat{\Psi}_0(\mathcal{L})f\|_{L^\infty} = \|f * \Psi_0\|_{L^\infty} \leq \|\Psi_0\|_{L^1(G)} \|f\|_{L^\infty(G)} \lesssim \|f\|_{\mathcal{C}^\sigma(G)}.$$

For $j = 1, 2, \dots$, by the vanishing moment condition on Ψ , we have

$$(4.19) \quad \begin{aligned} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty} &= 2^{j\sigma} \sup_{x \in G} |f * \Psi_j(x)| \\ &\leq 2^{j\sigma} \sup_{x \in G} \int_G |f(xy^{-1}) - f(x)| |\Psi_j(y)| \, dy \\ &\leq \|f\|_{\mathcal{C}^\sigma(G)} \int_G |2^j y|^\sigma |\Psi_j(y)| \, dy \\ &= \|f\|_{\mathcal{C}^\sigma(G)} \int_G |y|^\sigma |\Psi(y)| \, dy \lesssim \|f\|_{\mathcal{C}^\sigma(G)}. \end{aligned}$$

Combining (4.18) and (4.19) gives (1.5).

To see the converse statement, we need first to show that every distribution $f \in \mathcal{S}'(G)$ that satisfies $\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty$ coincides with a bounded continuous function on G . Indeed, by (1.3), (1.4) and Lemma 3.4, for any $f \in \mathcal{S}'(G)$ we have

$$(4.20) \quad f = \sum_{j=0}^{\infty} \widehat{\Psi}_j(\mathcal{L}) \widehat{\Psi}_j(\mathcal{L}) f = \sum_{j=0}^{\infty} f * \Psi_j * \Psi_j$$

with convergence in $\mathcal{S}'(G)$. Hence, if $\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty$, then for any $N_1, N_2 \in \mathbb{N} \cup \{0\}$ with $N_1 < N_2$ we have

$$(4.21) \quad \begin{aligned} \sum_{j=N_1}^{N_2} |f * \Psi_j * \Psi_j(x)| &\leq \sum_{j=N_1}^{N_2} \int_G |f * \Psi_j(y)| |\Psi_j(y^{-1}x)| \, dy \\ &\leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \sum_{j=N_1}^{N_2} 2^{-j\sigma} \int_G |\Psi_j(y)| \, dy \\ &\lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \sum_{j=N_1}^{N_2} 2^{-j\sigma} \rightarrow 0 \quad \text{as } N_1, N_2 \rightarrow \infty, \end{aligned}$$

which shows that the partial sum of the series $\sum_{j=0}^{\infty} f * \Psi_j * \Psi_j(x)$ is uniformly Cauchy in the variable $x \in G$. Thus the series converges uniformly to $f(x)$. Since every $f * \Psi_j * \Psi_j$ is continuous on G , the sum function f is also continuous on G . Moreover, it follows from (4.21) that

$$\|f\|_{L^\infty(G)} \lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

To establish (1.6) it remains to show that for all $x \in G$ and $y \in G \setminus \{0\}$ we have

$$(4.22) \quad |f(xy) - f(x)| \lesssim |y|^\sigma \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

To this end, by (4.20) with convergence uniformly on G we have

$$\begin{aligned} |f(xy) - f(x)| &\leq \sum_{j=0}^{\infty} \int_G |f * \Psi_j(z)| |\Psi_j(z^{-1}xy) - \Psi_j(z^{-1}x)| dz \\ &\leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \sum_{j=0}^{\infty} 2^{-j\sigma} \int_G |\Psi_j(zxy) - \Psi_j(zx)| dz. \end{aligned}$$

Hence, if $|y| \geq 1$, we deduce easily from the above estimate that

$$\begin{aligned} |f(xy) - f(x)| &\leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \sum_{j=0}^{\infty} 2^{-j\sigma} \int_G |\Psi_j(zxy)| + |\Psi_j(zx)| dz \\ &\lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} \leq |y|^\sigma \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}. \end{aligned}$$

Now suppose $|y| < 1$. Let l be the unique nonnegative integer such that $2^{-l-1} \leq |y| < 2^{-l}$. If $j \leq l$, then $|y| \lesssim 2^{-j}$, and hence by Proposition 2.1 and (1.1) we have

$$\begin{aligned} |\Psi_j(zxy) - \Psi_j(zx)| &\lesssim \sup_{\substack{|w| \leq b|y| \\ 1 \leq k \leq n_1}} |y| |(X_k \Psi_j)(zw)| \\ &= 2^{j(Q+1)} \sup_{\substack{1 \leq k \leq n_1 \\ |w| \leq b|y|}} |y| |(X_k \Psi)(2^j(zw))| \\ &\lesssim 2^{j(Q+1)} \sup_{|w| \leq b|y|} |y| (1 + 2^j|zw|)^{-N} \\ &\sim 2^{j(Q+1)} |y| (1 + 2^j|z|)^{-N} \sim 2^{j(Q+1)} 2^{-l} (1 + 2^j|z|)^{-N}. \end{aligned}$$

