

SUMS OF MULTIPLICATIVE FUNCTION IN SPECIAL
ARITHMETIC PROGRESSIONS

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Abstract. We find, via the Selberg-Delange method, an asymptotic formula for the mean of arithmetic functions on certain APs. It generalizes a result due to Cui and Wu (2014).

Keywords: Selberg-Delange method; multiplicative function; arithmetic progressions

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1. INTRODUCTION

Many number-theoretic problems lead to the study of mean values of arithmetic functions. In [1] Cui and Wu obtain mean values of certain arithmetic functions over short intervals by using the Selberg-Delange method and zero density estimates of the Riemann zeta function.

In order to state their result, it is necessary to introduce some notations. From [10], Theorem II.5.1, the function

$$Z(s; z) := \frac{((s-1)\zeta(s))^z}{s} \quad (z \in \mathbb{C})$$

is holomorphic in disc $|s-1| < 1$ and admits the Taylor series expansion

$$(1.1) \quad Z(s; z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z)}{j!} (s-1)^j,$$

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where $\gamma_j(z)$ are entire functions of z that satisfy for all $B > 0$ and $\varepsilon > 0$ the estimate

$$\frac{\gamma_j(z)}{j!} \ll_{B,\varepsilon} (1 + \varepsilon)^j \quad (j \geq 0, |z| \leq B).$$

The following is the result due to Cui and Wu.

Theorem 1.1. *Let $f(n) \ll_\varepsilon n^\varepsilon$ be a multiplicative function and let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $M > 0$ be some constants. Suppose that the Dirichlet series $\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ is of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$. Then for any $\varepsilon > 0$ we have*

$$(1.2) \quad \sum_{x < n \leq x+y} f(n) = y(\log x)^{\kappa-1} \left(\sum_{l=0}^N \frac{\lambda_l(\kappa, w)}{(\log x)^l} + O(R_N(x, y)) \right)$$

uniformly for

$$x \geq y \geq x^{\theta(\kappa, \delta) + \varepsilon} \geq 2, \quad N \geq 0, \quad 0 \leq \kappa \leq B, \quad |w| \leq B,$$

where $\theta(\kappa, \delta) := (5\kappa + 15\delta + 21)/(5\kappa + 15\delta + 36)$, $\lambda_l(\kappa, w) := g_l(\kappa, w)/\Gamma(\kappa - l)$,

$$g_l(\kappa, w) := \frac{1}{l!} \sum_{j=0}^l \binom{l}{j} \frac{\partial^{l-j} (G(s; \kappa, w) \zeta(2s)^{-w})}{\partial s^{l-j}} \Big|_{s=1} \gamma_j(\kappa),$$

where $\gamma_j(\kappa)$ is defined by (1.1) and

$$R_N(x, y) := \frac{y}{x} \sum_{l=1}^{N+1} \frac{l |\lambda_{l-1}(\kappa, w)|}{(\log x)^l} + M \left(\left(\frac{c_1 N + 1}{\log x} \right)^{N+1} + \frac{(c_1 N + 1)^{N+1}}{e^{c_2 (\log x)^{1/3} (\log_2 x)^{-1/3}}} \right)$$

for some constants $c_1 > 0$ and $c_2 > 0$. The implied constant in the O -term depends only on A, B, α, δ and ε .

Note. Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ is said to be of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$ if the following conditions are verified:

(a) for any $\varepsilon > 0$ we have

$$(1.3) \quad |f(n)| \ll_\varepsilon n^\varepsilon \quad (n \geq 1),$$

where the implied constant depends only on ε ;

(b) we have

$$\sum_{n=1}^{\infty} |f(n)|n^{-\sigma} \leq (\sigma - 1)^{-\alpha} \quad (\sigma > 1);$$

(c) the Dirichlet series

$$(1.4) \quad G(s; \kappa, w) := \mathcal{F}(s)\zeta(s)^{-z}\zeta(2s)^w$$

can be analytically continued to a holomorphic function in (an open set containing) $\sigma \geq \frac{1}{2}$ and, in this region, $G(s; \kappa, w)$ satisfies the bound

$$(1.5) \quad |G(s; \kappa, w)| \leq M(|\tau| + 1)^{\max(\delta(1-\sigma), 0)} \log^A(|\tau| + 1)$$

uniformly for $0 < \kappa \leq B$ and $|w| \leq B$.

