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A CLASS OF FERMIONIC NOVIKOV SUPERALGEBRAS WHICH IS A CLASS OF NOVIKOV SUPERALGEBRAS

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Abstract. We construct a special class of fermionic Novikov superalgebras from linear functions. We show that they are Novikov superalgebras. Then we give a complete classification of them, among which there are some non-associative examples. This method leads to several new examples which have not been described in the literature.

Keywords: left symmetric algebra; Novikov superalgebra; fermionic Novikov superalgebra

MSC 2010: 17A30, 17A70

1. Introduction

The notion of Novikov algebras arises from mathematical physics and is closely related to the Poisson brackets of hydrodynamic type (see [2], [4], [5]) and the Hamiltonian operators in the formal variational calculus (see [6], [7], [8], [9], [10]). In [6], [7], Gel'fand and Dikii gave a bosonic formal variational calculus and in [10] Xu gave a fermionic formal variational version. Combining the bosonic theory of Gel'fand-Dikii and the fermionic theory, Xu gave in [11] a formal variational calculus of supervariables. Fermionic Novikov algebras are also related to the Hamiltonian superoperator in terms of this theory.

The notion of a fermionic Novikov superalgebra is a super-version of that of a fermionic Novikov algebra, which is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with a bilinear product $(x, y) \mapsto xy$ satisfying the condition that for any $x \in A_i$, $y \in A_j$, $z \in A$, we have

$$(xy)z - x(yz) = (-1)^{ij}((yx)z - y(xz)), \tag{1.1}$$

$$(zx)y = -(-1)^{ij}(zy)x. \tag{1.2}$$

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The even part of a fermionic Novikov superalgebra is exactly a fermionic Novikov algebra. The supercommutator

$$[x, y] = xy - (-1)^{ij} yx \quad \forall\, x \in A_i,\ y \in A_j$$

makes any fermionic Novikov superalgebra A into a Lie superalgebra.

Fermionic Novikov superalgebras constitute a special class of left symmetric superalgebras which are defined by the identity (1.1). For algebras the most important problem is to study their structure and classification. However, up to now there has not been a suitable structure and classification theory for general fermionic Novikov superalgebras due to their non-associativity. Since known fermionic Novikov superalgebras are rather rare, it is important to construct new interesting and explicit examples.

The main goal of this article is to produce a class of new examples of fermionic Novikov superalgebras from linear functions. We will show that all of our examples are also Novikov superalgebras. Our method is inspired by [1], in which the author constructed a class of interesting left symmetric algebras (including some fermionic Novikov algebras) from linear functions. Meanwhile, in the proof of our main results, we will take use of some results in [3].

2. The main results

Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded vector space over the complex field $\mathbb{C}$. A bilinear function $h\colon\ A \times A \to \mathbb{C}$ is said to be supersymmetric if

$$h(x, y) = (-1)^{ij} h(y, x) \quad \forall\, x \in A_i,\ y \in A_j.$$

Let $f, g\colon\ A \to \mathbb{C}$ be two linear functions. Define a product on A by

$$x * y = f(x)y + g(y)x + h_0(x, y)c_0 + h_1(x, y)c_1 \quad \forall\, x, y \in A, \tag{2.1}$$

where h_0, h_1 are supersymmetric bilinear functions, $c_0 \in A_0$, $c_1 \in A_1$ and $c_0 + c_1 \neq 0$. Suppose $\dim A_0 = n$, $n \geqslant 2$ and $\dim A_1 = m$, $m \geqslant 1$. Let $\{e_1, \ldots, e_n\}$ be a basis of A_0, and $\{f_1, \ldots, f_m\}$ a basis of A_1. The main theorem of this paper can be stated as follows:

Theorem 2.1. *Keep the above notation. Then the product defined in* (2.1) *makes A a fermionic Novikov superalgebra. Moreover, any fermionic Novikov superalgebra defined by* (2.1) *must be isomorphic to one of the following algebras:*

(1) $F_1^{(k)} = f_{2j-1} * f_{2j} = e_1 = -f_{2j} * f_{2j-1}$, $1 \leqslant j \leqslant k$, $2k \leqslant m$;

(2) $F_2 = e_i * f_i = f_1 = f_i * e_i$, $e_1 * f_j = h_1(e_1, f_j)f_1 = f_j * e_1$, $2 \leqslant i \leqslant k$, $2 \leqslant j \leqslant m$ *for any* $k \leqslant \min\{m, n\}$;

