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EVEN FACTOR OF BRIDGELESS GRAPHS CONTAINING TWO SPECIFIED EDGES

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Abstract. An even factor of a graph is a spanning subgraph in which each vertex has a positive even degree. Let G be a bridgeless simple graph with minimum degree at least 3. Jackson and Yoshimoto (2007) showed that G has an even factor containing two arbitrary prescribed edges. They also proved that G has an even factor in which each component has order at least four. Moreover, Xiong, Lu and Han (2009) showed that for each pair of edges e_1 and e_2 of G, there is an even factor containing e_1 and e_2 in which each component containing neither e_1 nor e_2 has order at least four. In this paper we improve this result and prove that G has an even factor containing e_1 and e_2 such that each component has order at least four.

Keywords: bridgeless graph; components of an even factor; specified edge

MSC 2010: 05C70

1. Introduction

We use [1] for terminology and notation. A spanning subgraph of a graph $G = (V(G), E(G))$ is called a *factor* of G. An *even factor* of G is a factor of G, in which each vertex has even positive degree. A 2-*factor* (1-*factor*) of G is a factor of G such that every vertex has degree 2 (degree 1). The set of components of G and the minimum order of components of G are denoted by $C(G)$ and $\sigma(G)$, respectively.

In 2007, Jackson and Yoshimoto proved the following theorem in a bridgeless graph with minimum degree at least 3.

Theorem 1.1 (Jackson and Yoshimoto, [2]). *Every bridgeless simple graph with* $\delta(G) \geqslant 3$ *has an even factor in which every component has order at least four.*

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In 2009, they also proved the next theorem.

Theorem 1.2 (Jackson and Yoshimoto, [3]). *If G is a bridgeless graph with $\delta(G) \geqslant 3$, then for each pair of edges e_1 and e_2 of G, there is an even factor of G containing e_1 and e_2.*

It is shown in [5] that the following result extends Theorem 1.1.

Theorem 1.3 (Xiong, Lu and Han, [5]). *Let G be a bridgeless simple graph with $\delta(G) \geqslant 3$. Then for each given edge e, G has an even factor F in which every component has order at least four such that F does not contain e.*

Now, if G is a bridgeless simple graph with at most two vertices of degree two, then we can add a new edge e connecting these two vertices and by Theorem 1.3, we have the following corollary.

Corollary 1.1. *If G is a bridgeless simple graph with at most two vertices of degree two, then G has an even factor in which every component has order at least four.*

There is another result for even factors of a bridgeless simple graph containing two given edges.

Theorem 1.4 (Xiong, Lu and Han, [5]). *Let G be a bridgeless simple graph with $\delta(G) \geqslant 3$. Then for each pair of edges e_1 and e_2 of G, there is an even factor of G containing e_1 and e_2 in which every component containing neither e_1 nor e_2 has order at least four.*

As our main result we improve the above result and prove the following theorem.

Theorem 1.5. *Let G be a bridgeless simple graph with $\delta(G) \geqslant 3$. Then for each pair of edges e_1 and e_2 of G, there is an even factor F of G containing e_1, e_2 in which $\sigma(F) \geqslant 4$.*

The set of edges incident to a vertex v of G, and the set of vertices which are joined to the vertex v are denoted by $E_G(v)$ and $N_G(v)$, respectively. Let $e = vx$ and $f = vy$ be two incident edges of G. The graph obtained from $G - \{e, f\}$ by adding a new vertex v' and new edges $v'x$ and $v'y$ is denoted by G_v^{ef}. For a connected subgraph H of G, the graph obtained from G by contracting every edge of H is denoted by G/H. Similarly, G/e is defined for an edge e of G. In the graph G/H the vertex corresponding to H is denoted by h^*. A *bond* of a graph is a minimal edge cut.

To prove our main theorem we need the following two lemmas.

