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NECESSARY AND SUFFICIENT CONDITIONS FOR THE L^1-CONVERGENCE OF DOUBLE FOURIER SERIES

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Abstract. We extend the results of paper of F. Móricz (2010), where necessary conditions were given for the L^1-convergence of double Fourier series. We also give necessary and sufficient conditions for the L^1-convergence under appropriate assumptions.

Keywords: double Fourier series; L^1-convergence; logarithm bound variation double sequences

MSC 2010: 42B05, 42B99

1. Introduction

Let $f = f(x, y)\colon\ \mathbb{T}^2 = [-\pi, \pi)\times[-\pi, \pi) \to \mathbb{C}$ be an integrable function in Lebesgue's sense, shortly $f \in L^1(\mathbb{T}^2)$, which has the double Fourier series of the form

$$f(x,y) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\mathrm{e}^{\mathrm{i}(jx+ky)}, \quad (x,y)\in\mathbb{T}^2, \tag{1.1}$$

where $\{c_{jk}\}_{j,k=0}^{\infty} \subset \mathbb{C}$ are the Fourier coefficients of f:

$$c_{jk} = \frac{1}{4\pi^2}\iint_{\mathbb{T}^2} f(x,y)\mathrm{e}^{-\mathrm{i}(jx+ky)}\,\mathrm{d}x\,\mathrm{d}y, \quad (j,k)\in\mathbb{N}^2,$$

$\mathbb{N} := \{0, 1, 2, \ldots\}$. In other words, we suppose that the coefficients of at least one negative index are zeros. We use the usual notations for the rectangular sums of the double series in (1.1):

$$s_{mn}(f) = s_{mn}(f;x,y) := \sum_{j=0}^{m}\sum_{k=0}^{n} c_{jk}\mathrm{e}^{\mathrm{i}(jx+ky)}, \quad (m,n)\in\mathbb{N}^2$$

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and for the L^1-norm:

$$\|f\|_1 = \iint_{\mathbb{T}^2} |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.$$

Our goal is to give conditions for the convergence of the rectangular sums in L^1-norm in terms of the coefficients. For one-variable functions this problem is well-studied, see for example papers [1], [5]. In the two-variable case, necessary conditions were given by Móricz in [4], from which we have:

Theorem A ([4]). *Suppose $f \in L^1(\mathbb{T}^2)$ and*

$$\|s_{mn} - f\|_1 \to 0 \quad \text{as } m, n \to \infty \text{ independently of one another.} \tag{1.2}$$

Then

$$\sum_{j=[m/2]}^{2m} \sum_{k=[n/2]}^{2n} \frac{|c_{jk}|}{(|j-m|+1)(|k-n|+1)} \to 0 \quad \text{as } m, n \to \infty.$$

Moreover,

$$\frac{\ln m \ln n}{mn} \sum_{j=[m/2]}^{2m} \sum_{k=[n/2]}^{2n} |c_{jk}| \to 0 \quad \text{as } m, n \to \infty.$$

To give sufficient conditions for the convergence in L^1-norm we need the following notations for the variations of the coefficients, $j, k \geqslant 0$:

$$\begin{aligned}
\Delta_{10} c_{jk} &:= c_{jk} - c_{j+1,k},\\
\Delta_{01} c_{jk} &:= c_{jk} - c_{j,k+1},\\
\Delta_{11} c_{jk} &:= \Delta_{10}(\Delta_{01} c_{jk}) = \Delta_{01}(\Delta_{10} c_{jk}) = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}.
\end{aligned}$$

Theorem B ([3]). *Let $f \in L^1(\mathbb{T}^2)$, and $\{c_{jk}\}_{j,k=0}^{\infty} \subset \mathbb{C}$ be its Fourier coefficients. If*

$$\sum_{k=0}^{\infty} |\Delta_{01} c_{mk}| \ln m \ln(k+2) \to 0 \quad \text{as } m \to \infty, \tag{1.3}$$

$$\sum_{j=0}^{\infty} |\Delta_{10} c_{jn}| \ln(j+2) \ln n \to 0 \quad \text{as } n \to \infty, \tag{1.4}$$

$$\lim_{\lambda \downarrow 1} \limsup_{m \to \infty} \sum_{k=0}^{\infty} \sum_{j=m}^{[\lambda m]} |\Delta_{11} c_{jk}| \ln j \ln(k+2) = 0, \tag{1.5}$$

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{j=0}^{\infty} \sum_{k=n}^{[\lambda n]} |\Delta_{11} c_{jk}| \ln(j+2) \ln k = 0, \tag{1.6}$$

then (1.2) *holds.*

We note that the previous theorems were stated and proved in a more general context, namely, when it is not supposed that the Fourier coefficients of at least one negative index are zeros.