It is natural to consider the analogous result of (1.2) for arithmetic progressions mod q . Nevertheless, the problem in arithmetic progressions is more difficult for large moduli q , partly because of the possible Siegel zero, and partly because of a zero-free region for the L -function (excluding the Siegel zero). So we only consider some special arithmetic progressions following the idea of Gallagher [4]. In this note, we prove the following results.

Theorem 1.2. *Let p_0 be a fixed odd prime, $q = p_0^r$ be an integer with r an integer and l be an integer such that $(l, q) = 1$ and let $f(n)$ be a multiplicative function such that $f(p) = \alpha$ ($0 < \alpha < 1$), $f(p^\nu) \ll p^{\delta\nu}$ ($\nu \geq 2$) for some $\delta < 0$. For any $\varepsilon > 0$, $0 < \theta < 1$ and $q \leq x^{15/(2(10\alpha+21))-\varepsilon}$,*

$$(1.6) \quad \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} f(n) = \frac{x}{\varphi(q)} \left(\sum_{0 \leq k \leq N} \frac{a_k}{(\log x)^{k+1-\alpha}} + O(R_N(x)) \right)$$

uniformly for $x \geq 2$, $N \geq 0$, where

$$a_k := \frac{1}{\Gamma(\alpha - k)} \sum_{h+j=k} \frac{G^{(h)}(1, \chi_0) \gamma_j(\alpha)}{h! j!},$$

$$G(s, \chi_0) := \prod_p \left(1 + \frac{\chi_0(p)f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu)f(p^\nu)}{p^{\nu s}} \right) \left(1 - \frac{\chi_0(p)}{p^s} \right)^\alpha,$$

$$R_N(x) := O\left(\left(\frac{1}{\log x} \right)^{N+2-\alpha} + e^{-\log^{(1-\theta)/2} x} \right)$$

and the implied constant depends only on ε , $\gamma_j(z)$ is defined by (1.1).

Note that for generally large moduli, at present there seems to be little hope for a proof of this, the reason being the Grand Riemman Hypothesis.

2. NOTATIONS AND PRELIMINARIES

Throughout the paper we will use the following notations: $s = \sigma + i\tau$, ε always denotes a sufficiently small positive number, p denotes a prime number, the parameters T and x are sufficiently large real numbers. When we write $f = O(g)$ or $f \ll g$ we will mean $|f| \leq Cg$ for some absolute constant C . When implied constants depend upon some parameters, we sometimes indicate that by a subscript.

Our work is inspired by Selberg-Delange method, which was developed by Selberg [9] and Delange [2], [3]. The method has been applied to some arithmetic problems, see [5], [6] and [7]. For more details, the reader is referred to the book by Tenenbaum [10].

3. SOME LEMMAS

Lemma 3.1. *Let $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_a . Suppose that there exists a real number $\eta > 0$ such that:*

- (i) $\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\eta} \quad (\sigma > \sigma_a)$,
and that $B(n)$ is a nondecreasing function satisfying
- (ii) $|a_n| \leq B(n) \quad (n \geq 1)$.

Then for $x \geq 2$, $T \geq 2$, $\sigma \leq \sigma_a$, $\kappa := \sigma_a - \sigma + 1/\log x$ we have

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa+iT}^{\kappa-iT} F(s+w) x^w \frac{dw}{w} + O\left(x^{\sigma_a - \sigma} \frac{(\log x)^\eta}{T} + \frac{B(2x)}{x^\sigma} \left(1 + x \frac{\log T}{T}\right)\right).$$