Lemma 1.1 (Jackson and Yoshimoto, [2]). *Let G be a 2-edge-connected graph, $v \in V(G)$ with $d(v) \geqslant 4$ and $e_1 \in E(v)$. Then*
(a) *there exists an edge $e_2 \in E(v) - e_1$ such that $G_v^{e_1e_2}$ is 2-edge-connected;*
(b) *if $d(v) = 4$, then there exists at most one edge $e_3 \in E(v) - e_1$ such that $G_v^{e_1e_3}$ is not 2-edge-connected.*

Lemma 1.2 (McKee, [4]). *Every bond of any even factor contains an even number of edges.*

2. Proof of main theorem

To prove our main theorem we will prove eleven claims. In Claims 2.3, 2.4 and 2.5 we will use some ideas of the proof of Theorem 1.1 in [2]. Note that in all these claims we construct some new graphs and choose suitable edges e'_1 and e'_2 of these graphs instead of e_1 and e_2, respectively. Actually, each edge of G corresponds to one edge on these graphs.

Proof. First, assume on the contrary that G is a counterexample to the statement such that $\Delta(G)$ is minimized and subject to the condition that the number of vertices of G with degree $\Delta(G)$ is minimized. Therefore, G has two edges e_1 and e_2 such that there is no even factor of G containing e_1 and e_2 in which every component has order at least four.

Claim 2.1. *The edges e_1 and e_2 are not adjacent.*

Proof. We prove by contradiction. Let $e_1 = ux$ and $e_2 = uy$. By Theorem 1.3, we have $d_G(u) \neq 3$. If $G_1 = G_u^{e_1e_2} + uu'$ is a bridgeless graph, then by Theorem 1.3, G_1 has an even factor F' containing e_1 and e_2 such that $\sigma(F') \geqslant 4$, since $d(u') = 3$. The even factor F' can be converted to a desired even factor F of G. Thus G_1 has a bridge e'. It is clear that $e' = uu'$. Let H_1 and H_2 be the components of $G_1 - e'$. By Corollary 1.1, H_1 has an even factor F_1 in which $\sigma(F_1) \geqslant 4$ and similarly H_2 has an even factor F_2 such that $\sigma(F_2) \geqslant 4$. We can suppose that $u' \in V(H_1)$ and $e_1, e_2 \in E(H_1)$. Since $d_{H_1}(u') = 2$, $e_1, e_2 \in E(F_1)$. By replacing u' with u in F_1, $F = F_1 \cup F_2$ is a desired even factor of G, contrary to the assumption. □

Claim 2.2. *G is not a cubic graph.*

Proof. Proceed by contradiction and let G be a cubic graph. By Theorem 1.2, G has a 2-factor F such that $e_1, e_2 \in E(F)$. Therefore, F contains a component $T = abca$ of order three. Let $G_2 = G/T$. Each edge of $G - T$ corresponds to one edge of G_2.

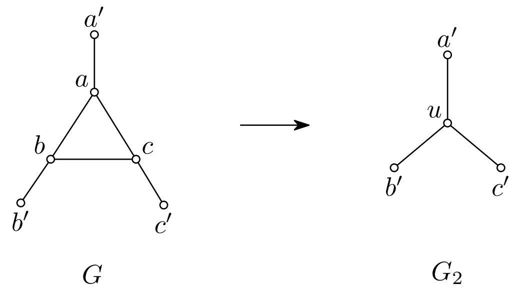

Figure 1. G and G_2

It is obvious that $G - F$ is a 1-factor and G_2 is a cubic bridgeless graph. By Claim 2.1, it suffices to consider for e_1 and e_2 the following two cases:

(1) $\{e_1, e_2\} \cap E(T) = \emptyset$. In this case suppose that $e_1' = e_1$ and $e_2' = e_2$.

(2) $\{e_1, e_2\} \cap E(T) = \{e_1\} = \{ab\}$. Consider $e_1' = uc'$ and $e_2' = e_2$ when $e_2 \neq cc'$, and consider $e_2' = uc'$ and $e_1' = ua'$ when $e_2 = cc'$.

Then G_2 has a desired 2-factor F' containing e_1' and e_2'. By symmetry, we can suppose that $ua', uc' \in E(F')$. Then $F = (F' - \{ua', uc'\}) \cup \{aa', ab, bc, cc'\}$ is a desired 2-factor of G. □

By Claim 2.2 we have $\Delta(G) \geqslant 4$. Assume that v is a vertex of G with degree $\Delta(G)$. If G is a complete graph, then it is clear that G has a hamiltonian cycle containing e_1 and e_2. Otherwise, the induced subgraph of G by $N_G(v)$ is not a complete graph, since G is a connected graph with maximum degree $\Delta(G)$. Therefore, there are two edges $e = vw$ and $f = vx$ such that $wx \notin E(G)$.