(3) $F_3^{(k),\alpha} = f_{2j} * f_{2j+1} = e_1 = -f_{2j+1} * f_{2j}$, $e_i * f_l = \alpha_{il} f_1 = f_l * e_i$, $j \leqslant k$, $2 \leqslant i \leqslant n$, $2 \leqslant l \leqslant m$, $2k + 1 \leqslant m$, $\alpha_{il} \in \mathbb{C}$;

(4) $F_4^{(k_1),(k_2)} = e_i * e_i = e_1$, $f_{2j-1} * f_{2j} = e_1 = -f_{2j} * f_{2j-1}$, $2 \leqslant i \leqslant k_1$, $j \leqslant k_2$, $2 \leqslant k_1 \leqslant n$, $2k_2 \leqslant m$;

(5) $F_5^{(k_1),(k_2),\alpha} = e_i * e_i = e_1$, $f_{2j} * f_{2j+1} = e_1 = -f_{2j+1} * f_{2j}$, $e_l * f_t = \alpha_{lt} f_1 = f_t * e_l$, $2 \leqslant i \leqslant k_1$, $1 \leqslant j \leqslant k_2$, $2 \leqslant l \leqslant n$, $2 \leqslant t \leqslant m$, $2 \leqslant k_1 \leqslant n$, $1 \leqslant 2k_2 + 1 \leqslant m$, $\alpha_{lt} \in \mathbb{C}$;

(6) $F_6 = e_1 * e_i = e_i$, $e_1 * f_j = f_j$, $2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, *where* $\{e_1, \ldots, e_n\}$ *and* $\{f_1, \ldots, f_m\}$ *are bases of* A_0 *and* A_1, *respectively.*

Remark 2.1.

(1) It is easy to see that A is non-associative if and only if A is isomorphic to F_6.

(2) $F_3^{(k),\alpha}$ and $F_5^{(k_1),(k_2),\alpha}$ are new examples of Novikov superalgebras, which have not been described in [3].

(3) Note that for different complex numbers α_1, α_2, it can occur that

$$F_3^{(k),\alpha_1} \simeq F_3^{(k),\alpha_2}.$$

For convenience, we discuss $F_3^{(k),\alpha}$ in some detail. Obviously, there is a one-to-one correspondence between the set $\{\alpha_{il},\ 2 \leqslant i \leqslant n,\ 2 \leqslant l \leqslant m\}$ and the matrix $M{=}(\alpha_{il})_{(n-1)\times(m-1)}$. Without loss of generality, we assume that the first column of M is a nonzero vector. Then by a linear transformation on $\{e_1, \ldots, e_n\}$, the matrix M can be reduced to the form

$$\begin{pmatrix} 1 & \cdots \\ 0 & \cdots \\ \vdots & \ddots \\ 0 & \cdots \end{pmatrix}.$$

If the second column is still a nonzero vector, then by a similar transformation the matrix M can be reduced to one of the following forms:

$$\text{(a)} \quad \begin{pmatrix} 1 & a & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \end{pmatrix}.$$

If all entries of the second column are zero, then we consider the third column. When the third column is a nonzero vector, the matrix M can be reduced to one of the following forms:

$$\text{(a)}\quad \begin{pmatrix} 1 & 0 & a & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \dots \end{pmatrix} \qquad \text{(b)}\quad \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \dots \end{pmatrix}.$$

If $m \leqslant n$, then we can repeat the above process to get a standard form which corresponds to a fermionic Novikov superalgebra $F_3^{(k),\alpha}$ up to isomorphisms. If $m > n$, then the first n columns can be reduced to a standard form in a similar way and the other entries of the matrix can be any complex numbers.

Corollary 2.1. *All fermionic Novikov superalgebras induced by* (2.1) *are Novikov superalgebras.*

Proof. It follows from Theorem 2.1 that $(x * y) * z = 0$ for all $x, y, z \in A$. Then the corollary follows. $\square$

3. Proof of the main theorem

In this section, we will prove Theorem 2.1. We first prove the following two lemmas.