Proof. See [10], Corollary II.2.2.1. □

Lemma 3.2. *Let $q > 2$ be an integer and χ be Dirichlet character modulo q . Then we have*

$$L(\sigma + i\tau, \chi) \ll q^{1-\sigma} (|\tau| + 1)^{1/6} \ln(|\tau| + 1).$$

Proof. See [8], Theorem 1, page 485. □

Lemma 3.3. *Let $q > 2$ be an integer and χ be a non principal Dirichlet character modulo q . For $s = \sigma + i\tau$, $0 < \varepsilon < \frac{1}{2}$, $\varepsilon \leq \sigma < 1$, $|\tau| + 2 \leq T$ we have*

$$L(\sigma + i\tau, \chi) \ll_\varepsilon (q^{1/2} T)^{1-\sigma+\varepsilon}.$$

Proof. The lemma is Exercise 241 of [11]. \square

Lemma 3.4. For $q = p_0^r$ (p_0 odd prime), the Dirichlet L -function to modulus q has no zeros in region

$$(3.1) \quad \sigma > 1 - \frac{c}{\log^\gamma(q(|t| + 2))}, \quad \gamma < 1.$$

Proof. See [4], Theorem 2. \square

Lemma 3.5. Let $f(n)$ be a multiplicative function such that $f(p) = \alpha$ ($0 < \alpha < 1$), $f(p^\nu) \ll p^{\delta\nu}$ ($\nu \geq 2$) for some $\delta < 0$ and χ_0 be the principal character to the modulus q . For any $\varepsilon > 0$,

$$(3.2) \quad \sum_{n \leq x} \chi_0(n) f(n) = \frac{x}{(\log x)^{1-\alpha}} \left(\sum_{0 \leq k \leq N} \frac{a_k}{(\log x)^k} + O_\varepsilon \left(\frac{1}{(\log x)^{N+1}} \right) \right)$$

holds for $x \geq 2$, where

$$a_k := \frac{1}{\Gamma(\alpha - k)} \sum_{h+j=k} \frac{G^{(h)}(1) \gamma_j(\alpha)}{h! j!},$$

$$G(s, \chi_0) := \prod_p \left(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}} \right) \left(1 - \frac{\chi_0(p)}{p^s} \right)^\alpha$$

and $\gamma_j(z)$ is defined by (1.1).

Proof. We write for $\operatorname{Re} s > 1$,

$$(3.3) \quad \begin{aligned} \mathcal{F}(s, \chi_0) &:= \sum_{n=1}^{\infty} \frac{\chi_0(n) f(n)}{n^s} = \prod_p \left(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}} \right) \\ &= \prod_p \left(1 + \frac{\chi_0(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi_0(p^\nu) f(p^\nu)}{p^{\nu s}} \right) \left(1 - \frac{\chi_0(p)}{p^s} \right)^\alpha \left(1 - \frac{\chi_0(p)}{p^s} \right)^{-\alpha} \\ &= L(s, \chi_0)^\alpha G(s, \chi_0) = (s-1)^{-\alpha} \cdot \prod_{p|q} \left(1 - \frac{1}{p^s} \right)^\alpha ((s-1)\zeta(s))^\alpha \cdot G(s, \chi_0) \\ &=: (s-1)^{-\alpha} \cdot Z(s; \alpha) \cdot G(s, \chi_0), \end{aligned}$$

where $Z(s; \alpha)$ is holomorphic and $O_q(M)$ in the disc $|s-1| \leq c$ ($0 < c < 1/10$) and $G(s, \chi_0)$ is expandable as a Dirichlet series

$$G(s, \chi_0) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

where $g(n)$ is a multiplicative function for which the values on primes' powers are determined by the identity

$$1 + \sum_{\nu \geq 1} g(p^\nu) \xi^\nu = (1 - \xi)^\alpha \sum_{\nu \geq 0} f(p^\nu) \xi^\nu \quad (|\xi| \leq 1).$$

In particular, we have $g(p) = 0$ and Cauchy inequality implies that $|g(p^\nu)| \ll_\varepsilon p^{\varepsilon\nu}$ ($\varepsilon > 0$). So we have shown that for $\sigma > \frac{1}{2} + \varepsilon$

$$\sum_p \sum_{\nu \geq 1} |g(p^\nu)| p^{-\nu\sigma} \leq \sum_p \frac{1}{p^{\sigma-\varepsilon}(p^{\sigma-\varepsilon} - 1)} \leq \frac{c}{(\sigma - \varepsilon) - 1/2},$$

where c is an absolute constant. Then we deduce that $G(s, \chi_0)$ is absolutely convergent for $\sigma > \frac{1}{2} + \varepsilon$ and $G(s, \chi_0) \ll_\varepsilon 1$. By using Selberg-Delange theorem [10], Theorem III. 5.3, we obtain (3.2). \square