Claim 2.3. $\Delta(G) = 4$.

P r o o f. Suppose on the contrary that $\Delta(G) \geqslant 5$. Consider G_v^{ef} and let G_3 be the graph obtained by removing v' from G_v^{ef} and adding wx. Each edge of G corresponds to one edge of G_3 (e and f correspond to wx). By Claim 2.1, there are the following two cases:

(1) If $\{e, f\} \cap \{e_1, e_2\} = \emptyset$, then $e_1' = e_1$ and $e_2' = e_2$.

(2) If $\{e, f\} \cap \{e_1, e_2\} = \{e_1\}$, then $e_1' = wx$ and $e_2' = e_2$.

If G_3 is a bridgeless graph, then G_3 has a required even factor containing e_1' and e_2'. It is easy to convert this even factor to a desired even factor of G containing e_1 and e_2. Thus, G_3 and hence G_v^{ef} has a bridge e_0. Let G_1' and G_2' be the components of $G_v^{ef} - e_0$. We may suppose that $w, x, v' \in V(G_1')$ and $v \in V(G_2')$. By symmetry, we can suppose that w is not incident with e_0. By Lemma 1.1, there is an edge $h = vz$ such that G_v^{eh} is bridgeless. We have $z \in V(G_2')$ and hence $wz \notin E(G)$. Let G_3' the graph obtained from G_v^{eh} by removing v' and adding wz. Then G_3' is a bridgeless simple graph and we can apply the preceding method for G'_3 to prove Claim 2.3, as above. □

Claim 2.4. *There are $z \in N_G(v) - w$ and $h = vz \in E(G) - e$ such that $G_v^{eh} + vv'$ is a bridgeless simple graph and $wz \notin E(G)$.*

Proof. The proof is similar to the proof of Claim 2 in [2]. We repeat this proof with a few changes.

According to the preceding discussion, $G_v^{ef} + vv'$ is simple. Suppose that $G_v^{ef} + vv'$ has a bridge e_0. It is clear that $e_0 = v'v$, and hence G_v^{ef} is disconnected. Let G'_1 and G'_2 be the components of G_v^{ef}. Since G is bridgeless, we may suppose that $w, x, v' \in V(G'_1)$ and $v \in V(G'_2)$. Choose $h = vz \in E_G(v)$ with $z \in V(G'_2)$. By Lemma 1.1 (b), G_v^{eh} is bridgeless. Clearly, $wz \notin E(G)$ and G_v^{eh} is simple. □

Relabelling f and h if necessary, $G' = G_v^{ef} + vv'$ is a bridgeless simple graph and $wx \notin E(G)$. Let $N_G(v) = \{w, x, y, z\}$. We can find two edges e'_1 and e'_2 of G' corresponding to e_1 and e_2 of G, respectively. The graph G' has an even factor F' with $\sigma(F') \geqslant 4$ such that $e'_1, e'_2 \in E(F')$. If $vv' \notin E(F')$, then F' is easily converted to a desired even factor of G containing e_1 and e_2. Hence, $vv' \in E(F')$. Since G is a counterexample and F'/vv' is an even factor of G, there is $D \in C(F')$ such that $vv' \in V(D)$ and D is a 4-cycle. Without loss of generality we may suppose that $T = D/vv' = vwyv$ is a triangle in G. Let H be the subgraph induced by $\{w, x, y, z\}$ and H^{c} be the complement of H.

Claim 2.5. *H^{c} has a 1-factor.*

Proof. Suppose on the contrary that H^{c} has no 1-factor. We already have $wx \in E(H^{\mathrm{c}})$. According to the assumption, $yz \notin E(H^{\mathrm{c}})$ and hence $yz \in E(G)$. Moreover, we have $yw \in E(G)$. Now, consider two cases:

(A) $xy \in E(G)$. In this case $d_G(y) = d_G(v) = 4$ and the edge vy is a chord in the 4-cycle $vzywv$ of G. Thus, $G - vy$ is a bridgeless simple graph and $\delta(G - vy) \geqslant 3$. If $vy \notin \{e_1, e_2\}$, then $G - vy$, and hence G has a desired even factor. Therefore, $vy \in \{e_1, e_2\}$. Consider $e_1 = vy$. Since e_1 and e_2 are not adjacent, $e_2 \neq yz$. Consider $e'_1 = yz$ and $e'_2 = e_2$. The graph $G - vy$ has an even factor F_1 such that $e'_1, e'_2 \in E(F_1)$ and $\sigma(F_1) \geqslant 4$. Since $d_{F_1}(v)$ is even, $d_{F_1}(v) = 2$. Hence $vv_0 \notin E(F_1)$ for some $v_0 \in \{x, z, w\}$. Then $\{v, v_0, y\}$ induces a triangle T' in G, and $e_2 \notin E(T')$ since e_1 and e_2 are not adjacent. Since $vv_0, vy \notin E(F_1)$, we obtain a desired even factor F of G such that $E(F) = E(F_1) \triangle E(T')$, where $\triangle$ denotes the symmetric difference.

(B) $xy \notin E(G)$. In this case $xy \in E(H^{\mathrm{c}})$. Since H^{c} has no 1-factor, $wz \notin E(H^{\mathrm{c}})$, and so $wz \in E(G)$. Hence, the induced subgraph by $\{v, w, y, z\}$ in G is isomorphic to K_4. Since G is a bridgeless graph, there is a vertex $u \in \{w, y, z\}$ adjacent to a vertex $s \in V(G) - \{v, w, y, z\}$. Let $v_1, v_2, v_3 \in N_G(v)$ such that $\{v_1, v_2, v_3\} = \{w, y, z\}$ and $u = v_3$. Then $d_G(v) = d_G(v_3) = 4$. The graph $G - vv_3$ is a bridgeless graph and $\delta(G - vv_3) \geqslant 3$. If $vv_3 \notin \{e_1, e_2\}$, then according to our hypotheses, $G - vv_3$ and hence G has a desired even factor F_1, a contradiction. Thus $vv_3 \in \{e_1, e_2\}$. Assume that $e_1 = vv_3$, and then consider $e_1' = v_2v_3$ and $e_2' = e_2$. Note that $e_2 \notin \{vv_1, vv_2, v_1v_3, v_2v_3\}$, since e_1 and e_2 are not adjacent in G. The graph $G - vv_3$ has an even factor F_1 containing e_1' and e_2'. If there exists $e^- \in \{vv_1, vv_2, v_1v_3, v_2v_3\} \setminus E(F_1)$, then we can find a triangle T' of G such that $e^-, vv_3 \in E(T')$. Since $e^- \neq e_2$, we obtain a desired even factor F such that $E(F) = E(F_1) \triangle E(T')$. Hence $\{vv_1, vv_2, v_1v_3, v_2v_3\} \subseteq E(F_1)$. If v_1v_2 is also an edge of F_1, then $(F_1 - \{vv_2, v_2v_3\}) \cup \{vv_3\}$ is a desired even factor of G. On the other hand, if $v_1v_2 \notin E(F_1)$, then $(F_1 - \{vv_1, v_2v_3\}) \cup \{vv_3, v_1v_2\}$ is a desired even factor of G. □

By Claim 2.5 and relabelling if necessary, we may assume that $wx, yz \notin E(G)$ and $T = vwyv$ is a triangle of G.

Claim 2.6. $wy \in \{e_1, e_2\}$.

P r o o f. By contradiction suppose that $yw \notin \{e_1, e_2\}$. We consider two cases for the edges e_1' and e_2' of $G_4 = (G - v) \cup \{yz, wx\}$:

(1) If $\{e_1, e_2\} \cap \{vx, vy, vz, vw\} = \{e_1\}$, then consider $e_1' = wx$ and $e_2' = e_2$.

(2) If $\{e_1, e_2\} \cap \{vx, vy, vz, vw\} = \emptyset$, then consider $e_1' = e_1$ and $e_2' = e_2$.