Lemma 3.1. *For any $x \in A_1$, we have $f(x) = g(x) = 0$.*

Proof. For any $x \in A_1$, $y \in A_0$, we have $x * y = f(x)y + g(y)x + h_0(x,y)c_0 + h_1(x,y)c_1 \in A_1$. This implies that $f(x)y + h_0(x,y)c_0 = 0$. Note that $\dim A_0 \geqslant 2$. Thus $f(x) = 0$. Similarly one can prove that $g(x) = 0$ for any $x \in A_1$. $\square$

Lemma 3.2. *Assume that $x * y = h_0(x,y)c_0$ for any $x, y \in A_1$. Then there exists a basis $\{f_1, \dots, f_m\}$ such that $f_{2j-1} * f_{2j} = c_0 = -f_{2j} * f_{2j-1}$, $j \leqslant k$, $2k \leqslant m$.*

Proof. The proof of this lemma is easy and can be omitted. $\square$

Proposition 3.1. *If $c_0 = 0$ and $c_1 \neq 0$, then A is isomorphic to F_2.*

Proof. By the assumption, we have

$$x * y = f(x)y + g(y)x + h_1(x,y)c_1 = f(x)y + g(y)x \quad \forall\, x, y \in A_0.$$

Note that the even part of A is a fermionic Novikov algebra. Thus by [1], A_0 is trivial and $f(x) = g(x) = 0$ for all $x \in A_0$. Then the product in A is given by

$$y * x = h_1(x, y)c_1 = x * y \quad \forall\, x \in A_0,\ y \in A_1,$$

and

$$x * y = 0 \quad \forall\, x, y \in A_1.$$

To verify $(x * y) * z = 0$ for any $x, y, z \in A$, we only need to show $(x * y) * z = 0$ for $x \in A_1,\ y, z \in A_0$. Since for $z \in A_0$,

$$(c_1 * z) * z = -(c_1 * z) * z,$$

we have $0 = (c_1 * z) * z = (h_1(c_1, z))^2 c_1$, which means $h_1(c_1, z) = 0$. Consequently, for $x \in A_1,\ y, z \in A_0$ we have

$$(x * y) * z = h_1(x, y)c_1 * z = h_1(x, y)h_1(c_1, z)c_1 = 0.$$

Therefore, by the assumption, A is a Novikov superalgebra and its structure is the same as described in Proposition 2.6 in [3]. Thus, A is isomorphic to F_2 (the notation in [3] is N_3^k). $\square$

Proposition 3.2. *If $c_0 \neq 0$, $c_1 = 0$ and $h_0(x, y) = 0$ for any $x, y \in A_0$, then A is isomorphic to $F_1^{(k)}$.*

P r o o f. By the assumption, $h_0(x, y) = 0$ for any $x, y \in A_0$. Then it follows from the discussions in [1] that A_0 must be trivial and $f(x) = g(x) = 0$ for all $x \in A_0$. Note that

$$x * y = g(y)x + h_0(x, y)c_0 \in A_1 \quad \forall\, x \in A_0,\ y \in A_1,$$

which means $x * y = 0$ for any $x \in A_0$, $y \in A_1$. Therefore, $(x * y) * z = 0$ for any $x, y, z \in A$, namely, A is a Novikov superalgebra and its product structure is the same as Case 1 in Proposition 2.4 in [3]. Therefore, A is isomorphic to $F_1^{(k)}$ (the notation in [3] is N_2^k). $\square$

Proposition 3.3. *If $c_0 \neq 0$, $c_1 = 0$ and there exists $x, y \in A_0$ such that $h_0(x, y) \neq 0$, then A is isomorphic to $F_4^{(k_1),(k_2)}$ or F_6.*

P r o o f. Similarly to Proposition 3.5, we can easily show that A_0 is isomorphic to one of the following algebras:

(1) $A_{(1)}^{(k)} = e_j e_j = e_1$, $2 \leqslant j \leqslant k+1$, $k = 0, \ldots, n-1$;

(2) $A_{(4)}^{0} = e_1 e_j = e_j$, $2 \leqslant j \leqslant n$.

Then the proof can be divided into two cases.

Case 1. $A_0 = A_{(1)}^{(k)}$. It is shown in [1] that

$$f(x) = g(x) = 0, \quad h_0(x, c_0) = 0 \quad \forall\, x \in A_0.$$

Thus we have

$$x * y = g(y)x + h_0(x, y)c_0 = 0 \quad \forall\, x \in A_0,\ y \in A_1.$$

Similarly, we can show that $y * x = 0$ for all $x \in A_0$ and $y \in A_1$. Then the nontrivial product in A must be given by

$$x * y = h_0(x, y)c_0 \quad \forall\, x, y \in A_0;$$
$$x * y = h_0(x, y)c_0 \quad \forall\, x, y \in A_1.$$

Therefore $(x*y)*z = 0$ for any $x, y, z \in A$, and A is a Novikov superalgebra with the same product structure as Case 1 of Proposition 2.5 in [3]. Therefore, A is isomorphic to $F_4^{(k_1),(k_2)}$ (the notation in [3] is $N_4^{k_1,k_2}$).

