Czechoslovak Mathematical Journal, 68 (143) (2018), 1091–1104

THE SPECTRAL DETERMINATIONS OF THE CONNECTED MULTICONE GRAPHS $K_w \bigtriangledown mP_{17}$ AND $K_w \bigtriangledown mS$

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Received March 8, 2017. Published online April 27, 2018.

Abstract. Finding and discovering any class of graphs which are determined by their spectra is always an important and interesting problem in the spectral graph theory. The main aim of this study is to characterize two classes of multicone graphs which are determined by both their adjacency and Laplacian spectra. A multicone graph is defined to be the join of a clique and a regular graph. Let K_w denote a complete graph on w vertices, and let m be a positive integer number. In A. Z. Abdian (2016) it has been shown that multicone graphs $K_w \bigtriangledown P_{17}$ and $K_w \bigtriangledown S$ are determined by both their adjacency and Laplacian spectra, where P_{17} and S denote the Paley graph of order 17 and the Schläfli graph, respectively. In this paper, we generalize these results and we prove that multicone graphs $K_w \bigtriangledown mP_{17}$ and $K_w \bigtriangledown mS$ are determined by their adjacency spectra as well as their Laplacian spectra.

Keywords: DS (determined by spectrum) graph; Schläfli graph; multicone graph; adjacency spectrum; Laplacian spectrum; Paley graph of order 17

MSC 2010: 05C50

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [8], [9], [13], [18], [30]. A graph consisting of k disjoint copies of an arbitrary graph G will be denoted by kG. The complement of a graph G is denoted by $\overline{G}$. The join of two graphs G and H is the graph obtained from the disjoint union of G and H by connecting any vertex of G to any vertex of H. The join of two graphs G and H is denoted by $G \bigtriangledown H$. We say that a graph G is an r-regular graph, if the degree of its regularity is r. Given a graph G, the cone over G is the graph formed by adjoining a vertex adjacent to every vertex of G. We say that a graph is a strongly regular graph if it is a connected regular graph

DOI: 10.21136/CMJ.2018.0098-17

with constants λ and μ such that every pair of vertices has λ or μ common neighbours if they are adjacent or non-adjacent, respectively. We use the notation $\mathrm{srg}(n, k, \lambda, \mu)$ to denote such graphs with degree k and n vertices. If G is regular and has precisely three distinct eigenvalues, then it is well-known that G must be strongly regular [24]. Let the matrix $A(G)$ be the $(0,1)$-adjacency matrix of G and d_k be the degree of the vertex v_k. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G, where $D(G)$ is the $n \times n$ diagonal matrix with $V = V(G) = \{d_1, \ldots, d_n\}$ as diagonal entries (and all other entries 0). Since both the matrices $A(G)$ and $L(G)$ are real and symmetric, all their eigenvalues are real numbers. Assume that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$ $(= 0)$ are, respectively, the adjacency eigenvalues and the Laplacian eigenvalues of a graph G. The adjacency spectrum of the graph G consists of the adjacency eigenvalues (together with their multiplicities), and the Laplacian spectrum of the graph G consists of the Laplacian eigenvalues (together with their multiplicities) and we denote them by $\mathrm{Spec}_A(G)$ and $\mathrm{Spec}_L(G)$, respectively. Two graphs G and H are said to be cospectral if they have the same spectrum (i.e., the same characteristic polynomial). If G and H are isomorphic, they are necessarily cospectral. Clearly, if two graphs are cospectral, they must possess the same number of vertices. We say that a graph G is determined by its adjacency (Laplacian) spectra (DS, for short), if for any graph H with $\mathrm{Spec}_A(G) = \mathrm{Spec}_A(H)$ ($\mathrm{Spec}_L(G) = \mathrm{Spec}_L(H)$), G is isomorphic to H. The Schläfli graph, named after Ludwig Schläfli, is a 10-regular undirected graph with 27 vertices and 135 edges. The Paley graph of order 17 is a 8-regular graph which has 17 vertices and 68 edges (see [25], page 262). The Paley graph of order 17 and the Schläfli graph are strongly regular graphs.

