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ON ALMOST EVERYWHERE DIFFERENTIABILITY OF THE METRIC PROJECTION ON CLOSED SETS IN $l^p(\mathbb{R}^n)$, $2 < p < \infty$

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Abstract. Let F be a closed subset of $\mathbb{R}^n$ and let $P(x)$ denote the metric projection (closest point mapping) of $x \in \mathbb{R}^n$ onto F in l^p-norm. A classical result of Asplund states that P is (Fréchet) differentiable almost everywhere (a.e.) in $\mathbb{R}^n$ in the Euclidean case $p = 2$. We consider the case $2 < p < \infty$ and prove that the ith component $P_i(x)$ of $P(x)$ is differentiable a.e. if $P_i(x) \neq x_i$ and satisfies Hölder condition of order $1/(p-1)$ if $P_i(x) = x_i$.

Keywords: normed space; uniform convexity; closed set; metric projection; l^p-space; Fréchet differential; Lipschitz condition

MSC 2010: 26E25, 46B20, 49J50

1. Introduction

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let F be a closed subset of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we let $P(x)$ denote the metric projection of x onto the set F, i.e. $P(x)$ is the set of points $\underline{P}(x)$ in F satisfying

$$\|\underline{P}(x) - x\| = \inf_{y \in F} \|y - x\| = \operatorname{dist}(x, F) \tag{1}$$

and $\operatorname{dist}(x, F)$is the distance from x to F. It was proved by Asplund [2] that $P(x)$ is Fréchet differentiable almost everywhere (a.e.) in $\mathbb{R}^n$ with the Euclidean norm. The key to the proof was Alexandroff's theorem in [10] stating that convex functions have second order differentials a.e. (Abazoglou in [1], Theorem 2, and Zajíček in [12], Theorem 4), extended this result to norms that are close to being Euclidean. In the two-dimensional case, P is known to be Fréchet differentiable a.e. for any strictly convex norm, see [12], Theorem 3, which includes the l^p-norm, $1 < p < \infty$. The present paper treats the l^p-norm, $2 < p < \infty$, in spaces of dimension at least three, which is not covered by the results mentioned above.

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The metric projection (closest point mapping) in finite dimensional spaces was studied at some length by Phelps in [8], [9]. The problem of the differentiability of $P(x)$ seems to first have been considered by Kruskal in [7]. He asked if the set of points $x \in \mathbb{R}^n$ and directions $v \in S^{n-1}$ such that $P(x)$ has a directional derivative at x in the direction v, is dense in $\mathbb{R}^n \times S^{n-1}$. See Shapiro [11] and the references contained there for more on directional differentiability of the metric projection. Differentiability of metric projections in general Hilbert spaces is studied in [5]. Asplunds result gives an affirmative answer to Kruskal's question in the Euclidean case. It is the purpose of this paper to give a partial extension of Asplund's result to $2 < p < \infty$. We prove that $P_i(x)$ is differentiable for a.e. x such that $P_i(x) - x_i \neq 0$ and that $P_i(x)$ satisfies Lipschitz condition if $P_i(x) - x_i = 0$, where $P_i(x)$ is the ith coordinate of $P(x)$. We state our result as follows.

Theorem 1. *Let $2 < p < \infty$, let F be a closed subset of $\mathbb{R}^n$ and let $P(x)$ denote the metric projection onto F defined in* (1) *by the l^p-norm. Then $P(x)$ is single valued and continuous for a.e. x in $\mathbb{R}^n$. Further,*

(a) *$P_i(x)$ is Fréchet differentiable for a.e. x such that $P_i(x) - x_i \neq 0$,*
(b) *$P_i(x)$ satisfies $P_i(x+h) - P_i(x) = O(\|h\|_p^{1/(p-1)})$, as $h \to 0$, for a.e. x such that $P_i(x) - x_i = 0$.*

Remark. It remains an open question if $P_i(x)$ is differentiable a.e. for closed sets F or at least for convex sets, when $P_i(x) - x_i = 0$.

The proof of the theorem uses Asplund's idea to define a convex auxiliary function whose differential is closely connected to $P(x)$. The new idea is the map $D_p\colon \mathbb{R}^n \to \mathbb{R}^n$ and its differentiability properties.

The organisation of this paper is as follows. Section 2 gives our notation, definitions and three propositions. The proof of the theorem is given in Section 4 after some lemmas have been proved in Section 3.