Case 2. $A_0 = A_{(4)}^{(0)}$. In this case, it is shown in [1] that $g(x) = 0$ for all $x \in A_0$ and there exists a basis $\{e_1, \ldots, e_n\}$ of A_0 such that $h_0(e_1, e_1) \neq 0$, $h_0(e_i, e_k) = 0$, $2 \leqslant i \leqslant n$, $1 \leqslant k \leqslant n$, $f(e_1) = 1$, $f(e_l) = 0$, $l \geqslant 2$, $f(c_0) \neq 0$ and $A_0 = e_1 * e_j = e_j$, $2 \leqslant j \leqslant n$.

Note that

$$x * y = f(x)y + g(y)x + h_0(x, y)c_0 = f(x)y \quad \forall\, x \in A_0,\ y \in A_1,$$
$$y * x = f(y)x + h_0(x, y)c_0 = 0 \quad \forall\, x \in A_0,\ y \in A_1,$$

and

$$x * y = h_0(x, y)c_0 \quad \forall\, x, y \in A_1.$$

Thus the product in A must be given by

$$x * y = f(x)y \quad \forall\, x \in A_0,\ y \in A_1,$$

and

$$x * y = h_0(x, y)c_0 \quad \forall\, x, y \in A_1.$$

So there exists a basis $\{f_1, \ldots, f_m\}$ of A_1 such that $A_1 = f_{2j-1} * f_{2j} = c_0 = f_{2j} * f_{2j-1}$, $j \leqslant k$, $2k \leqslant m$. We claim that $k = 0$. Otherwise, we have

$$(f_1 * f_2) * e_2 = c_0 * e_2 = f(c_0)e_2 \neq 0.$$

However, $(f_1 * e_2) * f_2 = 0$, which is a contradiction. Hence A is isomorphic to F_6. $\square$

Proposition 3.4. *Assume that $c_0 \neq 0$, $c_1 \neq 0$ and $h_0(x,y) = 0$ for any $x, y \in A_0$. Then A is isomorphic to $F_1^{(k)}$, F_2 or $F_3^{(k),\alpha}$.*

P r o o f. By the assumption that $h_0(x,y) = 0$ for any $x, y \in A_0$, we have

$$x * y = f(x)y + g(y)x \quad \forall\, x, y \in A_0.$$

Note that the even part of A is a fermionic Novikov algebra. Thus by [1], A_0 is trivial and $f(x) = g(x) = 0$ for all $x \in A_0$. Therefore, by Lemma 3.1, the nonzero product in A must be given by

$$x * y = h_1(x,y)c_1 = y * x \quad \forall\, x \in A_0,\ y \in A_1,$$

and

$$x * y = h_0(x,y)c_0 \quad \forall\, x, y \in A_1.$$

Note that

$$0 = (x * y) * z = -(x * z) * y = -h_1(x,z)h_1(c_1,y)c_1 \quad \forall\, x, y \in A_0,\ z \in A_1$$

and

$$0 = (x * y) * z = -(x * z) * y = -h_1(x,z)h_0(c_1,y)c_0 \quad \forall\, x, y \in A_1,\ z \in A_0.$$

If $h_1(x,y) = 0$ for any $x \in A_0$, $y \in A_1$, then by Lemma 3.2 we can find a basis $\{f_1, \ldots, f_m\}$ of A_1 such that A is isomorphic to $F_1^{(k)}$.

If there exist $x \in A_0$ and $y \in A_1$ such that $h_1(x,y) \neq 0$, then we have

$$h_1(c_1,x) = h_0(c_1,y) = 0 \quad \forall\, x \in A_0,\ y \in A_1.$$

Now let $e_1 = c_0$ and $f_1 = c_1$. Then by Lemma 3.2, there exists a basis $\{f_1, \ldots, f_m\}$ of A_1 such that $f_{2j} * f_{2j+1} = e_1 = -f_{2j+1} * f_{2j}$, $j \leqslant k$, $2k+1 \leqslant m$. Thus the discussion can be divided into the following two cases:

Case 1. $k = 0$. In this case it is easy to see that A is isomorphic to F_2 through the basis transformation.