So far numerous examples of cospectral but non-isomorphic graphs have been constructed by interesting techniques such as Seidel switching, Godsil-McKay switching, Sunada or Schwenk method. For more information, one may see [25], [26] and the references cited in them. Only a few graphs with very special structures have been reported to be determined by their spectra (DS, for short) (see [1], [2], [3], [5], [10], [11], [12], [14], [15], [16], [17], [19], [21], [25], [26], [27], [28] and the references cited in them). Recently Wei Wang and Cheng-Xian Xu have developed a new method in [24] to show that many graphs are determined by their spectrum and the spectrum of their complement. Van Dam and Haemers [25] conjectured that almost all graphs are determined by their spectra. Nevertheless, the set of graphs that are known to be determined by their spectra is too small. So, discovering classes of graphs that are determined by their spectra can be an interesting problem. The characterization of DS graphs goes back about half a century and it originated in Chemistry [16], [22]. About the background of the question "Which graphs are determined by their spectrum?", we refer to [25]. A spectral characterization of multicone graphs were studied in [27], [29]. In [29], Wang, Zhao and Huang investigated the spectral characterization of multicone graphs and they also claimed that friendship graphs F_n (which are special classes of multicone graphs) are DS with respect to their adjacency spectra. In addition, Wang, Belardo, Huang and Borovićanin [27] proposed such conjecture on the adjacency spectrum of F_n. This conjecture caused some activities on the spectral characterization of F_n. Das [12] claimed to have a proof, but some authors found a mistake [7]. In addition, these authors gave correct proofs in some special cases. Finally, Cioabă et al., [12] proved that if $n \neq 16$, then friendship graphs F_n are DS with respect to their adjacency spectra. Abdian and Mirafzal [5] characterized new classes of multicone graphs which were DS with respect to their spectra. Abdian [1] characterized two classes of multicone graphs and proved that the join of an arbitrary complete graph and the generalized quadrangle graph $GQ(2,1)$ or $GQ(2,2)$ is DS with respect to its adjacency spectra as well as its Laplacian spectra. This author also proposed four conjectures about adjacency spectrum of the complement and signless Laplacian spectrum of these multicone graphs. In [2], the author showed that multicone graphs $K_w \bigtriangledown P_{17}$ and $K_w \bigtriangledown S$ are DS with respect to their adjacency spectra as well as their Laplacian spectra, where P_{17} and S denote the Paley graph of order 17 and the Schläfli graph, respectively. Also, this author conjectured that these multicone graphs are DS with respect to their signless Laplacian spectra. In [3], the author proved that multicone graphs $K_w \bigtriangledown L(P)$ are DS with respect to both their adjacency and Laplacian spectra, where $L(P)$ denotes the line graph of the Petersen graph. He also proposed three conjectures about the signless Laplacian spectrum and the complement spectrum of these multicone graphs. For getting further information about characterizing some multicone graphs which are DS see [4], [6].

We believe that the proofs in [29] contain some gaps. In [29], the authors conjectured that if a graph is cospectral to a friendship graph, then its minimum degree is 2 (see Conjecture 1). In other words, they could not determine the minimum degree of graphs cospectral to a (bidegreed) multicone graph (see Conjecture 1). Hence, by their techniques [29] they cannot characterize new classes of multicone graphs that we want to characterize. Conjectures (Conjectures 1 and 2) which had been proposed by Wang, Zhao and Huang [29] are not true and there is a counterexample for them (see the first paragraph after Corollary 2 of [12]). In Theorem 3 (ii) of [29] first the minimum degree of a graph cospectral to a graph belonging to $\beta(n-1,\delta)$ (classes of bidegreed graphs with degree sequence δ and $n-1$, where n denotes the number of vertices) must be determined, since in general the minimum degree of a graph cannot be determined by its spectrum. Therefore, we think that the theorem without knowing the minimum degree of a graph cospectral with one of graphs $\beta(n-1,\delta)$ will not be effective and useful.