Lemma 3.6. *Let $f(n)$ be a multiplicative function such that $f(p) = \alpha$ ($0 < \alpha < 1$), $f(p^\nu) \ll p^{\delta\nu}$ ($\nu \geq 2$) for some $\delta < 0$ and χ be a nonprincipal Dirichlet character modulo q , where $q = p_0^r$ (p_0 odd prime). For any $0 < \varepsilon < \frac{1}{2}$, $0 < \theta < 1$ and $q \leq x^{15/(2(10\alpha+21))-\varepsilon}$,*

$$(3.4) \quad \sum_{\chi \neq \chi_0} \max_{y \leq x} \left| \sum_{n \leq y} \chi(n) f(n) \right| = O(x \exp(-\log^{(1-\theta)/2} x))$$

holds for $x \geq 2$, where the implied constant depends only on ε .

Proof. As in Lemma 3.5, we have

$$(3.5) \quad \begin{aligned} \mathcal{F}(s, \chi) &:= \sum_{n=1}^{\infty} \frac{\chi(n) f(n)}{n^s} = \prod_p \left(1 + \frac{\chi(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi(p^\nu) f(p^\nu)}{p^{\nu s}} \right) \\ &= \prod_p \left(1 + \frac{\chi(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi(p^\nu) f(p^\nu)}{p^{\nu s}} \right) \left(1 - \frac{\chi(p)}{p^s} \right)^\alpha \left(1 - \frac{\chi(p)}{p^s} \right)^{-\alpha} \\ &=: L(s, \chi)^\alpha G(s, \chi), \end{aligned}$$

where

$$G(s, \chi) = \prod_p \left(1 + \frac{\chi(p) f(p)}{p^s} + \sum_{\nu=2}^{\infty} \frac{\chi(p^\nu) f(p^\nu)}{p^{\nu s}} \right) \left(1 - \frac{\chi(p)}{p^s} \right)^\alpha.$$

As similarly shown in Lemma 3.5, we easily see that $G(s, \chi) \ll_\varepsilon 1$ for $\operatorname{Re} s > \frac{1}{2} + \varepsilon$.

We can apply Lemma 3.1 with the choice of parameters $\sigma_a = 1$, $B(n) = n^\varepsilon$, $\eta = \eta$, $\sigma = 0$ and $T = x^{15/(2(10\alpha+21))+\sqrt{\varepsilon}}$ to write

$$\sum_{n \leq y} \chi(n) f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \mathcal{F}(s, \chi) \frac{y^s}{s} ds + O\left(\frac{y^{1+\varepsilon}}{T}\right),$$

where $b = 1 + 2/\log x$ and $100 \leq T \leq x$ such that $L(\sigma + iT, \chi) \neq 0$ for $0 < \sigma < 1$.

Let \mathcal{L} be the boundary of the modified rectangle with vertices $(\frac{1}{2} + \varepsilon) \pm iT$ and $b \pm iT$, where

- ▷ $\varepsilon > 0$ is a small constant chosen so that $L(\frac{1}{2} + \varepsilon + i\gamma, \chi) \neq 0$ for $|\gamma| < T$, and
- ▷ the zeros of $L(s, \chi)$ of the form $\varrho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\varrho = \beta + i\gamma$.
- ▷ \mathcal{L}_1 and \mathcal{L}_2 denote horizontal segment $[(\frac{1}{2} + \varepsilon) \pm iT, (1 + 2/\log x) \pm iT]$, \mathcal{L}_3 and \mathcal{L}_4 denote vertical segment $[(\frac{1}{2} + \varepsilon), (\frac{1}{2} + \varepsilon) \pm iT] \setminus \{\varrho: \varrho = (\frac{1}{2} + \varepsilon) \pm i\gamma\}$, Γ_ϱ denotes horizontal segment $[(\frac{1}{2} + \varepsilon) \pm i\gamma, \beta \pm i\gamma]$.