2. Preliminaries

We consider $\mathbb{R}^n$ with points $x = (x_1, \dots, x_n)$ and let $l^p = l^p(\mathbb{R}^n)$, $1 < p < \infty$, denote $\mathbb{R}^n$ with the norm

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

Then l^p is an n-dimensional and uniformly convex Banach space with dual space l^q, $p^{-1} + q^{-1} = 1$, see [3] and [6]. Let F be a closed set in $\mathbb{R}^n$ and define the metric projection $P(x)$ onto F by (1). The map $P(x)$ is in general multiple valued and we denote by $\underline{P}(x)$ any choice of an element in $P(x)$. We say that $P(x)$ is continuous at the point x if $P(x)$ is single valued at x and $\underline{P}(x+h) = P(x) + o(1)$, as $h \to 0$, for all $\underline{P}(x+h)$ in $P(x+h)$. Similarily, $P(x)$ is Fréchet differentiable at x if $P(x)$ is single valued at x and there is an $n \times n$-matrix M such that

$$\underline{P}(x+h) = P(x) + M \cdot h + o(\|h\|_p), \quad \text{as} \quad h \to 0 \tag{2}$$

for all $\underline{P}(x+h)$ in $P(x+h)$. In the following, differentiability always means Fréchet differentiability in this sense. Before we prove Theorem 1 we prepare the way by a series of propositions, which we give in a more general form than is actually needed for the proof. In the following three propositions we assume that $\|\cdot\|$ is any uniformly convex norm on $\mathbb{R}^n$ and that $\|x\|$ is differentiable for $x \neq 0$. We denote any such space $(\mathbb{R}^n, \|\cdot\|)$ by B. Let $P(x)$ be the metric projection defined by (1) onto a closed set F and let $f(x) = \|P(x) - x\|$ be the distance between x and F. Then f satisfies Lipschitz condition $|f(x) - f(y)| \leqslant \|x - y\|$ and hence f is differentiable a.e. in $\mathbb{R}^n$ by the Rademacher-Stepanoff theorem in [4], p. 216.

Proposition 1. *For a.e. $x \in F$ we have $f'(x) = 0$ and $P'(x) = I$, the identity $n \times n$ matrix.*

P r o o f. Let $x \in F$ be a point where $f'(x)$ exists, then $0 \leqslant \|P(x+h) - x - h\| = f(x+h) - f(x) = f'(x)h + o(\|h\|)$ gives $f'(x) = 0$, $P(x+h) - x - h = o(\|h\|)$, as $h \to 0$, and $P'(x) = I$. □

Proposition 1 shows that $P(x)$ is differentiable a.e. on F. Let $B^\star$ be the dual space of B with norm $\|\cdot\|_\star$ and denote the pairing between B and $B^\star$ by $\langle \cdot, \cdot \rangle$. For any $x \neq 0$ in B there is $x^\star \in B^\star$ such that $\langle x^\star, x \rangle = \|x\|$ and $\|x^\star\|_\star = 1$ by the Hahn-Banach theorem. We call $x^\star$ the support functional of x. It is easy to prove that $x^\star$ is unique and is given by the formula

$$\langle x^\star, y \rangle = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \quad y \in \mathbb{R}^n.$$

An immediate consequence of Proposition 1 and the definition of $x^\star$ is the following formula connecting $f'(x)$ and $P(x)$, cf. [1], Lemma 4.

Proposition 2. *Let $x \in \mathbb{R}^n \setminus F$ be a point where f is differentiable. Then $P(x)$ is single valued and continuous at x and $f'(x) = (x - P(x))^\star$.*

P r o o f. It is proved in [1], Lemma 4 that $P(x)$ is single valued and $f'(x) = (x - P(x))^\star$. We show that $P(x)$ is continuous at x. Choose sequences $(x_i)_1^\infty$ and

$(z_i)_1^\infty$ such that $x_i \to x$ and $z_i \in P(x_i)$, as $i \to \infty$. Then $(z_i)_1^\infty$ is bounded and we may assume that $(z_i)_1^\infty$ converges to $z \in F$. We get

$$\begin{aligned}\|x - P(x)\| &\leqslant \|x - z\| \leqslant \|x - x_i\| + \|x_i - z_i\| + \|z_i - z\| \\ &= \|x - x_i\| + f(x_i) + \|z_i - z\| \to f(x) = \|x - P(x)\|,\end{aligned}$$

as $i \to \infty$. Thus $z = P(x)$, since $P(x)$ is single valued, and $P(x)$ is continuous at x. $\square$

3. Some technical lemmas

This section contains a number of technical lemmas. Lemma 6 and 7 constitute the basis for the proof of Theorem 1 in the next section. We begin with an elementary inequality.