In this paper, we present some techniques which enable us to characterize graphs that are DS with respect to their adjacency and Laplacian spectra.

The plan of this paper is as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that multicone graphs $K_w \bigtriangledown mP_{17}$ are determined by their adjacency spectrum. In Subsection 3.2, we prove that these graphs are DS with respect to their Laplacian spectrum. In Section 4 we characterize additional classes of graphs ($K_w \bigtriangledown mS$) and prove that these multicone graphs are DS with respect to their adjacency and Laplacian spectra. Subsections 4.1 and 4.2 are similar to Subsections 3.1 and 3.2, respectively. In Section 5, we recapitulate our results in this paper and propose four conjectures for further research.

2. Preliminaries

In this section we present some results which will play an important role throughout this paper.

Lemma 2.1 ([1], [2], [3], [5], [21], [25]). *Let G be a graph. For the adjacency matrix and the Laplacian matrix of G, the following can be obtained from the spectrum:*

(i) *The number of vertices.*
(ii) *The number of edges.*

For the adjacency matrix, the following follows from the spectrum:

(iii) *The number of closed walks of any length.*
(iv) *Whether G is regular, and the common degree.*
(v) *Being bipartite or not.*

For the Laplacian matrix, the following follows from the spectrum:

(vi) *The number of spanning trees.*
(vii) *The number of components.*
(viii) *The sum of squares of degrees of vertices.*

The adjacency spectra of graphs P_{17} and S are given below:

(ix) $\mathrm{Spec}_A(P_{17}) = \{[8]^1, [\frac{1}{2}(-1+\sqrt{17})]^8, [\frac{1}{2}(-1-\sqrt{17})]^8\}$ *(see* [25]*).*
(x) $\mathrm{Spec}_A(S) = \{[10]^1, [1]^{20}, [-5]^6\}$ *(see* [25]*).*

Theorem 2.1 ([5], [1], [2], [3], [13], [21], [29]). *If G_1 is r_1-regular with n_1 vertices and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by*

$$P_{G_1 \bigtriangledown G_2(y)} = \frac{P_{G_1}(y)P_{G_2}(y)}{(y-r_1)(y-r_2)}((y-r_1)(y-r_2)-n_1n_2).$$

The spectral radius of a graph Λ is the largest eigenvalue of the adjacency matrix of the graph Λ and is denoted by $\varrho(\Lambda)$. A graph is called bidegreed, if the set of degrees of its vertices consists of two elements.

For further information about the following inequality we refer the reader to [29] (see the first paragraph after Corollary 2.2 and also Theorem 2.1 of [29]). It is stated in [29] that if G is disconnected, then the equality in the following relation can also occur. However, in this paper we only consider *connected case* and we state the equality in this case.

Theorem 2.2 ([1], [2], [3], [5], [21], [29]). *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leqslant \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Theorem 2.3 ([1], [2], [3], [5], [20], [21]). *Let G and H be two graphs with Laplacian spectrum $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_m$, respectively. Then the Laplacian spectra of $\overline{G}$ and $G \bigtriangledown H$ are $n-\lambda_1, n-\lambda_2, \ldots, n-\lambda_{n-1}, 0$ and $n+m, m+\lambda_1, \ldots, m+\lambda_{n-1}, n+\mu_1, \ldots, n+\mu_{m-1}, 0$, respectively.*

Theorem 2.4 ([1], [2], [3], [5], [20], [21]). *Let G be a graph on n vertices. Then n is one of the Laplacian eigenvalues of G if and only if G is the join of two graphs.*

Theorem 2.5 ([18]). *For a graph G, the following statements are equivalent:*

(i) *G is d-regular.*

(ii) *$\varrho(G) = d_G$, the average vertex degree.*

(iii) *G has $v = (1,1,\ldots,1)^{\mathrm{T}}$ as an eigenvector for $\varrho(G)$.*

