Czechoslovak Mathematical Journal, 68 (143) (2018), 1125–1130

A REMARK CONCERNING PUTINAR'S MODEL OF HYPONORMAL WEIGHTED SHIFTS

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Received March 21, 2017. Published online December 13, 2017.

To the memory of my beloved father, Atanasie

Abstract. The question whether a hyponormal weighted shift with trace class self-commutator is the compression modulo the Hilbert-Schmidt class of a normal operator, remains open. It is natural to ask whether Putinar's construction through which he proved that hyponormal operators are subscalar operators provides the answer to the above question. We show that the normal extension provided by Putinar's theory does not lead to the extension that would provide a positive answer to the question.

Keywords: weighted shift operator; almost normal operator; hyponormal operator

MSC 2010: 47B20, 47B37

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_1(\mathcal{H})$ and $\mathcal{C}_2(\mathcal{H})$ (or simply $\mathcal{C}_1$ and $\mathcal{C}_2$) the trace class and the Hilbert-Schmidt class, respectively. For an operator $S \in L(\mathcal{H})$, D_S will denote the self-commutator of S, that is $[S^*, S]$. An operator $S \in L(\mathcal{H})$ is called *almost normal* when $D_S \in \mathcal{C}_1(\mathcal{H})$ and is called *hyponormal* when $D_S \geqslant 0$. The class of operators defined on $\mathcal{H}$ which are almost normal will be denoted by $AN(\mathcal{H})$.

Voiculescu's Conjecture 4 (C_4) (cf. [4] or [5]) states that for $T \in AN(\mathcal{H})$ there exists $S \in AN(\mathcal{H})$ such that $T \oplus S = N + K$, where N is a normal operator and K is a Hilbert-Schmidt operator. The conjecture remains unsolved even for arbitrary almost normal weighted shifts. The only result in this direction was obtained by Pasnicu in [2] that states that hyponormal weighted shifts $Te_n = w_{n+1}e_{n+1}$, $n \geqslant 0$, such that $w_n \uparrow w$ and $\sum (w - w_n)^p$ converges for some $p > 0$, satisfy (C_4).

Putinar in [3] proved that a hyponormal operator is a subscalar of order two, and particularly, it is a compression to a semi-invariant subspace of a normal operator.

It is natural to ask whether Putinar's construction leads to proving that weighted shifts operators that are simultaneously hyponormal and almost normal satisfy (C_4), see [1].

We review briefly Putinar's construction of how a hyponormal operator T in $L(\mathcal{H})$ is a compression of a normal operator to a *semi-invariant* subspace, that is, there exists a normal operator $N \in L(\mathcal{K})$ with $\mathcal{H} \subset \mathcal{K}$ such that the matrix representation of N is

$$N = \begin{pmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{pmatrix}.$$

Let $\mathbb{D}$ be an open disc that includes the spectrum $\sigma(T)$ of a hyponormal operator $T \in L(\mathcal{H})$. Let

$$L^2(\mathbb{D}, \mathcal{H}) = \{f\colon \mathbb{D} \to \mathcal{H}\colon \|f\|^2_{2,\mathbb{D}} := \int_{\mathbb{D}} \|f(z)\|^2 \,\mathrm{d}\lambda(z) < \infty\},$$

where $\mathrm{d}\lambda$ is the planar Lebesgue measure. Let $W^2(\mathbb{D}, \mathcal{H})$ consist of those f in $L^2(\mathbb{D}, \mathcal{H})$ such that $\overline{\partial} f$ and $\overline{\partial}^2 f$, in the sense of distributions, belong to $L^2(\mathbb{D}, \mathcal{H})$, where $\overline{\partial}$ is the operator $\partial/\partial\overline{z}$. Endowed with the norm

$$\|f\|^2_{W^2} := \sum_{k=0}^{2} \|\overline{\partial}^k f\|^2_{2,\mathbb{D}},$$

$W^2(\mathbb{D}, \mathcal{H})$ becomes a closed subspace of $L^2(\mathbb{D}, \mathcal{H})$ in which $C^\infty(\overline{\mathbb{D}}, \mathcal{H})$ is a dense subspace. Let $N\colon L^2(\mathbb{D}, \mathcal{H}) \to L^2(\mathbb{D}, \mathcal{H})$ be the normal operator defined by $(Nf)(z) = zf(z)$ and let M be the restriction of N to the invariant subspace $W^2(\mathbb{D}, \mathcal{H})$. Let $\mathcal{H}_1$ be $\overline{(T-z)W^2(\mathbb{D}, \mathcal{H})}$, where

$$T - z\colon W^2(\mathbb{D}, \mathcal{H}) \to W^2(\mathbb{D}, \mathcal{H})$$

is defined by

$$((T-z)f)(z) = T(f(z)) - zf(z)$$

and it is a bounded operator whose range is invariant for operator M. Let

$$\widetilde{M}\colon W^2(\mathbb{D}, \mathcal{H})/\overline{(T-z)W^2(\mathbb{D}, \mathcal{H})} \to W^2(\mathbb{D}, \mathcal{H})/\overline{(T-z)W^2(\mathbb{D}, \mathcal{H})}$$