Lemma 1. *Let $1 < p < \infty$. Then there is $C_p \geqslant 1$ such that*

$$2^{p-1} \cdot |t|^p + 2^{p-1} - |t+1|^p \leqslant C_p \cdot (t-1)^2, \quad -1 \leqslant t \leqslant 1,$$

where $C_p = p/2$, $1 < p < 2$ and $C_p = 2^{p-2} \cdot \binom{p}{2}$, $2 \leqslant p < \infty$. The constant C_p for $2 \leqslant p < \infty$ is best possible.

Proof. Define

$$g_p(t) = 2^{p-1} \cdot |t|^p + 2^{p-1} - |t+1|^p - C_p \cdot (t-1)^2, \quad -1 \leqslant t \leqslant 1.$$

We first let $2 < p < \infty$, noting that $g_2(t) \equiv 0$. If $0 < t < 1$, an easy calculation shows that $g_p''(t) < 0$ and $g_p'(t) \geqslant g_p'(1) = 0$. Hence, $g_p(t) \leqslant g_p(1) = 0$, for $0 \leqslant t \leqslant 1$. If $-1 < t < 0$, then $g_p^{(3)}(t) < 0$, $g_p''(t)$ has a unique zero at t_0 in $(-1, 0)$ and $g_p'(t)$ has its maximum at t_0. Since both $g_p'(0)$ and $g_p'(-1)$ are positive, we have $g_p'(t) > 0$ and hence $g_p(t) \leqslant g_p(0) < 0$ for $-1 \leqslant t \leqslant 0$. The case $1 < p < 2$ is proved in a similar way. To show that C_p is best possible for $2 \leqslant p < \infty$, take $t = 1 - h$ and let $h \to 0$. $\square$

Lemma 2. *Let $2 \leqslant p < \infty$. Then*

$$2^{p-1} \cdot |x|^p + 2^{p-1} \cdot |y|^p - |x+y|^p \leqslant C_p \cdot R^{p-2} \cdot |x-y|^2$$

for all real numbers x, y such that $|x| \leqslant R$ and $|y| \leqslant R$, where C_p is the constant in Lemma 1.

Proof. Assume that $|x| \leqslant |y|$ and take $t = x/y$ in Lemma 1. $\square$

For any nonzero vector x in $l^p(\mathbb{R}^n)$, $1 < p < \infty$, the dual vector $x^\star$ is given by $x^\star = \|x\|_p^{1-p} \cdot (|x_1|^{p-2} \cdot x_1, \ldots, |x_n|^{p-2} \cdot x_n)$. The following closely related map D_p will be used in our proof of Theorem 1. Let $1 < p < \infty$ and define a one-to-one map D_p of $\mathbb{R}^n$ onto $\mathbb{R}^n$ by $D_p(0) = 0$ and

$$D_p(x) = \Big(\frac{1}{p} \cdot \|x\|_p^p\Big)' = (|x_1|^{p-2} \cdot x_1, \ldots, |x_n|^{p-2} \cdot x_n)$$

for $x \neq 0$, where $|x_i|^{p-2} \cdot x_i = 0$ if $x_i = 0$. Note that D_2 is the identity map.

Lemma 3. *Let $1 < p < \infty$. Then D_p is an injective map of $\mathbb{R}^n$ onto $\mathbb{R}^n$ with inverse $D_p^{-1}(x) = D_q(x)$ and $\|D_p(x)\|_q^q = \|x\|_p^p$, where $1/p + 1/q = 1$.*

Proof. It is clear that $D_p(x) = 0$ if only if $x = 0$. Let $x \neq 0$ and $D_p(x) = y$, then

$$|x_i|^{p-2} \cdot x_i = y_i, \quad |x_i| = |y_i|^{q-1}$$

and $x_i = |y_i|^{q-2} \cdot y_i$, $1 \leqslant i \leqslant n$, i.e. $x = D_q(y)$. Further, $\|y\|_q^q = \|x\|_p^p$, which completes the proof of Lemma 3. □

Lemma 4. *Let $2 \leqslant p < \infty$. Then $D_p(x)$ is Fréchet differentiable for all $x \in \mathbb{R}^n$ and $D_p'(x)$ is given by the diagonal matrix*

$$D_p'(x) = (p-1) \cdot (|x_i|^{p-2} \cdot \delta_{i,j}),$$

where $\delta_{i,j}$ is the Dirac delta function.