Proposition 2.1 ([1], [2], [3], [5], [13], [21], [23]). *Let $G - j$ be the graph obtained from G by deleting the vertex j and all edges containing j. Then $P_{G-j}(y) = P_G(y) \sum\limits_{i=1}^{m} \alpha_{ij}^2/(y-\mu_i)$, where m and α_{ij} are the number of distinct eigenvalues and the main angles (see [23]) of the graph G, respectively.*

Proposition 2.2 ([26]). *Let G be a disconnected graph that is determined by its Laplacian spectrum. Then the cone over G, that is, the graph H obtained from G by adding one vertex that is adjacent to all vertices of G, is also determined by its Laplacian spectrum.*

Remark 2.1. For further information about the *adjacency spectrum of graphs S and P_{17}*, one can see [25]. For finding why graphs mP_{17} and mS are DS with respect to their adjacency spectra as well as their Laplacian spectra one can see Propositions 3 and 10 of [25].

3. Main results

The main goal of this section is to prove that any connected graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$ is DS with respect to its adjacency spectrum as well as its Laplacian spectrum.

3.1. Connected graphs cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$ with respect to adjacency and Laplacian spectra.

Proposition 3.1. *Let G be a graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$. Then*

$$\mathrm{Spec}_A(G) = \left\{ [-1]^{w-1}, [8]^{m-1}, \left[\frac{-1 \pm \sqrt{17}}{2}\right]^{8m}, \left[\frac{\theta \pm \sqrt{\theta^2 - 4\Gamma}}{2}\right]^{1} \right\},$$

where $\theta = w + 7$ and $\Gamma = 8(w-1) - 17mw$.

Proof. By Lemma 2.1 and Theorem 2.1 the proof is straightforward. □

Lemma 3.1. *Let G be a connected graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$. Then $\delta(G) = w + 8$.*

Proof. Suppose that $\delta(G) = w + 8 + x$, where x is an integer number. First, it is clear that in this case the equality in Theorem 2.2 occurs if and only if $x = 0$. We show that $x = 0$. By contrary, we suppose that $x \neq 0$. It follows from Theorem 2.2 and Proposition 3.1 that

$$\begin{aligned} \varrho(G) &= \frac{w + 7 + \sqrt{8k - 4l(w+8) + (w+9)^2}}{2} \\ &< \frac{w + 7 + x + \sqrt{8k - 4l(w+8) + (w+9)^2 + x^2 + (2w + 18 - 4l)x}}{2}, \end{aligned}$$

where the integer numbers k and l denote the number of edges and the number of vertices of the graph G, respectively. For convenience, we let $B = 8k - 4l(w+8) + (w+9)^2 \geqslant 0$ and $C = w + 9 - 2l$, and also let $g(x) = x^2 + (2w + 18 - 4l)x = x^2 + 2Cx$.

Then clearly

$$\sqrt{B} - \sqrt{B + g(x)} < x.$$

We consider two cases:

Case 1: $x < 0$.

It is easy and straightforward to see that $|\sqrt{B} - \sqrt{B + g(x)}| > |x|$, since $x < 0$. Transposing and squaring yields

$$2B + g(x) - 2\sqrt{B(B + g(x))} > x^2.$$

Replacing $g(x)$ by $x^2 + 2Cx$, we get

$$B + Cx > \sqrt{B(B + x^2 + 2Cx)}.$$

Obviously $Cx > 0$, since $x < 0$ and $C = w + 9 - 2l = w + 9 - 2(17m + w) = -34m + 9 - w < 0$. Squaring again and simplifying yields

$$C^2 > B.$$

Therefore,

$$k < \frac{l(l-1)}{2}.$$

So, if $x < 0$, then G cannot be a complete graph. In other words, if G is a complete graph, then $x > 0$. Or one can say that if G is a complete graph, then:

$$\delta(G) > w + 8. \tag{3.1}$$

Case 2: $x > 0$.