Case 2. $k \geqslant 1$. Note that

$$0 = (f_2 * f_i) * f_3 = (f_2 * f_3) * f_i = h_1(e_1, f_i)f_1, \quad i \neq 3$$

and

$$h_1(e_1, f_3)f_1 = (f_2 * f_3) * f_3 = -(f_3 * f_2) * f_3 = -(f_3 * f_3) * f_2 = 0.$$

Therefore $h_1(e_1, f_i) = 0$, $1 \leqslant i \leqslant m$. Hence $h_1(e_1, x) = 0$ for all $x \in A_1$. Thus A is isomorphic to $F_3^{(k),\alpha}$. □

Proposition 3.5. *Assume that $c_0 \neq 0$, $c_1 \neq 0$ and there exist $x, y \in A_0$ such that $h_0(x, y) \neq 0$. Then A is isomorphic to $F_4^{(k_1),(k_2)}$, $F_5^{(k_1),(k_2),\alpha}$ or F_6.*

Proof. By the assumption, there exist $x, y \in A_0$ such that $h_0(x, y) \neq 0$. Then it follows from the discussions in [1] that A_0 is isomorphic to one of the following algebras:

(1) $A_{(1)}^{(k)} = e_j e_j = e_1$, $2 \leqslant j \leqslant k+1$, $k = 0, \ldots, n-1$;

(2) $A_{(4)}^{0} = e_1 e_j = e_j$, $2 \leqslant j \leqslant n$.

Thus the proof can be divided into two cases.

Case 1. $A_0 = A_{(1)}^{(k)}$. By [1], we have $f(x) = g(x) = h_0(x, c_0) = 0$ for any $x \in A_0$. Then by Lemma 3.1, the product in A must be given by

$$x * y = h_1(x, y)c_1 = y * x \quad \forall\, x \in A_0,\ y \in A_1;$$
$$x * y = h_0(x, y)c_0 \quad \forall\, x, y \in A_1.$$

Note that for any $x \in A_0$ and $y, z \in A_1$, we have

$$(x * y) * z = h_1(x, y)c_1 * z = h_1(y, x)c_1 * z = (y * x) * z$$
$$= -(y * z) * x = -h_0(y, z)h_0(c_0, x)c_0 = 0$$

and

$$(x * y) * z = h_1(x, y)h_0(c_1, z)c_0.$$

This implies that

$$h_1(x, y)h_0(c_1, z) = 0 \quad \forall\, x \in A_0,\ y, z \in A_1.$$

If for any $x \in A_0$ and $y \in A_1$, $h_1(x, y) = 0$, then by Lemma 3.2 there exist bases $\{e_1 = c_0, \ldots, e_n\}$ of A_0 and $\{f_1, \ldots, f_m\}$ of A_1 such that A is isomorphic to $F_4^{(k_1),(k_2)}$.

If there exist $x \in A_0$ and $y \in A_1$ such that $h_1(x, y) \neq 0$, then $h_0(c_1, x) = 0$ for any $x \in A_1$. By the discussions in [1], one easily sees that $A_0 = e_i * e_i = e_1$, $i \geqslant 2$, where $\{e_1 = c_0, \ldots, e_n\}$ is a basis of A_0. Now set $f_1 = c_1$. Then for any $1 \leqslant i, l \leqslant n$, $1 \leqslant j \leqslant m$, we have

$$(e_i * f_j) * e_l = h_1(e_i, f_j)f_1 * e_l = h_1(e_i, f_j)h_1(f_1, e_l)f_1$$

and

$$(e_i * f_j) * e_l = -(e_i * e_l) * f_j = -h_0(e_i, e_l)e_1 * f_j = -h_0(e_i, e_l)h_1(e_1, f_j)f_1,$$

which gives the equation

$$h_1(e_i, f_j)h_1(f_1, e_l) = -h_0(e_i, e_l)h_1(e_1, f_j).$$

Setting $i = l = j = 1$, we get $h_1(f_1, e_1) = 0$. Similarly, setting $i = l$, $j = 1$, we get

$h_1(f_1, e_i) = 0$, $1 \leqslant i \leqslant n$; setting $i = l = 2$, we get $h_1(e_1, f_j) = 0$, $1 \leqslant j \leqslant m$. Thus, by Lemma 3.2, A is isomorphic to $F_5^{(k_1),(k_2),\alpha}$.