2. Main results

In the first two theorems we extend the results of Theorem A by establishing further necessary conditions for the convergence in L^1-norm defined in (1.2).

Theorem 2.1. *Suppose that $f \in L^1(\mathbb{T}^2)$, f is in the form* (1.1) *and* (1.2) *holds. Then*

$$\sum_{j=[m/2]}^{2m} \sum_{k=0}^{\infty} \frac{|c_{jk}|}{(|j-m|+1)(k+1)} \to 0 \quad \text{as } m \to \infty, \tag{2.1}$$

$$\sum_{j=0}^{\infty} \sum_{k=[n/2]}^{2n} \frac{|c_{jk}|}{(j+1)(|k-n|+1)} \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Theorem 2.2. *Suppose that* (2.1)–(2.2) *hold. Then we have*

$$\frac{\ln m}{m} \sum_{j=[m/2]}^{2m} \sum_{k=0}^{\infty} \frac{|c_{jk}|}{k+1} \to 0 \quad \text{as } m \to \infty, \tag{2.3}$$

$$\frac{\ln n}{n} \sum_{j=0}^{\infty} \sum_{k=[n/2]}^{2n} \frac{|c_{jk}|}{j+1} \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

Now we establish necessary and sufficient conditions for the convergence in L^1-norm in case of coefficients of special type. We use the concept of *logarithm bound variation double sequences*, see [2]. A double sequence $\{c_{jk}\}_{j,k=0}^{\infty} \subset \mathbb{R}_+ = [0,\infty)$ satisfying $c_{jk} \to 0$ as $j+k \to \infty$ is said to be in logarithm bound variation double sequences for some $N = (N_1, N_2)$ ($\mathrm{LBVDS_N}$), where $N_1, N_2 > 0$ are integers, if

$$\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} \left| \Delta_{11} \Big(\frac{c_{jk}}{\ln^{N_1}(j+2) \ln^{N_2}(k+2)} \Big) \right| \leqslant C_{\{c_{jk}\}} \frac{c_{mn}}{\ln^{N_1}(m+2) \ln^{N_2}(n+2)} \tag{2.5}$$

for all $(m,n) \in \mathbb{N}^2$.

Theorem 2.3. *Suppose that $f \in L^1(\mathbb{T}^2)$, f is in the form* (1.1) *and* $\{c_{jk}\}_{j,k=0}^{\infty} \in$ $\mathrm{LBVDS_N}$ *for some positive integer pair* $N = (N_1, N_2)$. *Then* (1.2) *is satisfied if and only if*

$$\sum_{k=0}^{\infty} \frac{c_{mk} \ln m}{k+1} \to 0 \quad \text{as } m \to \infty, \tag{2.6}$$

$$\sum_{j=0}^{\infty} \frac{c_{jn} \ln n}{j+1} \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

3. Proofs

First we draw a lemma which was seen in [4], Lemma 5, we just use c_{jk} in place of jk.

Lemma 3.1. *For all* $0 \leqslant m < \mu$ *and* $0 \leqslant n < \nu$ *we have*

$$\Big\| \sum_{j=m}^{\mu} \sum_{k=n}^{\nu} c_{jk} \mathrm{e}^{\mathrm{i}(jx+ky)} \Big\|_1 \geqslant \frac{1}{\pi^2} \max \Big\{ \sum_{j=m}^{\mu} \sum_{k=n}^{\nu} \frac{|c_{jk}|}{(j-m+1)(k-n+1)}, \sum_{j=m}^{\mu} \sum_{k=n}^{\nu} \frac{|c_{jk}|}{(\mu-j+1)(k-n+1)}, \sum_{j=m}^{\mu} \sum_{k=n}^{\nu} \frac{|c_{jk}|}{(j-m+1)(\nu-k+1)}, \sum_{j=m}^{\mu} \sum_{k=n}^{\nu} \frac{|c_{jk}|}{(\mu-j+1)(\nu-k+1)} \Big\}.$$

Now, we shall prove the main results.