In the same way as in Case 1, we can conclude that if G is a complete graph, then:

$$\delta(G) < w + 8. \tag{3.2}$$

But, Cases (3.1) and (3.2) cannot occur together. Hence we must have $x = 0$.

Therefore, the assertion holds. □

In the next lemma, we show that any connected graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$ must be bidegreed.

Lemma 3.2. *Let G be a connected graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$. Then G is bidegreed and any vertex of G is either of degree $w + 8$ or $w - 1 + 17m$.*

Proof. By Theorem 2.5 G cannot be regular. Now, by Lemma 3.1 and Theorem 2.2 the proof is completed. □

In the next theorem, we prove that any connected graph cospectral with a multicone graph $K_1 \bigtriangledown mP_{17}$, the cone of graphs mP_{17}, is DS with respect to its adjacency spectrum.

Lemma 3.3. *Any connected graph cospectral with a multicone graph $K_1 \bigtriangledown mP_{17}$ is DS with respect to its adjacency spectra.*

Proof. Let G be cospectral with the multicone graph $K_1 \bigtriangledown mP_{17}$. It follows from Lemma 3.2 that G is bidegreed and each of its vertices is of degree $17m$ or 9. We suppose that G has t vertex (vertices) of degree $17m$. Therefore, by Lemma 2.1 (ii) and due to the spectrum of the graph G, we deduce that $t(17m) + (17m + 1 - t)9 = 170m$. So, $t = 1$. This means that G has one vertex of degree $17m$, say j. By Proposition 2.1 $P_{G-j}(\lambda) = (\lambda - \mu_3)^{8m-1}(\lambda - \mu_4)^{8m-1}(\lambda - \mu_5)^{m-2}[\alpha_{1j}^2 Y_1 + \alpha_{2j}^2 Y_2 + \alpha_{3j}^2 Y_3 + \alpha_{4j}^2 Y_4 + \alpha_{5j}^2 Y_5]$, where $\mu_1 = \frac{1}{2}(8 + \sqrt{64 + 68m})$, $\mu_2 = \frac{1}{2}(8 - \sqrt{64 + 68m})$, $\mu_3 = \frac{1}{2}(-1 + \sqrt{17})$, $\mu_4 = \frac{1}{2}(-1 - \sqrt{17})$ and $\mu_5 = 8$,

$$
\begin{aligned}
Y_1 &= (\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5),\\
Y_2 &= (\lambda - \mu_1)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5),\\
Y_3 &= (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_4)(\lambda - \mu_5),\\
Y_4 &= (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_5),\\
Y_5 &= (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4).
\end{aligned}
$$

We know that $G - j$ has $17m$ eigenvalues. In other words, $P_{G-j}(\lambda)$ has $17m$ roots. Also, by removing the vertex j from graph G, the number of edges and triangles that are removed from graph G are $17m$ (the number of vertices of graph $G - j$) and $68m$ (the number of edges of graph $G - j$), respectively. Moreover, it follows from Lemma 3.2 that $G - j$ is regular and the degree of its regularity is 8. By Lemma 2.1 (iii) for the closed walks of lengths 1, 2 and 3, we have:

$$
\begin{aligned}
\alpha + \beta + \theta + 8 &= -((8m - 1)\mu_3 + (8m - 1)\mu_4 + (m - 2)\mu_5),\\
\alpha^2 + \beta^2 + \theta^2 + 64 &= 136m - ((8m - 1)\mu_3^2 + (8m - 1)\mu_4^2 + (m - 2)\mu_5^2),\\
\alpha^3 + \beta^3 + \theta^3 + 512 &= 408m - ((8m - 1)\mu_3^3 + (8m - 1)\mu_4^3 + (m - 2)\mu_5^3),
\end{aligned}
$$

where α, β and θ are the eigenvalues of graph $G - j$. The roots are $\alpha = \frac{1}{2}(-1 + \sqrt{17})$, $\beta = \frac{1}{2}(-1 - \sqrt{17})$ and $\theta = 8$. Hence $\mathrm{Spec}_A(G - j) = \mathrm{Spec}_A(mP_{17})$ and so $G - j \cong mP_{17}$. Therefore, the result follows. □

Up to now, we show that each connected graph cospectral with a multicone graph $K_1 \bigtriangledown mP_{17}$, the cone of graphs mP_{17}, is DS with respect to its adjacency spectrum.

