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# ON THE GYŐRY-SÁRKÖZY-STEWART CONJECTURE IN FUNCTION FIELDS

Igor E. Shparlinski, Sydney

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*Abstract.* We consider function field analogues of the conjecture of Győry, Sárközy and Stewart (1996) on the greatest prime divisor of the product $(ab+1)(ac+1)(bc+1)$ for distinct positive integers $a$, $b$ and $c$. In particular, we show that, under some natural conditions on rational functions $F, G, H \in \mathbb{C}(X)$, the number of distinct zeros and poles of the shifted products $FH+1$ and $GH+1$ grows linearly with $\deg H$ if $\deg H \geqslant \max\{\deg F, \deg G\}$. We also obtain a version of this result for rational functions over a finite field.

*Keywords*: shifted polynomial product; number of zeros

*MSC 2010*: 11R09, 11S05, 12E05

## 1. Motivation and main result

Let $P(k)$ denote the largest prime divisor of an integer $k \geqslant 2$. The conjecture of Győry, Sárközy and Stewart (see [12]) asserts that for integers $a > b > c \geqslant 1$ we have $P((ab+1)(ac+1)(bc+1)) \to \infty$ as $a \to \infty$. This conjecture has been established by Corvaja and Zannier in [7] and then refined and extended in various directions in [6], [14].

Here we apply some results of Corvaja and Zannier (see [8], [10]) to establish explicit and essentially optimal function field analogues of this conjecture for only two shifted pairwise products $FH+1$ and $GH+1$ for $F$, $G$ and $H$ from a wide class of rational functions over an arbitrary field, which, for example, includes all nontrivial polynomial functions. This result applies when the degree of $H$ is larger than the degrees of $F$ and $G$. Furthermore, we also use some results and ideas of Ostafe (see [17]) to study this problem without any restriction on the degrees of $F$, $G$

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and $H$. Finally, in a special case when $F$, $G$ and $H$ are polynomials and the degree of $H$ is much larger than the other two, we use a result of Bernstein (see [2]) to derive a stronger bound.

Namely, let $K$ be a field. We use $\operatorname{char} K$ to denote the characteristic of $K$.

For a rational function $Q \in K(X)$ we denote by $\mathcal{Z}(Q)$ the set of distinct zeros and poles of $Q$ in the algebraic closure of $K$ and also define

$$Z(Q) = \#\mathcal{Z}(Q)$$

as the number of zeros and poles of $Q$.

We write $Q = f/g$ for relatively prime polynomials $f, g \in K[X]$ (and always assume that $f$ is monic) and then define

$$\operatorname{num}(Q) = f \quad \text{and} \quad \operatorname{den}(Q) = g.$$

We say that two rational functions $Q, R \in K(X)$ are *disjoint* if the set of zeros and poles of each of them is disjoint from the set of zeros and poles of the other.

Furthermore, we say that a rational function $Q \in K(X)$ is *quasipolynomial* if $\deg \operatorname{num}(Q) > \deg \operatorname{den}(Q)$. In particular, for a quasipolynomial rational function,

$$\deg Q = \max\{\deg \operatorname{num}(Q), \deg \operatorname{den}(Q)\} = \deg \operatorname{num}(Q).$$

We define $\mathfrak{Q}_K$ as the set of quasipolynomial functions $Q \in K(X)$ (which is obviously closed under multiplication). For example, we obviously have $(K[X]\backslash K) \subseteq \mathfrak{Q}_K$.

Furthermore, for an integer $d \geqslant 1$ we define $N_d(K)$ as the smallest value of $Z(FH+1)+Z(GH+1)$ over all triples of pairwise distinct and disjoint nonconstant quasipolynomial functions $F, G, H \in K(X)$, when

$$d = \deg H \geqslant \deg F, \quad \deg G \geqslant 1, \tag{1}$$

that is

$$\begin{aligned} N_d(K) = \min\{Z(FH+1) + Z(GH+1)\colon & \\ & F, G, H \in \mathfrak{Q}_K \setminus K \text{ pairwise disjoint, satisfying (1)}\}. \end{aligned}$$

We note that such products of integers have been studied in [7].

