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GORENSTEIN PROJECTIVE COMPLEXES WITH RESPECT TO COTORSION PAIRS

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Abstract. Let $(\mathcal{A},\mathcal{B})$ be a complete and hereditary cotorsion pair in the category of left R-modules. In this paper, the so-called Gorenstein projective complexes with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$ are introduced. We show that these complexes are just the complexes of Gorenstein projective modules with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$. As an application, we prove that both the Gorenstein projective modules with respect to cotorsion pairs and the Gorenstein projective complexes with respect to cotorsion pairs possess stability.

Keywords: cotorsion pair; Gorenstein projective complex with respect to cotorsion pairs; stability of Gorenstein categories

MSC 2010: 18G25, 18G35

1. Introduction

Let $(\mathcal{A},\mathcal{B})$ be a complete and hereditary cotorsion pair in the category of left R-modules. Then there are two induced cotorsion pairs $(\widetilde{\mathcal{A}},\operatorname{dg}\widetilde{\mathcal{B}})$ and $(\operatorname{dg}\widetilde{\mathcal{A}},\widetilde{\mathcal{B}})$ ([8]), and both of them are complete and hereditary ([17], [14]). Recently, among others, the Gorenstein category $\mathcal{G}(\mathcal{A})$ with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$ was introduced and studied by Yang and Chen in [16], see Definition 2.3. In this paper, we generalize this notion to the category of complexes of left R-modules, namely, we introduce the Gorenstein projective complexes with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$, see Definition 3.1. The class of these complexes will be denoted by $\mathcal{G}(\widetilde{\mathcal{A}})$. It contains Gorenstein projective complexes [4], **F**-Gorenstein flat complexes, see [10], and Gorenstein flat complexes [7] over right coherent rings as its special cases. By us-

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ing the techniques of Bravo and Gillespie in [3], we prove the following result, see Theorem 3.5.

Theorem 1.1. *A complex C of left R-modules belongs to $\mathcal{G}(\widetilde{\mathcal{A}})$ if and only if each C_n belongs to $\mathcal{G}(\mathcal{A})$.*

Theorem 1.1 generalizes [10], Theorem 4.7. and [18], Theorems 2.2, 3.1, and it provides interesting relationships between the Gorestein projectivity with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ of a complex and that of all its terms. By this connection, we prove the following results, see Theorem 4.1 and Theorem 4.2, respectively.

Theorem 1.2. *A left R-module M belongs to $\mathcal{G}(\mathcal{A})$ if and only if there exists a $\operatorname{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$-exact exact sequence $\ldots \to G_1 \to G_0 \to G_{-1} \to \ldots$ in $\mathcal{G}(\mathcal{A})$ such that $M \cong \operatorname{Im}(G_0 \to G_{-1})$.*

Theorem 1.3. *A complex C of left R-modules belongs to $\mathcal{G}(\widetilde{\mathcal{A}})$ if and only if there exists a $\operatorname{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \operatorname{dg} \widetilde{\mathcal{B}})$-exact exact sequence $\ldots \to G^1 \to G^0 \to G^{-1} \to \ldots$ in $\mathcal{G}(\widetilde{\mathcal{A}})$ such that $C \cong \operatorname{Im}(G^0 \to G^{-1})$.*

These two results imply that the categories $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\widetilde{\mathcal{A}})$ possess stability, respectively.

The contents of this paper are summarized as follows. In Section 2, we review some basic notation and notions for use throughout the paper. Section 3 is devoted to introducing the notion of Gorenstein projective complexes with respect to cotorsion pairs and giving the proof of Theorem 1.1. By using Theorem 1.1, in Section 4 we give the proof of Theorem 1.2 and Theorem 1.3.

2. Preliminaries

Throughout this article, R denotes an associative ring with identity, modules are assumed to be unitary, and the default action of the ring is on the left. Right modules over R are hence treated as (left) modules over the opposite ring R°. We use R-Mod to denote the category of R-modules, $C(R)$ to denote the category of complexes of R-modules and $\mathcal{P}$, $\mathcal{F}, \mathcal{C}$ to denote the class of projective, flat, cotorsion R-modules, respectively.

A complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots$$

will be denoted by (C, δ) or simply C. The nth cycle, boundary, homology of C is denoted by $\mathrm{Z}_n(C)$, $\mathrm{B}_n(C)$, $\mathrm{H}_n(C)$, respectively. We will use superscripts to distinguish complexes. So if $\{C^i\}_{i\in I}$ is a family of complexes, C^i will be complex

$$\cdots \longrightarrow C^i_{n+1} \xrightarrow{\delta^i_{n+1}} C^i_n \xrightarrow{\delta^i_n} C^i_{n-1} \longrightarrow \cdots$$

Given an R-module M, we will denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with M in 1st and 0th degrees. Given a $C \in C(R)$ and an integer m, $C[m]$ denotes the complex such that $C[m]_n = C_{n-m}$ and whose boundary operators are $(-1)^m\delta_{n-m}$. Given $C, D \in C(R)$, we use $\mathrm{Hom}_{C(R)}(C, D)$ to present the group of all morphisms from C to D, and $\mathrm{Ext}^i_{C(R)}(C, D)$ denotes the groups one gets from the right derived functor of Hom for $i \geqslant 0$.