Clearly the function $\mathcal{F}(s, \chi)$ is analytic inside \mathcal{L} . By Cauchy residue theorem, we can write

$$(3.6) \quad \sum_{n \leq y} \chi(n) f(n) = I_1 + \dots + I_4 + \sum_{\substack{\beta > \frac{1}{2} + \varepsilon \\ |\gamma| < T}} I_\varrho + O_\varepsilon\left(\frac{y^{1+\varepsilon}}{T}\right),$$

where

$$I_j := \frac{1}{2\pi i} \int_{\mathcal{L}_j} \mathcal{F}(s, \chi) \frac{y^s}{s} ds$$

and

$$I_\varrho := \frac{1}{2\pi i} \int_{\Gamma_\varrho} \mathcal{F}(s, \chi) \frac{y^s}{s} ds.$$

A. Estimation of I_1 and I_2 . In view of (3.5) and Lemma 3.3, we have

$$(3.7) \quad \mathcal{F}(s, \chi) \ll (q^{1/2}T)^{\alpha(1-\sigma)+\varepsilon}.$$

Thus

$$(3.8) \quad |I_1| + |I_2| \ll \int_{1/2+\varepsilon}^{1+2/\log x} (q^{1/2}T)^{\alpha(1-\sigma)+\varepsilon} \cdot \frac{y^\sigma}{T} d\sigma \\ \ll \frac{y}{T} \int_{1/2+\varepsilon}^{1+2/\log x} \left(\frac{(q^{1/2}T)^\alpha}{y}\right)^{1-\sigma} d\sigma \ll \frac{y}{T} \cdot e^{(1/2-\varepsilon)\log((q^{1/2}T)^\alpha/y)}.$$

B. Estimation of I_3 and I_4 . For $s = (\frac{1}{2} + \varepsilon) + i\tau$ with $0 \leq |\tau| \leq T$, in view of (3.5) and Lemma 3.2, we have

$$(3.9) \quad \mathcal{F}(s, \chi) \ll q^{\alpha/2}(|\tau| + 1)^{\alpha/6+\varepsilon}.$$

Thus

$$(3.10) \quad |I_3| + |I_4| \ll \int_0^T (q^{\alpha/2}(|\tau| + 1)^{\alpha/6+\varepsilon}) \frac{y^{1/2+\varepsilon}}{|(1/2 + \varepsilon) + i\tau|} d\tau \ll y^{1/2+\varepsilon} q^{\alpha/2} T^{\alpha/6}.$$

C. Estimation of I_ϱ . For $s = \sigma + i\gamma$ with $\frac{1}{2} + \varepsilon < \sigma \leq \beta \leq 1 - \sigma_0(\gamma)$, in view of (3.5) and Lemma 3.2, we have

$$\mathcal{F}(s, \chi) \ll q^{\alpha(1-\sigma)} |\gamma|^{\alpha/6+\varepsilon}.$$

Then we deduce that

$$(3.11) \quad I_\varrho \ll \int_{1/2+\varepsilon}^{\beta} (q^{\alpha(1-\sigma)} |\gamma|^{\alpha/6+\varepsilon}) \frac{y^\sigma}{|\sigma + i\gamma|} d\sigma.$$

Denote by $N(\sigma, T, \chi)$ the number of zeros of $L(s, \chi)$ in the region $\operatorname{Re} s \geq \sigma$ and $|\operatorname{Im} s| \leq T$. Summing (3.11) over $|\gamma| < T$ and interchanging the summations, we have

$$\sum_{\substack{\beta > 1/2+\varepsilon \\ |\gamma| < T}} |I_\varrho| \ll \log T \max_{T_0 < T} \int_{1/2+\varepsilon}^{1-\sigma_0(T_0)} q^{\alpha(1-\sigma)} T_0^{\alpha/6+\varepsilon} \cdot \frac{y^\sigma}{T_0} \cdot N(\sigma, T_0, \chi) d\sigma.$$