Proof. This follows at once from the definition of $D_p(x)$. □

Remark. Clearly, $D_2'(x)$ is simply the $n \times n$ identity matrix and $D_p'(x)$ is invertible at x if and only if x has all its coordinates nonzero. If $x \in \mathbb{R}^n$ and $x_i \neq 0$, then for the ith coordinate we have

$$D_p(x+h)_i = D_p(x)_i + (p-1) \cdot |x_i|^{p-2} \cdot h_i + o(\|h\|_2),$$

for any $1 < p < \infty$.

Our main tool in the proof of Theorem 1 is the auxiliary function $K_p(x)$ defined as

$$K_p(x) = -\|x - P(x)\|_p^p + \lambda \cdot \|x\|_2^2, \tag{3}$$

where $\lambda > 0$ is to be defined below. Note the close relation between $K_p(x)$ and $P(x)$. Then $K_p(x)$ coincides with the corresponding auxiliary functions in [2] and [1] for

$p = 2$. The assumption on the norm in [1] is however not satisfied in the present case, since $(\|x\|_p^p)'' = p \cdot D_p'(x)$ is not always invertible. The main property of $K_p(x)$ is its local convexity for suitable choices of λ. More exactly, we have the following lemma.

Lemma 5. *Let $2 < p < \infty$. Then for every $R > 0$ there is a number $\lambda = \lambda(F, p, R) > 0$ such that $K_p(x)$ is convex for $\|x\|_p < R$.*

P r o o f. Since $K_p(x)$ is continuous, it is sufficient to prove that it is also midpoint convex. It turns out to be sufficient to prove that if $\|x\|_p < R$ and $\|y\|_p < R$, then

$$2^{p-1} \cdot \|x+z\|_p^p + 2^{p-1} \cdot \|y+z\|_p^p - \|x+y+2z\|_p^p \leqslant \lambda \cdot \|x-y\|_2^2, \tag{4}$$

where $z = -\frac{1}{2}P(x+y)$, provided λ is large enough, cf. [1] Lemma 6. Then (4) follows from Lemma 2 applied to each coordinate separately with $\lambda = C_p \cdot R^{p-2} \cdot n$, since $\|x\|_p \leqslant R$ implies $|x_i| \leqslant R$, $1 \leqslant i \leqslant n$. □

The next two lemmas on the connection between $K_p(x)$ and $P(x)$ are the keys to the proof of Theorem 1.

Lemma 6. *Let $1 < p < \infty$ and let U be an open set where $K_p(x)$ is convex and let $\underline{P}(y)$ be any choice for $P(y)$, $y \in U$. Define*

$$\underline{K}_p'(y) = 2\lambda \cdot y - p \cdot D_p(y - \underline{P}(y)).$$

Then $\underline{K}_p'(y)$ is a subdifferential of $K_p(y)$.

P r o o f. Let $y \in U$ be fixed and choose $\{y_j\}_1^\infty$ in U such that $y_j \to y$, as $j \to \infty$, and f is differentiable at y_j, $j \geqslant 1$. Then by the convexity of K_p,

$$K_p(y_j + h) \geqslant K_p(y_j) + K_p'(y_j) \cdot h$$

for all $j \geqslant 1$ and any sufficiently small h. Fix any such h, then clearly $K_p(y_j + h) \to K_p(y + h)$ and $K_p(y_j) \to K_p(y)$, as $j \to \infty$, by the continuity of K_p. The lemma follows from the continuity of D_p if for every $x \in \underline{P}(y)$ we can choose $\{y_j\}_1^\infty$ such that also $P(y_j) \to x$, as $j \to \infty$. We note that if $z = y + t(x-y)$, $0 < t < 1$, then $P(z) = x$ and any w close to z has projection close to x, by the uniform convexity of the norm. This completes the proof of Lemma 6. □

Lemma 7. *Let $1 < p < \infty$, let U be an open, convex set, where $K_p(y)$ is convex and let $\underline{K}'_p(y)$ be defined as in Lemma 7. Then if x is any point in U, where f is differentiable and $K''_p(x)$ exists, we have*

$$\underline{K}'_p(x+h) = K'_p(x) + K''_p(x)h + o(\|h\|_2), \quad \text{as } h \to 0.$$

The following proof is found in [1], p. 495 and is due to Fitzpatrick.