be defined by $\widetilde{M}\tilde{f} = \widetilde{Mf}$, where $\tilde{f} \in W^2(\mathbb{D}, \mathcal{H})/\overline{(T-z)W^2(\mathbb{D}, \mathcal{H})}$ is the equivalence class of an f in $W^2(\mathbb{D}, \mathcal{H})$. Relative to the orthogonal decomposition of

$$L^2(\mathbb{D}, \mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H}(\mathbb{D}) \oplus \mathcal{H}',$$

where

$$\mathcal{H}(\mathbb{D}) = W^2(\mathbb{D}, \mathcal{H})/\overline{(T-z)W^2(\mathbb{D}, \mathcal{H})}$$

and

$$\mathcal{H}' = L^2(\mathbb{D}, \mathcal{H}) \ominus W^2(\mathbb{D}, \mathcal{H}),$$

the matrix representation of N is

$$N = \begin{pmatrix} A & * & * \\ 0 & \widetilde{M} & * \\ 0 & 0 & * \end{pmatrix}.$$

The initial space $\mathcal{H}$ and the operator T can be recuperated from $\widetilde{M}$. More precisely, $\mathcal{H}(\mathbb{D}) = \mathcal{H} \oplus \mathcal{H}''$ and relative to this decomposition, the operator $\widetilde{M}$ has representation $\widetilde{M} = \begin{pmatrix} T & * \\ 0 & * \end{pmatrix}$. Denoting $\mathcal{H}_2 = \mathcal{H}' \oplus \mathcal{H}''$, then relative to the decomposition of

$$L^2(\mathbb{D}, \mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H} \oplus \mathcal{H}_2,$$

the matrix representation of N is

$$N = \begin{pmatrix} A & B & C \\ 0 & T & D \\ 0 & 0 & E \end{pmatrix}.$$

The following theorem (Theorem 2.1, [1]) makes use of the above notation.

Theorem 1.1. *Let T be a hyponormal operator in $AN(\mathcal{H})$. If the operator A belongs to $AN(\mathcal{H}_1)$, then T satisfies* (C_4).

2. Application

Although it is a negative statement, it is relevant to provide the proof that the subnormal operator A cannot be almost normal when T is a hyponormal weighted shift. Indeed, let $\{e_n\}_{n\geqslant 0}$ be an orthonormal basis of $\mathcal{H}$ such that $Te_n = w_{n+1}e_{n+1}$, $n \geqslant 0$. A weighted shift operator is hyponormal if and only if the sequence $\{|w_n|\}_{n\geqslant 1}$ is nondecreasing. We can further assume that $w_n \geqslant 0$ since such weighted shifts are unitarily equivalent, and that $w_n \to w$, $w > 0$. We can further assume that $\mathbb{D}$ is a disc centered at the origin since the space $\mathcal{H}(\mathbb{D})$ is not dependent on the open set $\mathbb{D}$ that includes $\sigma(T)$. Let

$$E_{i,j,k}(z, \overline{z}) = z^i \overline{z}^j e_k, \quad i, j, k \geqslant 0$$

and let

$$F_{i,j,k} = (T - z)E_{i,j,k} = w_{k+1}E_{i,j,k+1} - E_{i+1,j,k}.$$

With the disc $\mathbb{D}$ centered at the origin, we have

$$E_{i,j,k} \perp E_{r,s,t} \quad \text{when } (i,j,k) \neq (r,s,t);$$

recall that the inner product in $L^2(\mathbb{D}, \mathcal{H})$ is defined by

$$\langle f, g\rangle_{L^2(\mathbb{D},\mathcal{H})} = \int_{\mathbb{D}} \langle f(z), g(z)\rangle_{\mathcal{H}} \, \mathrm{d}\lambda(z).$$

Let $G_n = \{F_{i,j,k}\colon\ i + j + k = n\}$ listed in the following order

$$F_{n,0,0}, F_{n-1,1,0}, \ldots, F_{0,n,0}; F_{n-1,0,1}, F_{n-2,1,1}, F_{n-3,2,1}, \ldots, F_{0,n-1,1}; \ldots; F_{0,0,n}.$$

Any vector of G_m is orthogonal on any vector of G_n when $m \neq n$ (Lemma 3.1, [1]), and thus the space $\mathcal{H}_1$ can be decomposed as

$$\mathcal{H}_1 = \bigoplus_{n\geqslant 0} \operatorname{span}(G_n),$$

where $\operatorname{span}(G_n)$ denotes the linear span of all vectors in G_n.

Since the operator A is the restriction of the operator of multiplication by variable z,

$$AF_{i,j,k} = zF_{i,j,k} = z(w_{k+1}E_{i,j,k+1} - E_{i+1,j,k}) = F_{i+1,j,k},$$

and consequently $A(\operatorname{span}(G_n)) \subseteq \operatorname{span}(G_{n+1})$. Relative to the above decomposition of $\mathcal{H}_1$, the operator A can be written

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ A_{10} & 0 & 0 & \ddots \\ 0 & A_{21} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

with each $A_{n+1,n}\colon\ \operatorname{span}(G_n) \to \operatorname{span}(G_{n+1})$.