P r o o f of Theorem 2.1. Condition (2.1) holds true since by Lemma 3.1 and the fulfillment of (1.2) we have

$$\begin{aligned}
&\sum_{j=[m/2]}^{2m} \sum_{k=0}^{n} \frac{|c_{jk}|}{(|j-m|+1)(k+1)} \\
&\leqslant \sum_{j=[m/2]}^{m} \sum_{k=0}^{n} \frac{|c_{jk}|}{(m-j+1)(k+1)} + \sum_{j=m+1}^{2m} \sum_{k=0}^{n} \frac{|c_{jk}|}{(j-m)(k+1)} \\
&\leqslant \Big\| \sum_{j=[m/2]}^{m} \sum_{k=0}^{n} c_{jk} \mathrm{e}^{\mathrm{i}(jx+ky)} \Big\|_1 + \Big\| \sum_{j=m+1}^{2m} \sum_{k=0}^{n} c_{jk} \mathrm{e}^{\mathrm{i}(jx+ky)} \Big\|_1 \\
&\leqslant \max_{[m/2]-1 \leqslant \mu_1 < \mu_2} \|s_{\mu_2,n}(f) - s_{\mu_1,n}(f)\|_1 \to 0
\end{aligned}$$

as m and n tend to infinity. Relation (2.2) follows from the observation

$$\max_{[n/2]-1\leqslant\nu_1<\nu_2}\|s_{m,\nu_2}(f)-s_{m,\nu_1}(f)\|_1\to 0,\quad m,n\to\infty$$

in a similar way as we got (2.1). □

P r o o f of Theorem 2.2. We state that conditions (2.3) and (2.4) can be obtained using the known fact (see [1], page 746) that for any non-negative sequence $\{a_l\}$

$$\sum_{l=[n/2]}^{2n}\frac{a_l}{|l-n|+1}\to 0,\quad n\to\infty$$

implies

$$\frac{\ln n}{n}\sum_{l=n}^{2n}a_l\to 0,\quad n\to\infty.$$

Indeed, defining

$$a_l:=\sum_{k=0}^{n}\frac{|c_{lk}|}{k+1}\quad\text{and}\quad a_l:=\sum_{j=0}^{m}\frac{|c_{jl}|}{j+1},$$

respectively, (2.1) and (2.2) imply the validity of (2.3) and (2.4). □

Before we prove Theorem 2.3, we need an inequality. A similar inequality was proved in [2], Lemma 2, although we think their proof is incomplete and we hereby give a complete one.

Lemma 3.2. *If* $\{c_{jk}\}_{j,k=0}^{\infty}\in\mathrm{LBVDS_N}$ *for some* $N=(N_1,N_2)$, *then*

$$\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}|\Delta_{11}c_{jk}|\ln(j+2)\ln(k+2)\leqslant C_{\{c_{jk}\}}\sum_{j=[\sqrt{m_1}]}^{m_2}\sum_{k=[\sqrt{n_1}]}^{n_2}\frac{c_{jk}}{(j+1)(k+1)}\tag{3.1}$$

for any $0\leqslant m_1\leqslant m_2\leqslant\infty$, $0\leqslant n_1\leqslant n_2\leqslant\infty$.

P r o o f. For the sake of convenience, we will use the notation

$$\Delta\ln^{N_0}l:=\ln^{N_0}(l+1)-\ln^{N_0}l.$$

With a little calculation,

$$\begin{aligned}\Delta_{11}c_{jk}=\;&\ln^{N_1}(j+3)\ln^{N_2}(k+3)\Delta_{11}\Big(\frac{c_{jk}}{\ln^{N_1}(j+2)\ln^{N_2}(k+2)}\Big)\\&-\frac{\Delta_{01}c_{jk}(\Delta\ln^{N_1}(j+2))}{\ln^{N_1}(j+2)}-\frac{\Delta_{10}c_{jk}(\Delta\ln^{N_2}(k+2))}{\ln^{N_2}(k+2)}\\&-\frac{c_{jk}(\Delta\ln^{N_1}(j+2))(\Delta\ln^{N_2}(k+2))}{\ln^{N_1}(j+2)\ln^{N_2}(k+2)}.\end{aligned}$$