Let $C \in C(R^\circ)$ and $D \in C(R)$. The tensor product $C \otimes_R D$ is the $\mathbb{Z}$-complex whose underlying graded module is given by $(C \otimes_R D)_n = \bigoplus\limits_{i+j=n} C_i \otimes_R D_j$, and whose differential is defined by specifying its action on an elementary tensor of homogeneous elements as $\delta^{C\otimes_R D}(x \otimes y) = \delta^C(x) \otimes y + (-1)^{|x|}x \otimes \delta^D(y)$, where $|x|$ is the degree of x in C. Let $C\overline{\otimes}_R D = C \otimes_R D/\mathrm{B}(C \otimes_R D)$, that is, $C\overline{\otimes}_R D$ is the complex of abelian groups with nth entry $(C\overline{\otimes}_R D)_n = (C \otimes_R D)_n/\mathrm{B}_n(C \otimes_R D)$ and boundary map $\delta^{C\overline{\otimes}_R D}(\overline{x \otimes y}) = \overline{\delta^C(x) \otimes y}$, where $\overline{x \otimes y}$ is used to denote the coset in $(C \otimes_R D)_n/\mathrm{B}_n(C \otimes_R D)$. This gives us a new right exact bifunctor $-\overline{\otimes}_R-$ which has left derived functor $\overline{\mathrm{Tor}}_i(-,-)$.

For $C, D \in C(R)$, $\mathrm{Hom}_R(C, D)$ is the complex of abelian groups with the degree-n term $\mathrm{Hom}_R(C, D)_n = \prod\limits_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, D_{n+i})$, and its boundary operators are $\delta^{\mathrm{Hom}_R(C,D)}_n((f_i)_{i\in\mathbb{Z}}) = (\delta^D_{n+i}f_i - (-1)^n f_{i-1}\delta^C_i)_{i\in\mathbb{Z}}$ for any $(f_i)_{i\in\mathbb{Z}} \in \mathrm{Hom}_R(C, D)_n$. Let $\underline{\mathrm{Hom}}_R(C, D) = \mathrm{Z}(\mathrm{Hom}_R(C, D))$, that is, $\underline{\mathrm{Hom}}_R(C, D)$ is the complex of abelian groups with nth entry $\underline{\mathrm{Hom}}_R(C, D)_n = \mathrm{Z}_n(\mathrm{Hom}_R(C, D)) = \mathrm{Hom}_{C(R)}(C, D[-n])$ and boundary map $\delta^{\underline{\mathrm{Hom}}_R(C,D)}_n((f_i)_{i\in\mathbb{Z}}) = ((-1)^n\delta^D_{n+i}f_i)_{i\in\mathbb{Z}}$ for any $(f_i)_{i\in\mathbb{Z}} \in \mathrm{Hom}_R(C, D)_n$. Then we get new functors $\underline{\mathrm{Hom}}_R(C, -)$ and $\underline{\mathrm{Hom}}_R(-, D)$ which are both left exact. The book [7] is a standard reference for complexes.

Let $\mathcal{D}$ be an abelian category. A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects of $\mathcal{D}$ is called a cotorsion pair if $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{A} = {}^\perp\mathcal{B}$, where $\mathcal{A}^\perp = \{D \in \mathcal{D}\colon \mathrm{Ext}^1_{\mathcal{D}}(A, D) = 0$ for all $A \in \mathcal{A}\}$ and ${}^\perp\mathcal{B} = \{D \in \mathcal{D}\colon \mathrm{Ext}^1_{\mathcal{D}}(D, B) = 0$ for all $B \in \mathcal{B}\}$. A special $\mathcal{A}$-precover or special $\mathcal{B}$-preenvelope of an object $D \in \mathcal{D}$ is a short exact sequence $0 \to B \to A \to D \to 0$ or $0 \to D \to B' \to A' \to 0$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ or $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$, respectively. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be complete if every object $D \in \mathcal{D}$ has a special $\mathcal{A}$-precover and a special $\mathcal{B}$-preenvelope. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{D}$ is said to be hereditary if $\mathrm{Ext}^i_{\mathcal{D}}(A, B) = 0$ for all $A \in \mathcal{A}, B \in \mathcal{B}$ and all $i \geqslant 1$.

If we choose $\mathcal{D} = R$-Mod for some ring R, the most obvious example of a complete hereditary cotorsion pair is $(\mathcal{P}, R\text{-Mod})$. Perhaps one of the most useful complete hereditary cotorsion pairs is the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$. For a good reference on cotorsion pairs see [6].

Definition 2.1 ([8], Definition 3.3)**.** Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod and X an R-complex.

(1) X is called an $\mathcal{A}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{A}$ for all $n \in \mathbb{Z}$.

(2) X is called a $\mathcal{B}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{B}$ for all $n \in \mathbb{Z}$.

(3) X is called a dg-$\mathcal{A}$ complex if each $X_n \in \mathcal{A}$ and $\mathrm{Hom}_R(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.