The graph G_4 has an even factor F_4 containing e_1' and e_2' such that $\sigma(F_4) \geqslant 4$. In the case (1), $wx \in E(F_4)$. If $yz \in E(F_4)$, then $F = F_4 - \{yz, wx\} \cup \{vy, vw, vx, vz\}$ is a required even factor of G, and if $yz \notin E(F_4)$, then $F = F_4 - \{wx\} \cup \{vw, vx\}$ is a required even factor of G. In the case (2), if $zy, xw \in E(F_4)$, then similarly to the case (1), $F = F_4 - \{zy, xw\} \cup \{vy, vw, vx, vz\}$ is a desired even factor of G. Then $zy, xw \notin E(F_4)$. If $wy \notin E(F_4)$, then $F = F_4 \cup \{vy, wy, vw\}$ is a required even factor of G. Otherwise, $F = (F_4 - wy) \cup \{vy, wv\}$ is a desired even factor of G, a contradiction. Therefore $wy \in \{e_1, e_2\}$. □

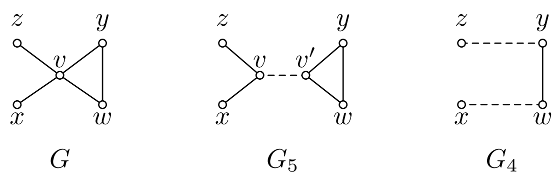

Figure 2. G, G_5 and G_4

By Claim 2.6, we can assume that $e_1 = wy$. Suppose that $h = vz$ and $f = vx$. Let $G_5 = G_v^{fh} + vv'$. If G_5 is a bridgeless graph, then G_5 has an even factor F_5 with $\sigma(F_5) \geqslant 4$ such that $e_1', e_2' \in E(F_5)$. Hence, $F = F_5$ (by replacing v' with v) or $F = F_5/vv'$ is an even factor of G containing e_1 and e_2 in which $\sigma(F) \geqslant 4$, since $wy \in E(F_5)$. Therefore, G_5 has a bridge e_0. It is obvious that $e_0 = vv'$ and v is a cut vertex of G.

Each vertex of G with degree $\Delta(G) = 4$ is similar to v and the following claim holds.

Claim 2.7. *If $u \in V(G)$ and $d_G(u) = 4$, then u is a cut vertex and the subgraph induced by $N_G(u)$ of G has at least one edge and the complement of subgraph induced by $N_G(u)$ has a 1-factor.*

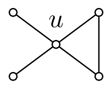

Figure 3. The vertex u of G with $d_G(u) = 4$.

Now, we continue the proof using these features of vertices with degree 4 of G.

Claim 2.8. *If u is a vertex with degree 4 of G such that $u_1, u_2 \in N_G(u)$ and $u_1u_2 \in E(G)$, then $u_1u_2 \in \{e_1, e_2\}$.*

Proof. This claim follows from Claims 2.6 and 2.7. □

Claim 2.9. *The vertices y and w do not have any common neighbour other than v.*

Proof. By contradiction, suppose that y and w have a common neighbour r other than v. There are two cases:

(1) $d_G(r) = 3$.

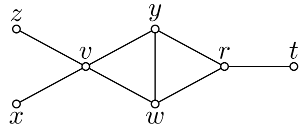

Figure 4. $d_G(r) = 3$

According to Claims 2.7 and 2.8, $d_G(y) = d_G(w) = 3$. Let C be the 4-cycle $vyrwv$ and $G_6 = G/C$.

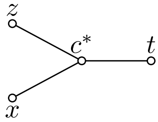

Figure 5. G_6

Since e_1 and e_2 are not adjacent and $e_1 = wy$, we have $e_2 \in E(G_6)$. Consider $e_1' = c^*t$ and $e_2' = e_2$. It is possible that $e_1' = e_2'$. The graph G_6 has an even factor F_6 containing e_1' and e_2' and $\sigma(F_6) \geqslant 4$. We may assume that $tc^*, xc^* \in E(F_6)$. Thus, $F = (F_6 - \{c^*t, xc^*\}) \cup \{xv, vw, wy, ry, rt\}$ is a required even factor of G.

(2) $d_G(r) = 4$.