Now we can estimate

$$\begin{aligned}
&\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}|\Delta_{11}c_{jk}|\ln(j+2)\ln(k+2)\\
&\quad\leqslant \sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\Big|\Delta_{11}\Big(\frac{c_{jk}}{\ln^{N_1}(j+2)\ln^{N_2}(k+2)}\Big)\Big|\ln^{N_1+1}(j+3)\ln^{N_2+1}(k+3)\\
&\qquad+C_{N_1}\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\frac{|\Delta_{01}c_{jk}|\ln(k+2)}{j+1}+C_{N_2}\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\frac{|\Delta_{10}c_{jk}|\ln(j+2)}{k+1}\\
&\qquad+C_N\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\frac{c_{jk}}{(j+1)(k+1)}=:I_1+I_2+I_3+I_4
\end{aligned}$$

since

$$\frac{\Delta\ln^{N_0}(l+2)}{\ln^{N_0-1}(l+2)}\leqslant\frac{C_{N_0}}{l+1}. \tag{3.2}$$

First, for the estimation of I_1, set

$$R_{mn}=\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Big|\Delta_{11}\Big(\frac{c_{jk}}{\ln^{N_1}(j+2)\ln^{N_2}(k+2)}\Big)\Big|.$$

Then

$$\begin{aligned}
I_1&=\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}(R_{jk}-R_{j+1,k}-R_{j,k+1}+R_{j+1,k+1})\ln^{N_1+1}(j+3)\ln^{N_2+1}(k+3)\\
&=\sum_{j=m_1}^{m_2-1}\sum_{k=n_1}^{n_2-1}R_{j+1,k+1}(\Delta\ln^{N_1+1}(j+3))(\Delta\ln^{N_2+1}(k+3))\\
&\quad+\sum_{j=m_1}^{m_2-1}R_{j+1,n_1}(\Delta\ln^{N_1+1}(j+3))\ln^{N_2+1}(n_1+3)\\
&\quad+\sum_{k=n_1}^{n_2-1}R_{m_1,k+1}\ln^{N_1+1}(m_1+3)(\Delta\ln^{N_2+1}(k+3))\\
&\quad-\sum_{j=m_1}^{m_2-1}R_{j+1,n_2+1}(\Delta\ln^{N_1+1}(j+3))\ln^{N_2+1}(n_2+3)\\
&\quad-\sum_{k=n_1}^{n_2-1}R_{m_2+1,k+1}\ln^{N_1+1}(m_2+3)(\Delta\ln^{N_2+1}(k+3))\\
&\quad+R_{m_1n_1}\ln^{N_1+1}(m_1+3)\ln^{N_2+1}(n_1+3)\\
&\quad-R_{m_2+1,n_1}\ln^{N_1+1}(m_2+3)\ln^{N_2+1}(n_1+3)
\end{aligned}$$

$$- R_{m_1,n_2+1} \ln^{N_1+1}(m_1+3) \ln^{N_2+1}(n_2+3)$$
$$+ R_{m_2+1,n_2+1} \ln^{N_1+1}(m_2+3) \ln^{N_2+1}(n_2+3).$$

Using (2.5) and (3.2) we get

$$\begin{aligned} I_1 \leqslant C_{\{c_{jk}\}} \Big(& \sum_{j=m_1+1}^{m_2} \sum_{k=n_1+1}^{n_2} \frac{c_{jk}}{(j+1)(k+1)} \\ & + \sum_{j=m_1+1}^{m_2} \frac{c_{jn_1}}{j+1} \ln(n_1+2) + \sum_{k=n_1+1}^{n_2} \frac{c_{m_1k}}{k+1} \ln(m_1+2) \\ & + \sum_{j=m_1+1}^{m_2} \frac{c_{jn_2}}{j+1} \ln(n_2+2) + \sum_{k=n_1+1}^{n_2} \frac{c_{m_2k}}{k+1} \ln(m_2+2) \\ & + c_{m_1n_1} \ln(m_1+2)\ln(n_1+2) + c_{m_2n_1} \ln(m_2+2)\ln(n_1+2) \\ & + c_{m_1n_2} \ln(m_1+2)\ln(n_2+2) + c_{m_2n_2} \ln(m_2+2)\ln(n_2+2) \Big) \end{aligned}$$

and since for any non-negative integer n

$$\ln(n+2) \leqslant C \sum_{l=\sqrt{n}}^{n} \frac{1}{l+1},$$

we can obtain

$$I_1 \leqslant C_{\{c_{jk}\}} \sum_{j=[\sqrt{m_1}]}^{m_2} \sum_{k=[\sqrt{n_1}]}^{n_2} \frac{c_{jk}}{(j+1)(k+1)}.$$