Hence, putting $A := \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}$, we have

$$\begin{aligned}
& |f(xy) - f(x)| \\
& \leq \sum_{j=0}^{\infty} \int_G |f * \Psi_j(z)| |\Psi_j(z y) - \Psi_j(z)| \, dz \\
& \leq A \sum_{j=0}^{\infty} 2^{-j\sigma} \int_G |\Psi_j(z y) - \Psi_j(z)| \, dz \\
& \leq A \left(\sum_{j=0}^l 2^{-j\sigma} \int_G |\Psi_j(z y) - \Psi_j(z)| \, dz + \sum_{j=l+1}^{\infty} 2^{-j\sigma} \int_G |\Psi_j(z y)| + |\Psi_j(z)| \, dy \right) \\
& \lesssim A \left(\sum_{j=0}^l 2^{-j\sigma} \int_G 2^{j(Q+1)} 2^{-l} (1 + 2^j |z|)^{-N} \, dz + \sum_{j=l+1}^{\infty} 2^{-j\sigma} \right) \\
& = A \left(2^{-l} \sum_{j=0}^l 2^{j(1-\sigma)} \int_G (1 + |z|)^{-N} \, dz + \sum_{j=l+1}^{\infty} 2^{-j\sigma} \right) \\
& \sim A(2^{-l} 2^{l(1-\sigma)} + 2^{-l\sigma}) \sim A 2^{-l\sigma} \sim A |y|^\sigma.
\end{aligned}$$

Therefore, for all $x \in G$ and $y \in G \setminus \{0\}$, (4.22) is valid. Thus the assertion of the theorem is true in the case $0 < \sigma < 1$.

Case II: $\sigma = k + \sigma'$ where $k = 1, 2, \dots$ and $0 < \sigma' < 1$. Suppose $f \in \mathcal{S}'(G)$ is such that $\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty$. Then it follows from Lemma 4.6 that for all $I \in \mathcal{I}(n_1)$ with $|I| \leq k$

$$\begin{aligned}
\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma'} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} & \leq \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j(\sigma - |I|)} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} \\
& \lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} < \infty.
\end{aligned}$$

From this and the remarks in Case I we see that for every $I \in \mathcal{I}(n_1)$ with $|I| \leq k$, $X_I f$ coincides with a bounded continuous function on G . Moreover, for such I we have

$$\|X_I f\|_{\mathcal{C}^{\sigma'}(G)} \lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma'} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} \lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

Hence

$$\|f\|_{\mathcal{C}^\sigma(G)} = \sum_{I \in \mathcal{I}(n_1), |I| \leq k} \|X_I f\|_{\mathcal{C}^{\sigma'}(G)} \lesssim \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)}.$$

Now we prove the converse. Suppose $f \in \mathcal{C}^\sigma(G)$. For every $I \in \mathcal{I}(n_1)$ with $|I| \leq k$, by what we proved in Case I, we have

$$\sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma'} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} \lesssim \|X_I f\|_{\mathcal{C}^{\sigma'}(G)}.$$

From this and Lemma 4.7 it follows that

$$\begin{aligned} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma} \|\widehat{\Psi}_j(\mathcal{L})f\|_{L^\infty(G)} &\sim \sum_{I \in \mathcal{I}(n), |I| \leq k} \sup_{j \in \mathbb{N} \cup \{0\}} 2^{j\sigma'} \|\widehat{\Psi}_j(\mathcal{L})(X_I f)\|_{L^\infty(G)} \\ &\lesssim \sum_{I \in \mathcal{I}(n), |I| \leq k} \|X_I f\|_{\mathcal{C}^{\sigma'}(G)} = \|f\|_{\mathcal{C}^\sigma(G)}. \end{aligned}$$

Thus the assertion of the theorem is true in the case $\sigma \in (0, \infty) \setminus \mathbb{N}$.

Case III: $\sigma = 1, 2, \dots$. In view of (1.2) and (4.14), it suffice to consider the case $\sigma = 1$. However, by Lemma 4.8 and Lemma 4.9, we can reduce the case $\sigma = 1$ to the cases $\sigma \in (0, 1)$ and $\sigma \in (1, 2)$. Hence, by the discussions in Case I and Case II, we are done.

Therefore, the proof of Theorem 1.1 is completed. \square

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