After orthonormalization of each subspace $\operatorname{span}(G_n)$ and redenoting its new vectors by

$$G_{n,0,0}, G_{n-1,1,0}, \ldots, G_{0,n,0}; G_{n-1,0,1}, G_{n-2,1,1}, G_{n-3,2,1}, \ldots, G_{0,n-1,1}; \ldots; G_{0,0,n},$$

each operator $A_{n+1,n}$ has a matrix representation $\tilde{A}_{n+1,n}$.

The operator A is almost normal if and only if

$$\operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}) \leqslant m < \infty, \quad n \geqslant 0$$

(see [1], Theorem 3.2).

Relative to conjecture (C_4), one can assume that the operator T has the norm less than 1 since multiplication by a constant preserves both hyponormality and (C_4), and thus one can choose the disc $\mathbb{D}$ to have radius 1.

Let the orthonormalized vectors of G_n be split into subgroups L_0^n, $L_1^n, \dots, L_n^n$, with each L_k^n consisting of

$$G_{n-k,0,k},\ G_{n-k-1,1,k}, \dots, G_{0,n-k,k}.$$

Since all vectors $F_{n-k,k,0}$ of the group L_0^n are orthogonal on each other,

$$G_{n-k,k,0} = \frac{F_{n-k,k,0}}{\|F_{n-k,k,0}\|_2}, \quad k = 0,1,\dots,n.$$

Thus, for $k = 0,1,\dots,n$,

$$AG_{n-k,k,0} = z\frac{F_{n-k,k,0}}{\|F_{n-k,k,0}\|_2} = \frac{F_{n-k+1,k,0}}{\|F_{n-k,k,0}\|_2} = \frac{\|F_{n-k+1,k,0}\|_2}{\|F_{n-k,k,0}\|_2}G_{n-k+1,k,0},$$

that is, the kth vector of subgroup L_0^n is mapped into the kth vector of the subgroup L_0^{n+1}.

Theorem 2.1. *With above notation,* $\operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}) \to \infty$, $n \to \infty$, *and consequently the operator A is not almost normal.*

P r o o f. According to the above calculations,

$$\begin{aligned}\operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}) &= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\|A_{n+1,n}G_{n-k-j,j,k}\|_2^2 \\ &\geqslant \sum_{j=0}^{n}\|A_{n+1,n}G_{n-j,j,0}\|_2^2 = \sum_{j=0}^{n}\frac{\|F_{n-j+1,j,0}\|_2^2}{\|F_{n-j,j,0}\|_2^2}.\end{aligned}$$

On other hand,

$$\begin{aligned}\|F_{i,j,0}\|_2^2 &= \int_{\mathbb{D}}\|w_1z^i\overline{z}^je_1 - z^{i+1}\overline{z}^je_0\|^2\,\mathrm{d}\lambda(z) = \int_{\mathbb{D}}(w_1^2|z^i\overline{z}^j|^2 + |z^{i+1}\overline{z}^j|^2)\,\mathrm{d}\lambda(z) \\ &= 2\pi\int_0^1[w_1^2r^{2(i+j)} + r^{2(i+1+j)}]r\,\mathrm{d}r = 2\pi\Big[w_1^2\frac{1}{2(i+j+1)} + \frac{1}{2(i+j+2)}\Big],\end{aligned}$$

thus

$$\frac{\|F_{n-j+1,j,0}\|_2^2}{\|F_{n-j,j,0}\|_2^2} = \frac{n+1}{n+3}\,\frac{n(w_1^2+1)+3w_1^2+2}{n(w_1^2+1)+2w_1^2+1} \to 1,$$

and consequently, $\operatorname{tr}(\tilde{A}^*_{n+1,n}\tilde{A}_{n+1,n}) \to \infty$. $\square$

References

[1] *V. Lauric*: Remarks on hyponormal operators and almost normal operators. Matematiche (Catania) *72* (2017), 3–8. MR

[2] *C. Pasnicu*: Weighted shifts as direct summands mod $\mathcal{C}_2$ of normal operators. Dilation Theory, Toeplitz operators, and Other Topics. 7th Int. Conf. Oper. Theory, Timisoara, 1982, Oper. Theory, Adv. Appl. *11* (1983), pp. 275–281. zbl MR

[3] *M. Putinar*: Hyponormal operators are subscalar. J. Oper. Theory *12* (1984), 385–395. zbl MR

[4] *D. Voiculescu*: Hilbert space operators modulo normed ideals. Proc. Int. Congr. Math. Warszawa, 1983 *2* (1984), 1041–1047. zbl MR

[5] *D. V. Voiculescu*: Almost normal operators mod Hilbert-Schmidt and the K-theory of the Banach algebras $E\Lambda(\Omega)$. J. Noncommut. Geom. *8* (2014), 1123–1145. zbl MR doi

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