The natural question is; what happens for multicone graphs $K_w \bigtriangledown mP_{17}$? The next theorem answers this question.

Theorem 3.1. *Any connected graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$ is DS with respect to its adjacency spectrum.*

Proof. We solve the problem by induction on w. If $w = 1$, by Lemma 3.3 the proof is clear. Let the claim be true for w; that is, if $\mathrm{Spec}_A(G_1) = \mathrm{Spec}_A(K_w \bigtriangledown mP_{17})$, then $G_1 \cong K_w \bigtriangledown mP_{17}$, where G_1 is an arbitrary graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$. We show that the claim is true for $w+1$; that is, if $\mathrm{Spec}_A(G) = \mathrm{Spec}_A(K_{w+1} \bigtriangledown mP_{17})$, then $G \cong K_{w+1} \bigtriangledown mP_{17}$, where G is an arbitrary graph cospectral with a multicone graph $K_{w+1} \bigtriangledown mP_{17}$. It follows from Lemma 3.2 that G_1 has w vertices of degree $17m + w - 1$ and $17m$ vertices of degree $w + 8$. Also, this lemma implies that G has $w+1$ vertices of degree $17m+w$ and $17m$ vertices of degree $w + 9$. On the other hand, G has one vertex and $w + 17m$ edges more than G_1. So, we must have $G \cong K_1 \bigtriangledown G_1$. Now, the inductive hypothesis yields the result. $\square$

3.2. Connected graphs cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$ with respect to Laplacian spectrum.

In this subsection, we show that multicone graphs $K_w \bigtriangledown mP_{17}$ are DS with respect to their Laplacian spectrum.

Theorem 3.2. *Multicone graphs $K_w \bigtriangledown mP_{17}$ are DS with respect to their Laplacian spectrum.*

Proof. We perform mathematical induction on w. If $w = 1$, by Proposition 2.2 the proof is clear. Let the claim be true for w; that is, if

$$\begin{aligned}\mathrm{Spec}_L(G_1) &= \mathrm{Spec}_L(K_w \bigtriangledown mP_{17})\\ &= \left\{[0]^1, [17m+w]^w, [w]^{m-1}, \left[\frac{\sqrt{17}+17}{2}+w\right]^{8m}, \left[\frac{-\sqrt{17}+17}{2}+w\right]^{8m}\right\},\end{aligned}$$

then $G_1 \cong K_w \bigtriangledown mP_{17}$, where G_1 is an arbitrary graph cospectral with a multicone graph $K_w \bigtriangledown mP_{17}$. We show that the theorem is true for $w + 1$; that is, we show that it follows from

$$\begin{aligned}&\mathrm{Spec}_L(G)\\ &= \left\{[0]^1, [17m+w+1]^{w+1}, [w+1]^{m-1}, \left[\frac{\sqrt{17}+19}{2}+w\right]^{8m}, \left[\frac{-\sqrt{17}+19}{2}+w\right]^{8m}\right\}\end{aligned}$$

that $G \cong K_{w+1} \bigtriangledown mP_{17}$, where G is a graph. By Theorem 2.4 G_1 and G are the joins of two graphs. In addition, it follows from Theorem 2.3 that $\mathrm{Spec}_L(G) =$

$\mathrm{Spec}_L(K_1 \bigtriangledown G_1)$. On the other hand, G has one vertex and $w + 17m$ edges more than G_1. Therefore, we must have $G \cong K_1 \bigtriangledown G_1$. Now, the induction hypothesis yields the assertion. □

From now on, we characterize other new classes of multicone graphs that are DS with respect to their adjacency and Laplacian spectra.