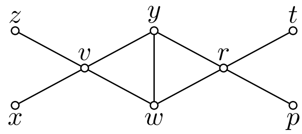

Figure 6. $d_G(r) = 4$

If $xz, pt \in E(G)$, then we can consider $pt \neq e_2$ (or $xz \neq e_2$). According to Claim 2.8, it is a contradiction. Therefore, xz or pt does not belong to $E(G)$. We may assume that $pt \notin E(G)$. In this case consider the graph $G_7 = (G - \{v, r, y, w\}) \cup \{zt, px\}$. By considering $e_1' = zt$ and a suitable edge e_2', the graph G_7 has an even factor F_7 containing e_1' and e_2' such that $\sigma(F_7) \geqslant 4$. By Lemma 1.2, $tz, px \in E(F_7)$. Therefore $F = (F_7 - \{tz, px\}) \cup \{rt, rp, wy, vy, vw, vz, vx\}$ is an even factor of G containing e_1 and e_2 and $\sigma(F) \geqslant 4$, because $pt \notin E(G)$ and $e_2 \neq ry, rw$. □

Claim 2.10. *$xz \in E(G)$ and $e_2 = xz$.*

P r o o f. First, suppose that $xz \notin E(G)$. Since v is a cut vertex, it is clear that $xy, wz \notin E(G)$ and $G_4 = (G - v) \cup \{yz, xw\}$ is a bridgeless graph. If we consider $e_2' = e_2$ and $e_1' = yz$, then G_4 has an even factor F_4 in which $\sigma(F_4) \geqslant 4$ and $e_2', e_1' \in E(F_4)$.

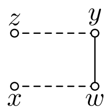

Figure 7. G_4

By Lemma 1.2, $yz, wx \in E(F_4)$. Let $F = F_4 - \{yz, wx\} \cup \{vy, vw, vx, vz\}$ be the even factor of G corresponding to F_4. If $yw \in E(F)$, then F is a required even factor of G. Otherwise, since y and w have no common neighbour other than v and $xz \notin E(G)$, $(F - \{vy, vw\}) \cup wy$ is a required even factor of G. It is a contradiction. Hence, $xz \in E(G)$ and by Claim 2.8, it is clear that $e_2 = xz$. □

Now, according to all the above discussions, we have the following note:

Note 2.1. Each vertex v with degree 4 of the counterexample G is a cut vertex and if $N_G(v) = \{x, y, w, z\}$, then we can assume that $yz, wx \notin E(G)$ and $wy, xz \in E(G)$. Moreover, y and w have no common neighbour other than v and similarly z and x have no common neighbour other than v and $e_1 = wy$ and $e_2 = xz$.

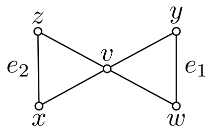

Figure 8. G and the edges e_1 and e_2

Claim 2.11. *G has only one vertex v with degree 4.*

P r o o f. There is only one choice for e_1 and e_2 and by Note 2.1, they join two neighbours of one vertex of degree 4, see Figure 8. Now, the claim is clear. □

According to Claim 2.11, we have $d_G(x) = d_G(y) = d_G(z) = d_G(w) = 3$. Now, consider $G'' = (G - \{v, z, x, y, w\}) \cup \{y'z', w'x'\}$ such that $xx', yy', zz', ww' \in E(G)$ (Since v is a cut vertex, $y'z', w'x' \notin E(G)$ and since x, y, z, w have no common neighbour, $z' \neq x'$ and $y' \neq w'$).

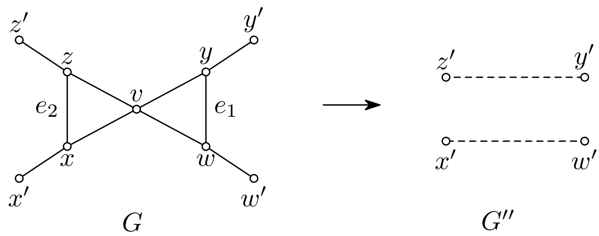

Figure 9. G and G''

The graph G'' is a bridgeless simple graph with $\delta(G) \geqslant 3$ and $\Delta(G'') < \Delta(G)$. Therefore, G'' has an even factor F'' in which $\sigma(F'') \geqslant 4$ and $N_{G''}(z') - \{y'z'\} \subseteq E(F'')$, since G is a counterexample to the statement such that $\Delta(G)$ is minimized. Then $y'z' \notin E(F'')$ and by Lemma 1.2, $w'x' \notin E(F'')$. Hence, $F = F'' \cup \{xz, vz, vx, vy, vw, yw\}$ is an even factor of G containing e_1 and e_2 and $\sigma(F) \geqslant 4$. It is a contradiction and we are done. □

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