(4) X is called a dg-$\mathcal{B}$ complex if each $X_n \in \mathcal{B}$ and $\mathrm{Hom}_R(A, X)$ is exact whenever A is an $\mathcal{A}$ complex.

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $\mathrm{dg}\,\widetilde{\mathcal{A}}$. Similarly, the $\mathcal{B}$ complexes are denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes is denoted by $\mathrm{dg}\,\widetilde{\mathcal{B}}$. Sometimes $\mathcal{A}$ complexes and dg-$\mathcal{A}$ complexes are called complexes by the name of the class $\mathcal{A}$. For example, the projective and dg-projective complexes are actually the $\mathcal{P}$ and dg-$\mathcal{P}$ complexes, respectively. It follows from [8], Proposition 3.6, that $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ and $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are cotorsion pairs in $C(R)$. Moreover, by [8], Corollary 3.13, [17], Theorem 2.4 and Corollary 2.7, or [14], Theorem 3.5, we have the following facts.

Lemma 2.2. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in R-Mod. Then the induced cotorsion pairs $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ and $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ in $C(R)$ are both complete and hereditary. Furthermore, $\mathrm{dg}\,\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$ and $\mathrm{dg}\,\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$, where $\mathcal{E}$ is the class of exact complexes.*

Let $\mathcal{D}$ be an abelian category and $\mathcal{H}$ a full subcategory of $\mathcal{D}$. Recall that a sequence $\mathbf{S}$ in $\mathcal{D}$ is $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{H})$-exact (resp., $\mathrm{Hom}_{\mathcal{D}}(\mathcal{H}, -)$-exact) if the sequence $\mathrm{Hom}_{\mathcal{D}}(\mathbf{S}, H)$ (resp., $\mathrm{Hom}_{\mathcal{D}}(H, \mathbf{S})$) is exact for any $H \in \mathcal{H}$.

Definition 2.3 ([16], Definition 3.1)**.** Let $(\mathcal{A}, \mathcal{B})$ be a complete and hereditary cotorsion pair in R-Mod. An R-module M is called Gorenstein projective with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ if there exists a $\mathrm{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$-exact exact sequence $\ldots \to A_1 \to A_0 \to A_{-1} \to \ldots$ with each $A_i \in \mathcal{A}$, such that $M \cong \mathrm{Im}(A_0 \to A_{-1})$. We let $\mathcal{G}(\mathcal{A})$ be the class of Gorenstein projective R-modules with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$.

Remark 2.4. (1) By completeness of the cotorsion pair $(\mathcal{A}, \mathcal{B})$, an R-module M is in $\mathcal{G}(\mathcal{A})$ if and only if $\mathrm{Ext}_R^{i \geqslant 1}(M, N) = 0$ for any $N \in \mathcal{A} \cap \mathcal{B}$ and there exists a $\mathrm{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$-exact exact sequence $0 \to M \to A_0 \to A_{-1} \to \dots$ with each $A_i \in \mathcal{A}$.

(2) This definition unifies the following notions: Gorenstein projective modules [5], [9] (in the case $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, R\text{-Mod})$); **F**-Gorenstein flat modules [10] (when $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$); and Gorenstein flat modules [7] (when $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$ and R is a right coherent ring), see [10], Lemma 3.2.

In what follows, we always assume that $(\mathcal{A}, \mathcal{B})$ is a complete and hereditary cotorsion pair in R-Mod.

3. Gorenstein projective complexes with respect to cotorsion pairs

Definition 3.1. An R-complex C is called Gorenstein projective with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ if there exists a $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence $\dots \to A^1 \to A^0 \to A^{-1} \to \dots$ with each $A^i \in \widetilde{\mathcal{A}}$ such that $C \cong \mathrm{Im}(A^0 \to A^{-1})$.

We denote the class of Gorenstein projective R-complexes with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ by $\mathcal{G}(\widetilde{\mathcal{A}})$.

Remark 3.2. (1) It is clear that $\widetilde{\mathcal{A}} \subseteq \mathcal{G}(\widetilde{\mathcal{A}})$. If $\mathbf{A} = \dots \to A^1 \to A^0 \to A^{-1} \to \dots$ is a $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence of complexes in $\widetilde{\mathcal{A}}$, then by symmetry, all the images, the kernels and the cokernels of $\mathbf{A}$ are in $\mathcal{G}(\widetilde{\mathcal{A}})$.

(2) If $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, R\text{-Mod})$, then Gorenstein projective complexes with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ are exactly the Gorenstein projective complexes in [4].

(3) If $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$, the flat cotorsion pair, then Gorenstein projective complexes with respect to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ are just **F**-Gorenstein flat complexes in [10].