4. Connected graphs cospectral with a multicone graph $K_w \bigtriangledown mS$ with respect to adjacency and Laplacian spectra

In this section we prove that any connected graph cospectral with a multicone graph $K_w \bigtriangledown mS$ is DS with respect to its adjacency spectrum as well as its Laplacian spectrum.

Proposition 4.1. *Let G be a graph cospectral with a multicone graph $K_w \bigtriangledown mS$. Then*

$$\mathrm{Spec}_A(G) = \left\{[-1]^{w-1}, [10]^{m-1}, [1]^{20m}, [-5]^{6m}, \left[\frac{\Lambda + \sqrt{\Lambda^2 - 4\Gamma}}{2}\right]^1, \left[\frac{\Lambda - \sqrt{\Lambda^2 - 4\Gamma}}{2}\right]^1\right\},$$

where $\Lambda = 9 + w$ and $\Gamma = 10(w-1) - 27mw$.

P r o o f. By Theorem 2.1 and Lemma 2.1 the proof is completed. □

Similarly to Lemma 3.2 we have the following lemma.

Lemma 4.1. *Let G be a connected graph cospectral with a multicone graph $K_w \bigtriangledown mS$. Then G is bidegreed and any vertex of G is either of degree $w + 10$ or $w - 1 + 27m$.*

4.1. Connected graphs cospectral with the multicone graph $K_1 \bigtriangledown mS$ with respect to adjacency spectra.

In this subsection, we show that any connected graph cospectral with a multicone graph $K_1 \bigtriangledown mS$ is DS with respect to its adjacency spectrum.

Lemma 4.2. *Any conncted graph cospectral with a multicone graph $K_1 \bigtriangledown mS$ is DS with respect to its adjacency spectrum.*

P r o o f. Let G be cospectral with multicone graph $K_1 \bigtriangledown mS$. By Lemma 4.1, it is easy to see that G has one vertex of degree 27, say l. On the other hand, it follows from Proposition 2.1 that $P_{G-l}(\lambda) = (\lambda - \mu_3)^{20m-1}(\lambda - \mu_4)^{6m-1}(\lambda - \mu_5)^{m-2}[\alpha_{1j}^2 D_1 + \alpha_{2j}^2 D_2 + \alpha_{3j}^2 D_3 + \alpha_{4j}^2 D_4 + \alpha_{5j}^2 D_5]$, where $\mu_1 = 5 + \frac{1}{2}\sqrt{100 + 108m}$, $\mu_2 = 5 - \frac{1}{2}\sqrt{100 + 108m}$, $\mu_3 = 1$, $\mu_4 = -5$ and $\mu_5 = 10$,

$$\begin{aligned}
D_1 &= (\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5),\\
D_2 &= (\lambda - \mu_1)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5),\\
D_3 &= (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_4)(\lambda - \mu_5),\\
D_4 &= (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_5),\\
D_5 &= (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4).
\end{aligned}$$

Now, by computing the closed walks of lengths 1, 2 and 3 belonging to $G - j$ we have

$$\begin{aligned}
\chi + \xi + \kappa + 10 &= -((20m-1)\mu_3 + (6m-1)\mu_4 + (m-2)\mu_5)),\\
\chi^2 + \xi^2 + \kappa^2 + 100 &= 270m - (20m-1)\mu_3^2 + (6m-1)\mu_4^2 + (m-2)\mu_5^2),\\
\chi^3 + \xi^3 + \kappa^3 + 1000 &= 270m - (20m-1)\mu_3^3 + (6m-1)\mu_4^3 + (m-2)\mu_5^3),
\end{aligned}$$

where χ, ξ and κ are the eigenvalues of the graph $G - j$. By solving the above equations we obtain $\chi = 1$, $\xi = -5$ and $\kappa = 10$. Hence $\mathrm{Spec}_A(G - j) = \mathrm{Spec}_A(mS)$ and so $G - j \cong mS$. This completes the proof. □