Proof. Let $R > 0$ be arbitrary and choose λ as in Lemma 6 such that $K_p(x)$ is convex for $[[x]]_p < R$. Then $K_p(x)$ is a.e. twice differentiable for $\|x\|_p < R$ by Alexandrov's theorem. Fix any such point x, then for every $0 < \varepsilon < 1$ there is $\delta > 0$ such that if $\|y-x\|_2 < \delta$, then

$$(5) \qquad |K_p(y) - K_p(x) - \langle K'_p(x), y-x\rangle - \tfrac{1}{2}\langle K''_p(x)(y-x), y-x\rangle| \leqslant \varepsilon\|y-x\|_2^2.$$

Let $\|z\|_2 = \|w\|_2 = 1$, $0 < |t| < \delta$ and $\alpha = \sqrt{\varepsilon}\cdot t$. Then by properties of the subdifferential

$$\langle \underline{K}'_p(x+tw), \alpha z\rangle \leqslant K_p(x+tw+\alpha z) - K_p(x+tw)$$

and by (5) we get

$$K_p(x+tw+\alpha z) \leqslant 4\varepsilon|t|^2 + K_p(x) + \langle K'_p(x), tw+\alpha z\rangle + \tfrac{1}{2}\langle K''_p(x)(tw+\alpha z), tw+\alpha z\rangle$$

and

$$K_p(x+tw) \geqslant -\varepsilon|t|^2 + K_p(x) + \langle K'_p(x), tw\rangle + \tfrac{1}{2}K''_p(tw), tw\rangle.$$

Combining the last three inequalities we obtain

$$\begin{aligned}\langle \underline{K}'_p(x+tw), \alpha z\rangle &\leqslant 5\varepsilon|t|^2 + \langle K_p(x), \alpha z\rangle + \tfrac{1}{2}\langle K''_p(x)(tw), \alpha z\rangle \\ &\quad + \tfrac{1}{2}\langle K''_p(x)(\alpha z), tw\rangle + \tfrac{1}{2}\langle K''_p(x)(\alpha z), \alpha z\rangle + 5\varepsilon|t|^2 \\ &= \langle K'_p(x), \alpha z\langle + \rangle K''_p(x)(tw), \alpha z\rangle + 5\varepsilon|t|^2 + \tfrac{1}{2}\alpha^2\langle K''_p(x)(z), z\rangle.\end{aligned}$$

Since $\alpha = \sqrt{\varepsilon}|t|^2$, we have

$$\langle \underline{K}'_p(x+tw) - K'_p(x) - K''_p(x)(tw), z\rangle \leqslant 5\sqrt{\varepsilon}|t| + \tfrac{1}{2}\sqrt{\varepsilon}|t|\langle K''_p(x)(z), z\rangle$$

and equivalently

$$\|\underline{K}'_p(x+tw) - K'_p(x) - K''_p(x)(tw)\|_2 \leqslant \big(5 + \tfrac{1}{2}\|K''_p(x)\|_2\big)\sqrt{\varepsilon}|t| = o(|t|),$$

which proves Lemma 7. $\square$

4. Proof of Theorem 1

The proof of Theorem 1 is based on the properties of the auxiliary function $K_p(x)$ proved above and the map D_p.

P r o o f of Theorem 1. We let F be a closed set in $\mathbb{R}^n$ and let $P(x)$ denote the metric projection onto F. Since $P'(x) = I$ a.e. in F by Proposition 1, we assume that $x \in \mathbb{R}^n \setminus F$. Recall the distance function $f(x) = \|P(x) - x\|_p$ and the auxiliary function $K_p(x)$ defined by (3). Let $R > 0$ be arbitrary and choose λ as in Lemma 6 such that $K_p(x)$ is convex for $\|x\|_p < R$. Then $K_p(x)$ has a second order differentiable a.e. in $\|x\|_p < R$ by Alexandrov's theorem, see [10].