Recall from [7] that a short exact sequence $0 \to S \to C \to C/S \to 0$ in $C(R)$ is pure if the sequence $0 \to D\overline{\otimes}_R S \to D\overline{\otimes}_R C \to D\overline{\otimes}_R C/S \to 0$ is exact for any $D \in C(R^\circ)$. According to [7], an R-complex C is called Gorenstein flat if there exists an exact sequence of flat complexes $\dots \to F^1 \to F^0 \to F^{-1} \to \dots$ with $C \cong \mathrm{Im}(F^0 \to F^{-1})$ which remains exact after applying $I\overline{\otimes}_R-$ for any injective R°-complex I. The next result shows that Gorenstein projective complexes with respect to the cotorsion pair $(\mathcal{F}, \mathcal{C})$ over right coherent rings are just Gorenstein flat complexes.

Proposition 3.3. *If R is a right coherent ring, then C is an* **F***-Gorenstein flat complex if and only if C is Gorenstein flat.*

P r o o f. $\Rightarrow$: Assume that C is an **F**-Gorenstein flat complex. Then there exists a $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{F}} \cap \mathrm{dg}\,\widetilde{\mathcal{C}})$-exact exact sequence $\dots \to F^1 \to F^0 \to F^{-1} \to \dots$ of flat complexes such that $C \cong \mathrm{Im}(F^0 \to F^{-1})$. Let I be an injective complex of right R-modules. Then $I^+[-n] \in \widetilde{\mathcal{F}} \cap \mathrm{dg}\,\widetilde{\mathcal{C}}$ for any $n \in \mathbb{Z}$ since R is right coherent, where $I^+ = \underline{\mathrm{Hom}}_{\mathbb{Z}}(I, \overline{\mathbb{Q}/\mathbb{Z}})$. Thus the sequence

$$\begin{aligned} \dots &\to \mathrm{Hom}_{C(R)}(F^{-1}, I^+[-n]) \to \\ &\to \mathrm{Hom}_{C(R)}(F^0, I^+[-n]) \to \mathrm{Hom}_{C(R)}(F^1, I^+[-n]) \to \dots \end{aligned}$$

is exact for any $n \in \mathbb{Z}$, and so

$$\dots \to \underline{\mathrm{Hom}}_R(F^{-1}, I^+) \to \underline{\mathrm{Hom}}_R(F^0, I^+) \to \underline{\mathrm{Hom}}_R(F^1, I^+) \to \dots$$

is exact. Hence the sequence

$$\dots \to I \overline{\otimes}_R F^1 \to I \overline{\otimes}_R F^0 \to I \overline{\otimes}_R F^{-1} \to \dots$$

is exact by [7], Proposition 4.2.1 (1). Therefore C is Gorenstein flat.

$\Leftarrow$: Suppose that C is a Gorenstein flat complex. Then there is an exact sequence $\dots \to F^1 \to F^0 \to F^{-1} \to \dots$ of flat complexes with $C \cong \mathrm{Im}(F^0 \to F^{-1})$ which remains exact after applying $I \overline{\otimes}_R -$ for any injective R°-complex I. Let $K \in \widetilde{\mathcal{F}} \cap \mathrm{dg}\,\widetilde{\mathcal{C}}$. Then we have a pure exact sequence $0 \to K \to K^{++} \to K^{++}/K \to 0$ by [7], Proposition 5.1.4 (4). Since $K \in \widetilde{\mathcal{F}}$, we get $K^{++} \in \widetilde{\mathcal{F}}$. So $K^{++}/K \in \widetilde{\mathcal{F}}$ by [8], Lemma 4.7. Thus the sequence $0 \to K \to K^{++} \to K^{++}/K \to 0$ is split. By [7], Proposition 4.2.1 (1), we have the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & (K^+ \overline{\otimes}_R F^{-1})^+ \arrow[r] \arrow[d, "\cong"'] & (K^+ \overline{\otimes}_R F^0)^+ \arrow[r] \arrow[d, "\cong"'] & (K^+ \overline{\otimes}_R F^1)^+ \arrow[r] \arrow[d, "\cong"'] & \cdots \\ \cdots \arrow[r] & \underline{\mathrm{Hom}}_R(F^{-1}, K^{++}) \arrow[r] & \underline{\mathrm{Hom}}_R(F^0, K^{++}) \arrow[r] & \underline{\mathrm{Hom}}_R(F^1, K^{++}) \arrow[r] & \cdots, \end{tikzcd}$$

where the top row is exact since K^+ is injective. So the lower row is exact. Hence the sequence

$$\dots \to \underline{\mathrm{Hom}}_R(F^{-1}, K) \to \underline{\mathrm{Hom}}_R(F^0, K) \to \underline{\mathrm{Hom}}_R(F^1, K) \to \dots$$

is exact. In particular, the sequence

$$\dots \to \mathrm{Hom}_{C(R)}(F^{-1}, K) \to \mathrm{Hom}_{C(R)}(F^0, K) \to \mathrm{Hom}_{C(R)}(F^1, K) \to \dots$$

is exact. So C is an **F**-Gorenstein flat complex. $\square$

The following result will be used in the sequel.

Lemma 3.4. *Let* $\ldots \to X^1 \to X^0 \to X^{-1} \to \ldots$ *be a* $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-*exact sequence of complexes, then the sequence* $\ldots \to X^1_n \to X^0_n \to X^{-1}_n \to \ldots$ *is* $\mathrm{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$-*exact for any* $n \in \mathbb{Z}$.