Finally, we need estimations on I_2 and I_3. For this, we use that for any $\{c_{jk}\} \in \mathrm{LBVDS_N}$, we have the one-dimensional logarithm bound variation condition [6]

$$\sum_{l=n}^{\infty} \Big| \Delta\Big(\frac{a_l}{\ln^{N_0}(l+2)} \Big) \Big| \leqslant C_{\{a_l\}} \frac{a_n}{\ln^{N_0}(n+2)} \tag{3.3}$$

satisfied for all the row and column subsequences of $\{c_{jk}\}$ with the same constant $C_{\{c_{jk}\}}$. Indeed, by [2], Lemma 1,

$$\sum_{j=m}^{\infty} \Big| \Delta_{10}\Big(\frac{c_{jn}}{\ln^{N_1}(j+2)\ln^{N_2}(n+2)} \Big) \Big| \leqslant C_{\{c_{jk}\}} \frac{c_{mn}}{\ln^{N_1}(m+2)\ln^{N_2}(n+2)},$$
$$\sum_{k=n}^{\infty} \Big| \Delta_{01}\Big(\frac{c_{mk}}{\ln^{N_1}(m+2)\ln^{N_2}(k+2)} \Big) \Big| \leqslant C_{\{c_{jk}\}} \frac{c_{mn}}{\ln^{N_1}(m+2)\ln^{N_2}(n+2)},$$

and we have (3.3) for $a_l := c_{ln}/\ln^{N_2}(n+2)$ with $N_0 = N_1$ and the same time for $a_l := c_{ml}/\ln^{N_1}(m+2)$ with $N_0 = N_2$. Then we immediately get (3.3) for the row and column subsequences and we can say $\{c_{ln}\}_{l=0}^{\infty} \in \mathrm{LRBVS}_{\mathrm{N}_1}$ and $\{c_{ml}\}_{l=0}^{\infty} \in \mathrm{LRBVS}_{\mathrm{N}_2}$. Then, by [6], ineqality (8) and Theorem 4,

$$\sum_{l=n_1}^{n_2} |\Delta a_l| \ln(l+2) \leqslant C_{\{a_l\}} \sum_{l=[\sqrt{n_1}]}^{n_2} \frac{a_l}{l+1}$$

is satisfied for any $\{a_l\} \in \mathrm{LRBVS}_{\mathrm{N}_0}$, therefore

$$I_2 \leqslant C_{\{c_{jk}\}} \sum_{j=m_1}^{m_2} \sum_{k=[\sqrt{n_1}]}^{n_2} \frac{c_{jk}}{(j+1)(k+1)},$$
$$I_3 \leqslant C_{\{c_{jk}\}} \sum_{j=[\sqrt{m_1}]}^{m_2} \sum_{k=n_1}^{n_2} \frac{c_{jk}}{(j+1)(k+1)}.$$

Altogether this means (3.1) holds. □

P r o o f of Theorem 2.3. *Sufficiency.* Let us assume that conditions (2.6) and (2.7) are satisfied. By Theorem B, it is enough to see that the four conditions (1.3)–(1.6) hold. Since $\{c_{jk}\} \in \mathrm{LBVDS}_{\mathrm{N}}$, we have $\{c_{ln}\}_{l=0}^{\infty} \in \mathrm{LRBVS}_{\mathrm{N}_1}$ and $\{c_{ml}\}_{l=0}^{\infty} \in \mathrm{LRBVS}_{\mathrm{N}_2}$, moreover by [6], Theorem 4, for any non-negative $\mathrm{LRBVS}_{\mathrm{N}_0}$ sequence $\{a_l\}$,

$$\sum_{l=0}^{\infty} |\Delta a_l| \ln(l+2) \leqslant C_{\{a_l\}} \sum_{l=0}^{\infty} \frac{a_l}{l+1}.$$

If we substitute $a_l := c_{ml}$ with $N_0 = N_2$ and $a_l := c_{ln}$ with $N_0 = N_1$, we get (1.3) and (1.4):

$$\sum_{k=0}^{\infty} |\Delta_{01} c_{mk}| \ln m \ln(k+2) \leqslant C_{\{c_{jk}\}} \sum_{k=0}^{\infty} \frac{c_{mk} \ln m}{k+1} \to 0 \quad \text{as } m \to \infty,$$
$$\sum_{j=0}^{\infty} |\Delta_{10} c_{jn}| \ln(j+2) \ln n \leqslant C_{\{c_{jk}\}} \sum_{j=0}^{\infty} \frac{c_{jn} \ln n}{j+1} \to 0 \quad \text{as } n \to \infty.$$