Case 2. $A_0 = A^0_{(4)}$. It is shown in [1] that $g(x) = 0$ (for all $x \in A_0$), $h_0(e_1, e_1) \neq 0$, $h_0(e_i, e_k) = 0$, $2 \leqslant i \leqslant n$, $1 \leqslant k \leqslant n$, and $f(e_1) = 1$, $f(e_l) = 0$, $l \geqslant 2$, $f(c_0) \neq 0$. Consider the equation

$$x * y = h_0(x, y)c_0 \quad \forall\, x, y \in A_1.$$

By Lemma 3.2, we can choose a basis $\{f_1, \ldots, f_m\}$ of A_1 such that $A_1 = \langle f_{2j-1} * f_{2j} = c_0 = -f_{2j} * f_{2j-1},\ j \leqslant k\rangle$, $2k \leqslant m$. We claim that $k = 0$. Otherwise, for any $f_i, f_j, f_l \in A_1$, we have

$$(f_i * f_j) * f_l = h_0(f_i, f_j)c_0 * f_l = h_0(f_i, f_j)(f(c_0)f_l + h_1(c_0, f_l)c_1)$$

and

$$(f_i * f_l) * f_j = h_0(f_i, f_l)c_0 * f_j = h_0(f_i, f_l)(f(c_0)f_j + h_1(c_0, f_j)c_1).$$

Note that $(f_i * f_j) * f_l = (f_i * f_l) * f_j$. Thus we have

$$h_0(f_i, f_j)(f(c_0)f_l + h_1(c_0, f_l)c_1) = h_0(f_i, f_l)(f(c_0)f_j + h_1(c_0, f_j)c_1).$$

Now, setting $i = j = 1$, $l = 2$, we have $f(c_0)f_1 + h_1(c_0, f_1)c_1 = 0$. Since $f(c_0) \neq 0$, f_1 and c_1 are linearly dependent. Considering the case $i = j = 2$, $l = 1$, we can similarly deduce that f_2 and c_1 are linearly dependent, which is a contradiction.

Now setting $f_1 = c_1$, we have

$$(e_1 * f_k) * e_1 = (f_k + h_1(e_1, f_k)f_1) * e_1 = (h_1(f_k, e_1) + h_1(e_1, f_k)h_1(f_1, e_1))f_1.$$

Moreover, setting $k = 1$ and taking into account the fact that $(e_1 * e_1) * f_k = 0$, we obtain $h_1(f_1, e_1) + h_1(e_1, f_1)h_1(f_1, e_1) = 0$, which implies that $h_1(e_1, f_1) = 0$ or -1.

If $h_1(e_1, f_1) = 0$, then we have $h_1(e_1, f_k) = 0$, $1 \leqslant k \leqslant m$. Note that for $2 \leqslant j \leqslant n$, we have

$$(e_j * e_j) * f_1 = 0 \quad \text{and} \quad (e_j * f_1) * e_j = h_1(e_j, f_1)h_1(f_1, e_j)f_1,$$

which implies that $h_1(e_j, f_1)h_1(f_1, e_j) = 0$. It follows that $h_1(e_j, f_1) = 0$ for all $j \geqslant 2$. On the other hand, for $2 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$, we have

$$(e_1 * e_j) * f_k = e_j * f_k = h_1(e_j, f_k)f_1,$$

and

$$(e_1 * f_k) * e_j = (f_k + h_1(e_1, f_k)f_1) * e_j = (h_1(f_k, e_j) + h_1(e_1, f_k)h_1(f_1, e_j))f_1.$$

Therefore $-2h_1(e_j, f_k) = h_1(e_1, f_k)h_1(f_1, e_j) = 0$. It follows that $h_1(e_j, f_k) = 0$ for $2 \leqslant j \leqslant n$, $1 \leqslant k \leqslant m$. Hence A is isomorphic to F_6.

Finally, if $h_1(e_1, f_1) = -1$, then we have

$$(f_1 * e_1) * e_1 = h_1(f_1, e_1) f_1 * e_1 = (h_1(e_1, f_1))^2 f_1 = f_1.$$

On the other hand, we have $(f_1 * e_1) * e_1 = -(f_1 * e_1) * e_1$. Thus $(f_1 * e_1) * e_1 = 0$, which is a contradiction. This completes the proof. $\square$

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