P r o o f. Let $K \in \mathcal{A} \cap \mathcal{B}$ and $n \in \mathbb{Z}$. Then $\overline{K}[n] \in \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}}$ by [8], Lemma 3.4. So we have the exact sequence

$$\ldots \to \mathrm{Hom}_{C(R)}(X^{-1}, \overline{K}[n]) \to \mathrm{Hom}_{C(R)}(X^0, \overline{K}[n]) \to \mathrm{Hom}_{C(R)}(X_1, \overline{K}[n]) \to \ldots$$

Using the standard adjunction of [8], Lemma 3.1 (2), we get the exact sequence

$$\ldots \to \mathrm{Hom}_R(X^{-1}_n, K) \to \mathrm{Hom}_R(X^0_n, K) \to \mathrm{Hom}_R(X^1_n, K) \to \ldots$$

This completes the proof. □

Now, we are in position to prove our main result, which gives a characterization of complexes in $\mathcal{G}(\widetilde{\mathcal{A}})$ and unifies [10], Theorem 4.7.

Theorem 3.5. *Let* C *be an* R-*complex. Then* $C \in \mathcal{G}(\widetilde{\mathcal{A}})$ *if and only if* $C_n \in \mathcal{G}(\mathcal{A})$ *for any* $n \in \mathbb{Z}$.

P r o o f. $\Rightarrow$: Assume that $C \in \mathcal{G}(\widetilde{\mathcal{A}})$. Then there exists a $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence $\ldots \to A^1 \to A^0 \to A^{-1} \to \ldots$ with each $A^i \in \widetilde{\mathcal{A}}$ such that $C \cong \mathrm{Im}(A^0 \to A^{-1})$. Now for any but fixed $n \in \mathbb{Z}$, by Lemma 3.4, we have the $\mathrm{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$-exact exact sequence of modules in $\mathcal{A}$

$$\ldots \to A^1_n \to A^0_n \to A^{-1}_n \to A^{-2}_n \to \ldots$$

such that $C_n = \mathrm{Im}(A^0_n \to A^{-1}_n)$. Hence $C_n \in \mathcal{G}(\mathcal{A})$.

$\Leftarrow$: Suppose that $C_n \in \mathcal{G}(\mathcal{A})$ for all $n \in \mathbb{Z}$. Then for any $n \in \mathbb{Z}$ there exists an exact sequence

$$0 \to C_n \to A_n \to L_n \to 0,$$

where $A_n \in \mathcal{A}$ and $L_n \in \mathcal{G}(\mathcal{A})$. These exact sequences induce a short exact sequence of complexes

$$0 \to \bigoplus_{n\in\mathbb{Z}} \overline{C_n}[n] \to \bigoplus_{n\in\mathbb{Z}} \overline{A_n}[n] \to \bigoplus_{n\in\mathbb{Z}} \overline{L_n}[n] \to 0.$$

Put $A^{-1} = \bigoplus\limits_{n\in\mathbb{Z}} \overline{A_n}[n]$. It is easy to see that $A^{-1} \in \widetilde{\mathcal{A}}$. On the other hand, there is an obvious (degreewise split) short exact sequence

$$0 \longrightarrow C \xrightarrow{\binom{1}{\delta}} \bigoplus_{n\in\mathbb{Z}} \overline{C_n}[n] \xrightarrow{(-\delta,1)} C[1] \longrightarrow 0,$$

where δ is the differential of C. Now let $\alpha\colon C \to A^{-1}$ be the composite

$$C \to \bigoplus_{n\in\mathbb{Z}} \overline{C_n}[n] \to \bigoplus_{n\in\mathbb{Z}} \overline{A_n}[n].$$

Then α is monoic since it is the composite of two monomorphisms. Denote $\operatorname{Coker}\alpha$ by C^{-1}. Then by the Snake lemma, we have a short exact sequence

$$0 \to C[1] \to C^{-1} \to \bigoplus_{n\in\mathbb{Z}} \overline{L_n}[n] \to 0.$$

Since each degree of $\bigoplus\limits_{n\in\mathbb{Z}} \overline{L_n}[n]$ and $C[1]$ is in $\mathcal{G}(\mathcal{A})$, each degree of C^{-1} belongs to $\mathcal{G}(\mathcal{A})$ by [16], Proposition 3.3 (1). Let $K \in \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}}$. Then $K \in \widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}} = \widetilde{\mathcal{A}\cap\mathcal{B}}$ by [8], Theorem 3.12. Thus $K \cong \prod\limits_{n\in\mathbb{Z}} \overline{\mathrm{Z}_n(K)}[n]$ by [11], Lemma 4.1. Hence

$$\operatorname{Ext}^1_{C(R)}(C^{-1}, K) \cong \prod_{n\in\mathbb{Z}} \operatorname{Ext}^1_{C(R)}(C^{-1}, \overline{\mathrm{Z}_n(K)}[n]) \cong \prod_{n\in\mathbb{Z}} \operatorname{Ext}^1_R(C^{-1}_n, \mathrm{Z}_n(K)) = 0,$$

where the second isomorphism follows from [8], Lemma 3.1 (2), and the last equality follows from Remark 2.4 (1). This implies that $0 \to C \to A^{-1} \to C^{-1} \to 0$ is $\operatorname{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact. Notice that C^{-1} has the same property as C, so we can use the same procedure to construct a $\operatorname{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence of complexes

$$(\dagger) \qquad 0 \to C \to A^{-1} \to A^{-2} \to \dots,$$

where each A^i is an $\mathcal{A}$-complex.