Furthermore, from (3.1), we have (for any $\lambda < m$) that

$$\sum_{k=0}^{\infty} \sum_{j=m}^{[\lambda m]} |\Delta_{11} c_{jk}| \ln j \ln(k+2) \leqslant C_{\{c_{jk}\}} \sum_{k=0}^{\infty} \sum_{j=[\sqrt{m}]}^{[\lambda m]} \frac{c_{jk}}{(j+1)(k+1)}$$
$$\leqslant C_{\{c_{jk}\}} \sum_{k=0}^{\infty} \max_{[\sqrt{m}] \leqslant j \leqslant [\lambda m]} \frac{c_{jk} \ln[\lambda m]}{k+1} \leqslant C_{\{c_{jk}\}} \sum_{k=0}^{\infty} \max_{[\sqrt{m}] \leqslant j \leqslant [\lambda m]} \frac{c_{jk} \ln j}{k+1}$$

and similarly

$$\sum_{j=0}^{\infty}\sum_{k=n}^{[\lambda n]}|\Delta_{11}c_{jk}|\ln(j+2)\ln k \leqslant C_{\{c_{jk}\}}\sum_{j=0}^{\infty}\max_{[\sqrt{n}]\leqslant k\leqslant[\lambda n]}\frac{c_{jk}\ln k}{j+1}.$$

Hence (1.5) and (1.6) are obtained:

$$\lim_{\lambda\downarrow 1}\limsup_{m\to\infty}\sum_{k=0}^{\infty}\sum_{j=m}^{[\lambda m]}|\Delta_{11}c_{jk}|\ln j\ln(k+2) \leqslant C_{\{c_{jk}\}}\limsup_{m\to\infty}\sum_{k=0}^{\infty}\frac{c_{mk}\ln m}{k+1}=0,$$

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\sum_{j=0}^{\infty}\sum_{k=n}^{[\lambda n]}|\Delta_{11}c_{jk}|\ln(j+2)\ln k \leqslant C_{\{c_{jk}\}}\limsup_{n\to\infty}\sum_{j=0}^{\infty}\frac{c_{jn}\ln n}{j+1}=0.$$

Necessity. Let us suppose that (1.2) holds. By Theorems 2.1 and 2.2 we get (2.3)–(2.4). Moreover, we have $\{c_{ln}\}_{l=0}^{\infty}\in \mathrm{LRBVS}_{\mathrm{N}_1}$ and $\{c_{ml}\}_{l=0}^{\infty}\in \mathrm{LRBVS}_{\mathrm{N}_2}$. It was proved in [6] that for any non-negative $\{a_l\}\in \mathrm{LRBVS}_{\mathrm{N}_0}$,

$$a_n\leqslant C_{\{a_l\}}a_l \quad \text{for } [\sqrt{n}]\leqslant l\leqslant n,$$

consequently

$$a_n\leqslant \frac{C_{\{a_l\}}}{n}\sum_{l=[n/2]}^{n}a_l.$$

If we substitute $a_l:=c_{lk}$ and $a_l:=c_{jl}$, then we get

$$c_{mk}\leqslant \frac{C_{\{c_{jk}\}}}{m}\sum_{j=[m/2]}^{m}c_{jk} \quad\text{and}\quad c_{jn}\leqslant \frac{C_{\{c_{jk}\}}}{n}\sum_{k=[n/2]}^{n}c_{jk}.$$

Finally we obtain (2.6) and (2.7):

$$\sum_{k=0}^{\infty}\frac{c_{mk}\ln m}{k+1}\leqslant C_{\{c_{jk}\}}\frac{\ln m}{m}\sum_{j=[m/2]}^{2m}\sum_{k=0}^{\infty}\frac{c_{jk}}{k+1}\to 0 \quad \text{as } m\to\infty,$$

$$\sum_{j=0}^{\infty}\frac{c_{jn}\ln n}{j+1}\leqslant C_{\{c_{jk}\}}\frac{\ln n}{n}\sum_{j=0}^{\infty}\sum_{k=[n/2]}^{2n}\frac{c_{jk}}{j+1}\to 0 \quad \text{as } n\to\infty.$$

□

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