We remark that the example of the functions

$$F(X) = -X^d + 1, \quad G(X) = -X^d - 1, \quad H(X) = 1/X^d, \tag{2}$$

for which

$$FG+1 = X^{2d}, \quad FH+1 = 1/X^d, \quad GH+1 = -1/X^d,$$

shows that at least some conditions on the degrees of the numerators and denominators of $F$, $G$ and $H$ are necessary. In particular, the set $\mathfrak{Q}_K$ seems to be a natural family of functions to consider.

## 2. Main results

First, using results of Corvaja and Zannier (see [8], [10]), we establish the following general estimate.

**Theorem 1.** *There is an absolute constant $c_0 > 0$ such that for any algebraically closed field $K$ we have $N_d(K) \geqslant c_0 d$ for any $d$ if $\operatorname{char} K = 0$ and for $d \leqslant c_0 p$ if $p = \operatorname{char} K > 0$.*

We also consider an analogue of $N_d(K)$, where we relax condition (1) as

$$d = \deg H > \deg F - \deg G \geqslant 0 \tag{3}$$

if all functions $F, G, H \in K[X]$ are polynomials, and drop it completely if at least one of these functions has a finite pole, but instead we have to fix the set of possible zeros and poles.

We consider this only in the case of the field $K = \mathbb{C}$ of complex numbers.

**Theorem 2.** *For any fixed finite set $\mathcal{X} \subseteq \mathbb{C}$ there is a constant $D(\mathcal{X})$ such that for any pairwise disjoint $F, G, H \in \mathfrak{Q}_K \setminus K$, which in the case where $F, G, H \in K[X]$ also satisfy* (3), *if*

$$\mathcal{Z}(FH+1), \mathcal{Z}(GH+1) \subseteq \mathcal{X},$$

*then $\deg H \leqslant D(\mathcal{X})$.*

The example of the set $\mathcal{X} = \{0\}$ and polynomials

$$F(X) = X^{2d} + X^d + 1, \quad G(X) = X^d + 1, \quad H(X) = X^d - 1, \tag{4}$$

which are disjoint, but for any $d$ we have

$$\mathcal{Z}(FH+1) = \mathcal{Z}(GH+1) = \mathcal{X},$$

shows that some conditions of the type (3) are necessary in order to have a variant of Theorem 2 for polynomials $F, G, H \in K[X]$.

Finally, we use a result of Bernstein from [2] to obtain an explicit lower bound on $Z((FH+1)(GH+1))$ for polynomials $F, G, H \in K[X]$ over an arbitrary field $K$ of characteristic zero in the case where $\deg H$ is substantially larger than $\deg F$ and $\deg G$. This bound is of the same flavour as that of Stewart and Tijdeman in [19], Theorem 1.

**Theorem 3.** *Let* $\operatorname{char} K = 0$. *Then for arbitrary pairwise distinct polynomials* $F, G, H \in K[X]$ *we have*

$$Z((FH+1)(GH+1)) > \deg H - \max\{\deg F, \deg G\}.$$

Note that $Z(FH+1) + Z(GH+1) \geqslant Z((FH+1)(GH+1))$.

## 3. Preliminary results

As usual, we use $\mathbb{G}_m^k$ to denote the $k$-dimensional multiplicative torus, that is, $\mathbb{G}_m^k = (\mathbb{C}^*)^k$. We need a result on the finiteness of the number of points on the intersection of a curve with algebraic subgroups of $\mathbb{G}_m^k$ of codimension at least 2, which is due to Bombieri, Masser and Zannier in [5] for curves over $\mathbb{C}$, which also extends the previous result of Maurin in [16] that applies only to curves over $\overline{\mathbb{Q}}$ (that is, over the algebraic closure of $\mathbb{Q}$). We also refer to [1], [3], [4], [13] for several related results and further references. Although we apply it only to straight lines, we present it in full generality.