Since $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ is a complete cotorsion pair, we have a short exact sequence $0 \to C^1 \to A^0 \to C \to 0$, where $A^0 \in \widetilde{\mathcal{A}}$ and $C^1 \in \mathrm{dg}\,\widetilde{\mathcal{B}}$. Note that $C_n \in \mathcal{G}(\mathcal{A})$ for any $n \in \mathbb{Z}$, this sequence is $\operatorname{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact by a discussion similar to the above. Also, it follows from the exact sequence and [16], Proposition 3.3 (1), that each $C^1_n \in \mathcal{G}(\mathcal{A})$ for any $n \in \mathbb{Z}$. Thus we can continuously use the same method to construct a $\operatorname{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence

$$(\ddagger) \qquad \dots \to A^1 \to A^0 \to C \to 0,$$

where each A^i is an $\mathcal{A}$-complex.

Finally, gluing the sequences $(\dagger)$ and $(\ddagger)$ together, one has a $\operatorname{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence of complexes

$$\dots \to A^1 \to A^0 \to A^{-1} \to A^{-2} \to \dots$$

with all $A^i \in \widetilde{\mathcal{A}}$ such that $C \cong \operatorname{Im}(A^0 \twoheadrightarrow A^{-1})$. Hence $C \in \mathcal{G}(\widetilde{\mathcal{A}})$. $\square$

Let $\mathcal{D}$ be an abelian category with enough projective objects and injective objectives. Recall that a class $\mathcal{X}$ of objects of $\mathcal{D}$ is said to be projectively resolving or injectively resolving if it is closed under extensions and kernels of surjections or cokernels of injections, and contains all projective or injective objects of $\mathcal{D}$, respectively.

Corollary 3.6. *$\mathcal{G}(\widetilde{\mathcal{A}})$ is projectively resolving.*

Proof. Clearly, $\widetilde{\mathcal{P}} \subseteq \widetilde{\mathcal{A}} \subseteq \mathcal{G}(\widetilde{\mathcal{A}})$. Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of complexes with $C'' \in \mathcal{G}(\widetilde{\mathcal{A}})$. Then for any $n \in \mathbb{Z}$, in the exact sequence $0 \to C'_n \to C_n \to C''_n \to 0$, $C''_n \in \mathcal{G}(\mathcal{A})$ by Theorem 3.5. So $C'_n \in \mathcal{G}(\mathcal{A})$ if and only if $C_n \in \mathcal{G}(\mathcal{A})$ by [16], Proposition 3.3 (1). Hence $C' \in \mathcal{G}(\widetilde{\mathcal{A}})$ if and only if $C \in \mathcal{G}(\widetilde{\mathcal{A}})$ by Theorem 3.5. Now the result follows. □

Corollary 3.7. *Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence of complexes. If C', C belong to $\mathcal{G}(\widetilde{\mathcal{A}})$, then $C'' \in \mathcal{G}(\widetilde{\mathcal{A}})$ if and only if $\mathrm{Ext}^1_{C(R)}(C'', K) = 0$ for any $K \in \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}}$.*

Proof. $\Rightarrow$: It is obvious.

$\Leftarrow$: Let $n \in \mathbb{Z}$. Consider the exact sequence of R-modules

$$0 \to C'_n \to C_n \to C''_n \to 0.$$

By Theorem 3.5, C'_n, C_n belong to $\mathcal{G}(\mathcal{A})$. Let $K \in \mathcal{A} \cap \mathcal{B}$. Then $\overline{K}[n] \in \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}}$. Thus $\mathrm{Ext}^1_R(C''_n, K) \cong \mathrm{Ext}^1_{\mathcal{C}(R)}(C'', \overline{K}[n]) = 0$ by [8], Lemma 3.1 (2), and the hypothesis. Hence $C''_n \in \mathcal{G}(\mathcal{A})$ by [16], Proposition 3.3 (2). Therefore $C'' \in \mathcal{G}(\widetilde{\mathcal{A}})$ by Theorem 3.5. □

By Proposition 3.3, Theorem 3.5 and [10], Lemma 3.2, we immediately get:

Corollary 3.8 ([18], Theorem 3.1). *Let C be an R-complex. If R is a right coherent ring, then C is Gorenstein flat if and only if C_n is a Gorenstein flat R-module for any $n \in \mathbb{Z}$.*