In the following, we let x denote any point in $\mathbb{R}^n \setminus F$, where f is differentiable, $K_p(x)$ has a second order differential and $\|x\|_p < R$. Then by Lemma 3 and Lemma 6 we have

$$\underline{P}(x+h) = x + h - D_q\Big(\frac{2\lambda}{p} \cdot (x+h) - \frac{1}{p} \cdot \underline{K}'_p(x+h)\Big) \tag{6}$$

and further

$$\begin{aligned} \underline{P}(x+h) &= x + h - D_q\Big(\frac{2\lambda}{p} \cdot (x+h) - \frac{1}{p} \cdot K'_p(x) - \frac{1}{p} \cdot K''_p(x)h + o(\|h\|_2)\Big) \\ &= x + h - D_q\Big(\frac{2\lambda}{p} \cdot x - \frac{1}{p} \cdot K'_p(x) + \frac{2\lambda}{p} \cdot h - \frac{1}{p} \cdot K''_p(x)h + o(\|h\|_2)\Big) \end{aligned} \tag{7}$$

as $h \to 0$, by Lemma 4 and Lemma 7. Now assume that $P_i(x) - x_i \neq 0$, then the ith coordinate of $2\lambda \cdot x - K'_p(x)$ is nonzero by Lemma 6. Let $z = (2\lambda x - K'_p(x))/p$, then the ith coordinate of the last term in (7) equals

$$D_q(z)_i + (q-1) \cdot |z_i|^{q-1} \cdot \Big(\frac{2\lambda}{p} \cdot h_i - \frac{1}{p} \cdot (K''_p(x)h)_i\Big) + o(\|h\|_2),$$

by the remark following Lemma 4. It follows that there exists a vector L_i in $\mathbb{R}^n$ such that

$$\underline{P}_i(x+h) = P_i(x) + \langle L_i, h\rangle + \varphi_i(h), \tag{8}$$

where $\varphi_i(h) = o(\|h\|_2)$. More exactly, L_i is a linear combination of the ith unit row vector in $\mathbb{R}^n$ and the ith row vector in $K''_p(x)$. Hence, P_i is differentiable at x, which proves statement (a) in Theorem 1. If $P_i(x) = x_i$, (7) only gives the weaker result $P_i(x+h) - P_i(x) = O(\|h\|_p^{q-1})$, as $h \to 0$, which proves (b). □

Remark. It is tempting to guess that $P_i(x+h) - x_i - h_i = o(\|h\|_p)$, as $h \to 0$, when x is a density point of the set E where $P_i(x) = x_i$ for some $1 \leqslant i \leqslant n$. We have only the weaker result that the set $\{h\colon |\underline{P}_i(x+h) - x_i - h_i| \leqslant \varepsilon \cdot \|h\|_p\}$ has density one at $h = 0$ for every $\varepsilon > 0$. This is usually called an approximate derivative of P_i at x.

References

[1] *T. Abatzoglou*: Finite-dimensional Banach spaces with a.e. differentiable metric projection. Proc. Am. Math. Soc. *78* (1980), 492–496. zbl MR doi

[2] *E. Asplund*: Differentiability of the metric projection in finite dimensional metric spaces. Proc. Am. Math. Soc. *38* (1973), 218–219. zbl MR doi

[3] *J. Clarkson*: Uniformly convex spaces. Trans. Am. Math. Soc. *40* (1936), 396–414. zbl MR doi

[4] *H. Federer*: Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 153, Springer, Berlin, 1969. zbl MR

[5] *S. Fitzpatrick, R. R. Phelps*: Differentiability of metric projection in Hilbert space. Trans. Am. Math. Soc. *270* (1982), 483–501. zbl MR doi

[6] *O. Hanner*: On the uniform convexity of L^p and l^p. Ark. Mat. *3* (1956), 239–244. zbl MR doi

[7] *J. B. Kruskal*: Two convex counterexamples: A discontinuous envelope function and a non-differentiable nearest point mapping. Proc. Am. Math. Soc. *23* (1969), 697–703. zbl MR doi

[8] *R. R. Phelps*: Convex sets and nearest points. Proc. Am. Math. Soc. *8* (1957), 790–797. zbl MR doi

[9] *R. R. Phelps*: Convex sets and nearest points II. Proc. Am. Math. Soc. *9* (1958), 867–873. zbl MR doi

[10] *Ju. G. Rešetnyak*: Generalized derivatives and differentiability almost everywhere. Mat. Sb., N. Ser. *75(117)* (1968), 323–334. (In Russian.) zbl MR

[11] *A. Shapiro*: Differentiability properties of metric projection onto convex sets. J. Optim. Theory Appl. *169* (2016), 953–964. zbl MR doi

[12] *L. Zajíček*: On differentiation of metric projections in finite dimensional Banach spaces. Czech. Math. J. *33* (1983), 325–336. zbl MR

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