**Lemma 4.** *Let* $\mathcal{C} \subseteq \mathbb{C}^k$, $k \geqslant 2$, *be an irreducible curve over* $\mathbb{C}$. *Assume that for every nonzero vector* $(r_1, \ldots, r_k) \in \mathbb{Z}^k$ *the monomial* $X_1^{r_1} \ldots X_k^{r_k}$ *is not identically* 1 *on* $\mathcal{C}$. *Then there are only finitely many points* $(x_1, \ldots, x_k) \in \mathcal{C}$ *for which there exist linearly independent vectors* $(a_1, \ldots, a_k), (b_1, \ldots, b_k)$ *in* $\mathbb{Z}^k$ *such that*

$$x_1^{a_1} \ldots x_k^{a_k} = x_1^{b_1} \ldots x_k^{b_k} = 1.$$

We also use a very special case of the bound of Ostafe, see [17], Lemma 2.9, on the multiplicity of zeros in shifts of rational functions, which in turn is based on the polynomial ABC theorem of Stothers in [20], see also [15], [18].

**Lemma 5.** *For any rational function* $Q(X) \in \mathbb{C}(X)$, *the largest multiplicity of the zeros of* $Q(X) - 1$ *is at most* $Z(Q)$.

We say that two rational functions $U, V \in K(X)$ are *multiplicatively independent in* $K(X)/K^*$ if the function $U^k V^m$ with $k, m \in \mathbb{Z}$ is a constant only for $k = m = 0$.

We first recall the following result of Corvaja and Zannier in [8], Corollary 2.3, see also [9], Theorem CZ, or [21], Theorem 2.2. Furthermore, Zannier in [21], Section 2.4, also gives several applications of this result.

**Lemma 6.** *Let* $\operatorname{char} K = 0$. *Then for any two rational functions* $U, V \in K(X)$ *that are multiplicatively independent in* $K(X)/K^*$, *we have*

$$\deg \gcd (\operatorname{num}(U-1), \operatorname{num}(V-1)) \leqslant C \sqrt[3]{\deg U \deg V (Z(U) + Z(V))},$$

*where* $C$ *is some absolute constant.*

We also use the following simplified version of [10], Theorem 2:

**Lemma 7.** *Let* $\operatorname{char} K = p$. *Then for any two rational functions* $U, V \in K(X)$ *that are multiplicatively independent in* $K(X)/K^*$, *we have*

$$\begin{aligned} &\deg \gcd \, (\operatorname{num}(U-1), \operatorname{num}(V-1)) \\ &\qquad \leqslant C \max\Big\{ \sqrt[3]{\deg U \deg V (Z(U) + Z(V))}, \frac{\deg U \deg V}{p} \Big\}, \end{aligned}$$

*where* $C$ *is some absolute constant.*

In order to apply Lemmas 6 and 7 we need to show that the rational functions $FH + 1$ and $GH + 1$ are multiplicatively independent in $K(X)/K^*$.

**Lemma 8.** *If* $F, G, H \in \mathfrak{Q}_K \setminus K$ *are pairwise disjoint and in the case where* $F, G, H \in K[X]$ *they also satisfy* (3), *then the rational functions* $FH+1$ *and* $GH+1$ *are multiplicatively independent in* $K(X)/K^*$.

Proof. Assume that $FH+1$ and $GH+1$ are multiplicatively dependent. Then for some $\alpha, \beta \in K^*$ and a rational function $Q \in K(X)$ we have

$$FH + 1 = \alpha Q^{\lambda} \quad \text{and} \quad GH + 1 = \beta Q^{\mu}.$$

First we note that $\lambda \neq \mu$. Indeed, otherwise we have $\alpha \neq \beta$ and the relation

$$(\beta F - \alpha G) H = \alpha - \beta \in K^*$$

contradicts the disjointness of $F$, $G$ and $H$ and the fact that $H \in \mathfrak{Q}_K$ and thus has a zero.

Clearly $FH+1, GH+1 \in \mathfrak{Q}_K$. Hence, we can assume that $Q \in \mathfrak{Q}_K$ and also $\lambda > \mu > 0$.