4. Stability of Gorenstein categories with respect to cotorsion pairs

The stability of Gorenstein categories was initiated by Sather-Wagstaff, Sharif and White [12]. They proved that if R is a commutative ring, then an R-module M

is a Gorenstein projective or injective module if and only if there exists an exact sequence of Gorenstein projective or injective R-modules $G = \ldots \longrightarrow G_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} G_{-1} \longrightarrow \ldots$ such that the complexes $\mathrm{Hom}_R(H, G)$ and $\mathrm{Hom}_R(G, H)$ are exact for each Gorenstein projective or injective R-module H, respectively, and $M = \mathrm{Im}\delta_0$. This was developed by Bouchiba [1], Xu and Ding [13], respectively. They showed, via different methods, that over any ring R, an R-module M is Gorenstein projective or injective if and only if there exists an exact sequence of Gorenstein projective or injective R-modules $G = \ldots \longrightarrow G_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} G_{-1} \longrightarrow \ldots$ such that the complex $\mathrm{Hom}_R(G, H)$ or $\mathrm{Hom}_R(H, G)$ is exact for any projective or injective R-module H, respectively, and $M = \mathrm{Im}\ \delta_0$. For more details, see [1]. The stabiltity of Gorenstein flat R-module has been treated by Bouchiba and Khaloui [2], Xu and Ding [13], Yang and Liu [15], respectively. By using totally different techniques, they showed that over a left GF-closed ring R (a ring R over which the class of the Gorenstein flat R-modules is closed under extensions), an R-module M is Gorenstein flat if and only if there exists an exact sequence of Gorenstein flat R-modules $G = \ldots \longrightarrow G^1 \xrightarrow{\delta^1} G^0 \xrightarrow{\delta^0} G^{-1} \longrightarrow \ldots$ such that the complex $I \otimes_R G$ is exact for each Gorenstein injective (or injective) R°-module I and $M = \mathrm{Im}\ \delta^0$. By using Theorem 3.5, in this section we investigate the stability of $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\widetilde{\mathcal{A}})$.

The next result shows that the category $\mathcal{G}(\mathcal{A})$ possesses stability, which is a generalization of [12], Theorem A, [13], Theorem A, and [10], Theorem 3.8.

Theorem 4.1. *Let M be an R-module. Then the following statements are equivalent:*

(1) $M \in \mathcal{G}(\mathcal{A})$.

(2) *There exists a both $\mathrm{Hom}_R(\mathcal{G}(\mathcal{A}), -)$-exact and $\mathrm{Hom}_R(-, \mathcal{G}(\mathcal{A}))$-exact exact sequence*

$$\cdots \to G_1 \to G_0 \to G_{-1} \to \cdots$$

in $\mathcal{G}(\mathcal{A})$ so that $M \cong \mathrm{Im}(G_0 \to G_{-1})$.

(3) *There exists a $\mathrm{Hom}_R(-, \mathcal{G}(\mathcal{A}))$-exact exact sequence*

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\mathcal{A})$ such that $M \cong \mathrm{Im}(G_0 \to G_{-1})$.

(4) *There exists a $\mathrm{Hom}_R(-, \mathcal{A})$-exact exact sequence*

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\mathcal{A})$ such that $M \cong \mathrm{Im} G_0 \to G_{-1})$.

(5) *There exists a* $\mathrm{Hom}_R(-,\mathcal{A}\cap\mathcal{B})$*-exact exact sequence*

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\mathcal{A})$ *such that* $M \cong \mathrm{Im}(G_0 \to G_{-1})$.

Proof. (1)⇒(2)⇒(3)⇒(4)⇒(5) are clear.

(5)⇒(1): Assume that there is a $\mathrm{Hom}_R(-,\mathcal{A}\cap\mathcal{B})$-exact exact sequence

$$G = \ldots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$

in $\mathcal{G}(\mathcal{A})$ such that $M \cong \mathrm{Z}_{-1}(G)$. Then $G \in \mathcal{G}(\widetilde{\mathcal{A}})$ by Theorem 3.5. Thus there exists a $\mathrm{Hom}_{C(R)}(-,\widetilde{\mathcal{A}}\cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence

$$\cdots \longrightarrow A^1 \xrightarrow{\sigma^1} A^0 \xrightarrow{\sigma^0} A^{-1} \xrightarrow{\sigma^{-1}} A^{-2} \longrightarrow \cdots$$

with each $A^i \in \widetilde{\mathcal{A}}$ such that $G \cong \mathrm{Ker}\sigma^{-1}$. Set $K^i = \mathrm{Ker}\sigma^i$ for $i \in \mathbb{Z}$. Then $K^i \in \mathcal{G}(\widetilde{\mathcal{A}})$ and K^i is exact for any $i \in \mathbb{Z}$ since $K^{-1} = G$ and all A^i are exact. So, by [11], Lemma 4.15 (1), we have the exact sequence

$$(\natural) \qquad \cdots \longrightarrow \mathrm{Z}_{-1}(A^1) \xrightarrow{\mathrm{Z}_{-1}(\sigma^1)} \mathrm{Z}_{-1}(A^0) \xrightarrow{\mathrm{Z}_{-1}(\sigma^0)} \mathrm{Z}_{-1}(A^{-1}) \longrightarrow \cdots$$

with each $\mathrm{Z}_{-1}(A^i) \in \mathcal{A}$, such that $M \cong \mathrm{Z}_{-1}(G) = \mathrm{Ker}(\mathrm{Z}_{-1}(\sigma^{-1}))$. To show $M \in \mathcal{G}(\mathcal{A})$, we need only to show that the sequence $(\natural)$ is $\mathrm{Hom}_R(-,\mathcal{A}\cap\mathcal{B})$-exact.