It is well-known that

$$(3.12) \quad N(\sigma, T, q) := \sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (qT)^{(12/5)(1-\sigma)} (\ln qT)^9$$

and in view of Lemma 3.4,

$$\sigma_0(t) = \frac{c}{\log^\gamma(q(|t| + 2))}, \quad \gamma < 1,$$

for $\frac{1}{2} + \varepsilon \leq \sigma \leq 1$ and $T \geq 2$. Thus

$$(3.13) \quad \begin{aligned} & \sum_{\chi \neq \chi_0} \sum_{\substack{\beta > 1/2+\varepsilon \\ |\gamma| < T}} |I_\varrho| \\ & \ll \log T \max_{T_0 < T} \int_{1/2+\varepsilon}^{1-\sigma_0(T_0)} q^{\alpha(1-\sigma)} T_0^{\alpha/6+\varepsilon} \cdot \frac{y^\sigma}{T_0} \cdot (qT_0)^{(12/5)(1-\sigma)} \log(qT_0)^9 d\sigma \\ & \ll y \log T \max_{T_0 < T} \log(qT_0)^9 \int_{1/2+\varepsilon}^{1-\sigma_0(T_0)} \left(\frac{q^{\alpha+12/5} T_0^{12/5}}{y} \right)^{1-\sigma} \cdot \frac{1}{T_0^{1-\alpha/6}} d\sigma \\ & \ll y \log T \max_{T_0 < T} \log(qT_0)^9 \int_{1/2+\varepsilon}^{1-\sigma_0(T_0)} \left(\frac{q^{\alpha+12/5} T_0^{2/5+\alpha/3}}{y} \right)^{1-\sigma} d\sigma \\ & \ll y \log T \max_{T_0 < T} \log(qT_0)^9 \left(\frac{q^{\alpha+12/5} T_0^{2/5+\alpha/3}}{y} \right)^{\sigma_0(T_0)} \\ & \ll y \log T \log(qT)^9 \left(\frac{q^{\alpha+12/5} T^{2/5+\alpha/3}}{y} \right)^{\sigma_0(T)}. \end{aligned}$$

Inserting (3.8), (3.10) and (3.13) into (3.6), we find that

$$\begin{aligned} \sum_{\chi \neq \chi_0} \max_{y \leq x} \left| \sum_{n \leq y} \chi(n) f(n) \right| &= \left(\frac{qx}{T} \cdot e^{(1/2-\varepsilon) \log((q^{1/2}T)^\alpha/x)} + q^{1+\alpha/2} x^{1/2} T^{\alpha/6+\varepsilon} \right. \\ &\quad \left. + x \log T \log(qT)^9 \left(\frac{q^{\alpha+12/5} T^{2/5+\alpha/3}}{x} \right)^{\sigma_0(T)} \right) + O_\varepsilon \left(\frac{qx^{1+\varepsilon}}{T} \right). \end{aligned}$$

For $q \leq x^{15/(2(10\alpha+21))-\varepsilon}$, noting $T = x^{15/(2(10\alpha+21))+\sqrt{\varepsilon}}$ and $0 < \alpha < 1$, we obtain (3.4). \square

4. PROOF OF THEOREM 1.2.

We are now ready to prove Theorem 1.2.

Proof. The starting point of the proof is the following observation.

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} f(n) &= \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) \sum_{n \leq x} \chi(n) f(n) \\ &= \frac{1}{\varphi(q)} \left(\sum_{n \leq x} \chi_0(n) f(n) + \sum_{\chi \neq \chi_0} \bar{\chi}(l) \sum_{n \leq x} \chi(n) f(n) \right). \end{aligned}$$

According to Lemma 3.5 and Lemma 3.6, we have the following result:

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} f(n) &= \frac{x}{\varphi(q)} \left(\sum_{0 \leq k \leq N} \frac{a_k}{(\log x)^{k+1-\alpha}} \right. \\ &\quad \left. + O_\varepsilon \left(\frac{1}{(\log x)^{N+2-\alpha}} \right) + O(e^{-\log^{(1-\theta)/2} x}) \right). \end{aligned}$$

This completes the proof of Theorem 1.2. \square

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