Theorem 4.1. *Any connected graph cospectral with a multicone graph $K_w \bigtriangledown mS$ is DS with respect to its adjacency spectrum.*

P r o o f. We will proceed by induction on w. For $w = 1$, the result follows from Lemma 4.2. Let the claim be true for w; that is, if $\mathrm{Spec}_A(G_1) = \mathrm{Spec}_A(K_w \bigtriangledown mS)$, then $G_1 \cong K_w \bigtriangledown mS$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, if $\mathrm{Spec}_A(G) = \mathrm{Spec}_A(K_{w+1} \bigtriangledown mS)$, then $G \cong K_{w+1} \bigtriangledown mS$, where G is a graph. It follows from Lemma 4.1 that G_1 has w vertices of degree $27m + w - 1$ and $27m$ vertices of degree $10 + w$. Also, this lemma implies that G has $w + 1$ vertices of degree $27m + w$ and $27m$ vertices of degree $11 + w$. On the other hand, G has one vertex and $w + 27m$ more than G_1. Hence we must have $G \cong K_1 \bigtriangledown G_1$. Now, the inductive hypothesis implies the assertion. □

4.2. Graphs cospectral with a multicone graph $K_w \bigtriangledown mS$ with respect to Laplacian spectrum.

In this subsection, we show that multicone graphs $K_w \bigtriangledown mS$ are DS with respect to their Laplacian spectra.

Theorem 4.2. *Multicone graphs $K_w \bigtriangledown mS$ are DS with respect to their Laplacian spectra.*

P r o o f. We solve the problem by induction on w. For $w = 1$, the result follows from Proposition 2.2. Let the claim be true for w; that is, if

$$\mathrm{Spec}_L(G_1) = \mathrm{Spec}_L(K_w \triangledown mS) = \{[w+27m]^w, [w]^{m-1}, [w+9]^{20m}, [w+15]^{6m}, [0]^1\},$$

then $G_1 \cong K_w \triangledown mS$, where G_1 is an arbitrary graph cospectral with a multicone graph $K_w \triangledown mS$. We show that the theorem is true for $w + 1$; that is, we show that

$$\mathrm{Spec}_L(G) = \{[w+1+27m]^{w+1}, [w+1]^{m-1}, [w+10]^{20m}, [w+16]^{6m}, [0]^1\}$$

implies that $G \cong K_{w+1} \triangledown mS$, where G is a graph. By Theorem 2.4 G_1 and G are the joins of two graphs. In addition, it follows from Theorem 2.3 that $\mathrm{Spec}_L(G) = \mathrm{Spec}_L(K_1 \triangledown G_1)$. On the other hand, G has one vertex and $w + 27m$ edges more than G_1. So, we must have $G \cong K_1 \triangledown G_1$. Now, the inductive hypothesis completes the proof. □

5. Concluding remarks and four problems

In this study, we proved that any connected graph cospectral with a multicone graph $K_w \triangledown mP_{17}$ or $K_w \triangledown mS$ is DS with respect to its adjacency and Laplacian spectra. Now, we pose the following conjectures.

Conjecture 1. *Graphs* $\overline{K_w \triangledown mP_{17}}$ *are DS with respect to their adjacency spectrum.*

Conjecture 2. *Multicone graphs* $K_w \triangledown mP_{17}$ *are DS with respect to their signless Laplacian spectrum.*

Conjecture 3. *Graphs* $\overline{K_w \triangledown mS}$ *are DS with respect to their adjacency spectrum.*

Conjecture 4. *Multicone graphs* $K_w \triangledown mS$ *are DS with respect to their signless Laplacian spectrum.*

Acknowledgment. The first author would like to thank Prof. W. H. Haemers for giving some useful suggestions. Also, the valuable suggestions of the anonymous referee of the Czech. Math. J. which greatly improved the presentation of the paper, are highly appreciated.

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