Let $H \in \mathcal{A}\cap\mathcal{B}$, it suffices to show that $\mathrm{Ext}^1_R(\mathrm{Z}_{-1}(K^i),H) = 0$ for all $i \in \mathbb{Z}$. Since each $K^i \in \mathcal{G}(\widetilde{\mathcal{A}})$, all $K^i_n \in \mathcal{G}(\mathcal{A})$ by Theorem 3.5. Thus, for any $i \in \mathbb{Z}$, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(K^{i-1},H) \longrightarrow \mathrm{Hom}_R(A^i,H) \longrightarrow \mathrm{Hom}_R(K^i,H) \longrightarrow 0$$

is exact. By the hypothesis, $\mathrm{Hom}_R(K^{-1},H)$ is exact. Note that $\mathrm{Hom}_R(A^i,H)$ is exact for each $i \in \mathbb{Z}$, then $\mathrm{Hom}_R(K^i,H)$ is exact for any $i \in \mathbb{Z}$. Hence $\mathrm{Ext}^1_R(\mathrm{Z}_{-1}(K^i),H) = 0$ since each $K^i_0 \in \mathcal{G}(\mathcal{A})$. Thus the sequence $(\natural)$ is $\mathrm{Hom}_R(-,\mathcal{A}\cap\mathcal{B})$-exact, as desired. □

Finally, by applying Theorem 3.5 and Theorem 4.1, we can achieve the following stability result for $\mathcal{G}(\widetilde{\mathcal{A}})$, which is a unification of [13], Theorem 3.1, and [10], Theorem 4.11.

Theorem 4.2. *Let C be a complex of R-modules. Then the following statements are equivalent:*

(1) $C \in \mathcal{G}(\widetilde{\mathcal{A}})$.

(2) *There exists a both* $\mathrm{Hom}_{C(R)}(\mathcal{G}(\widetilde{\mathcal{A}}), -)$*-exact and* $\mathrm{Hom}_{C(R)}(-, \mathcal{G}(\widetilde{\mathcal{A}}))$*-exact exact sequence*

$$\cdots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\widetilde{\mathcal{A}})$ *such that* $C \cong \mathrm{Im}(G^0 \to G^{-1})$.

(3) *There is a* $\mathrm{Hom}_{C(R)}(-, \mathcal{G}(\widetilde{\mathcal{A}}))$*-exact exact sequence*

$$\cdots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\widetilde{\mathcal{A}})$ *such that* $C \cong \mathrm{Im}(G^0 \to G^{-1})$.

(4) *There is a* $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}})$*-exact exact sequence*

$$\cdots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\widetilde{\mathcal{A}})$ *such that* $C \cong \mathrm{Im}(G^0 \to G^{-1})$.

(5) *There is a* $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\widetilde{\mathcal{B}})$*-exact exact sequence*

$$\cdots \longrightarrow G^1 \longrightarrow G^0 \longrightarrow G^{-1} \longrightarrow \cdots$$

in $\mathcal{G}(\widetilde{\mathcal{A}})$ *such that* $C \cong \mathrm{Im}(G^0 \to G^{-1})$.

Proof. (1)⇒(2)⇒(3)⇒(4)⇒(5) are trivial.

(5)⇒(1): Suppose that there exists a $\mathrm{Hom}_{C(R)}(-, \widetilde{\mathcal{A}} \cap \mathrm{dg}\,\widetilde{\mathcal{B}})$-exact exact sequence

$$\cdots \longrightarrow G^1 \xrightarrow{\sigma^1} G^0 \xrightarrow{\sigma^0} G^{-1} \xrightarrow{\sigma^{-1}} G^{-2} \longrightarrow \cdots$$

in $\mathcal{G}(\widetilde{\mathcal{A}})$ such that $C \cong \mathrm{Im}\,\sigma^0$. Then for any $n \in \mathbb{Z}$, by Lemma 3.4 we have the $\mathrm{Hom}_R(-, \mathcal{A} \cap \mathcal{B})$-exact exact sequence of modules

$$\cdots \longrightarrow G^1_n \xrightarrow{\sigma^1_n} G^0_n \xrightarrow{\sigma^0_n} G^{-1}_n \xrightarrow{\sigma^{-1}_n} G^{-2}_n \longrightarrow \cdots$$

such that $C_n \cong \mathrm{Im}\,\sigma^n_0$. By Theorem 3.5, $G^i_n \in \mathcal{G}(\mathcal{A})$ for each $i \in \mathbb{Z}$. Thus $C_n \in \mathcal{G}(\mathcal{A})$ by Theorem 4.1. Hence $C \in \mathcal{G}(\widetilde{\mathcal{A}})$ by Theorem 3.5. □

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