Since $F$ and $H$ are disjoint, we see from $FH+1 = \alpha Q^\lambda$ that the poles of $H$ are also poles of $Q^\lambda$ and appear with the same multiplicities as in $H$. However the same argument also applies to $GH+1 = \beta Q^\mu$. Since $\lambda \neq \mu$, this means that $H$ is a polynomial and furthermore, $H$ and $Q$ have no common zeros. Since $F$, $G$ and $H$ are disjoint, we see that $Q$ is a polynomial and so $F$ and $G$ are polynomials as well.

Hence, we can now assume that $F, G, H \in K[X]$ and thus satisfy (3).

Now, from $(F-G)H = Q^\mu \left(\alpha Q^{\lambda-\mu} - \beta\right)$ and since $H$ and $Q$ have no common zeros, we conclude that $Q^\mu$ divides $F-G$, which is impossible as

$$\deg Q^\mu = \deg G + \deg H > \deg(F-G)$$

due to assumption (3). $\square$

We remark that example (4) shows that some version of condition (3) is needed in the polynomial case of Lemma 8.

Finally, we recall an extension of the polynomial ABC theorem due to Bernstein in [2], Theorem 2.1:

**Lemma 9.** *Let* $\operatorname{char} K = 0$. *Then for arbitrary nonconstant polynomial* $H \in K[X]$ *pairwise distinct nonzero polynomials* $P, Q, R \in K[X]$ *without a common zero and with* $P \equiv Q \equiv R \pmod{H}$, *we have*

$$\max\{\deg P, \deg Q, \deg R\} > 2 \deg H - Z(PQR).$$

## 4. Proof of Theorem 1

Let $F, G, H \in \mathbb{C}(X)$ be nonconstant pairwise disjoint quasipolynomials with (1) and with

$$Z(FH+1) + Z(GH+1) = N_d.$$

By Lemma 8, since obviously (1) implies (3), we see that $FH+1$ and $GH+1$ are multiplicatively independent in $K(X)/K^*$.

If $\operatorname{char} K = p > 0$, then we can apply Lemma 7 with the rational functions $U = FH+1$ and $V = GH+1$. Hence, noticing that

$$\deg U \leqslant 2 \deg H, \quad \deg V \leqslant 2 \deg H$$

and

$$Z(U) + Z(V) \leqslant N_d(K)$$

and recalling that $H \in \mathfrak{Q}_K$, we derive:

$$\begin{aligned} d &= \deg \operatorname{num}(H) \\ &\leqslant C \max\Big\{ \sqrt[3]{\deg U \deg V (Z(U) + Z(V))}, \frac{\deg U \deg V}{p} \Big\} \\ &\leqslant C \max\Big\{ \sqrt[3]{4d^2 N_d(K)}, \frac{4d^2}{p} \Big\}. \end{aligned}$$

Now we see that if

$$\sqrt[3]{4d^2 N_d(K)} \leqslant \frac{4d^2}{p}, \tag{5}$$

then $d \leqslant 4Cd^2p^{-1}$. Hence, taking $c_0 < \frac{1}{4}C^{-1}$, we conclude that (5) is impossible.

Otherwise, that is if (5) fails, we have

$$C\sqrt[3]{4d^2 N_d(K)} \geqslant d.$$

Hence, taking $c_0 < \frac{1}{4}C^{-3}$, we obtain the result in the case of positive characteristic.

If $\operatorname{char} K = 0$, then we can apply Lemma 6 and follow the same argument except that we do not have to consider (5). □

## 5. Proof of Theorem 2

Let $F, G, H \in \mathbb{C}(X)$ be rational functions satisfying (3) and with $\mathcal{Z}(FH+1)$, $\mathcal{Z}(GH+1) \subseteq \mathcal{X}$. Denote $M = \#\mathcal{X}$.

Our goal is to estimate $d = \deg H$ in terms of $\mathcal{X}$. We do this in two steps: we first estimate the number $Z$ of distinct zeros of $H$ and then we estimate the largest multiplicity $\mu$ of zeros of $H$.

Write

$$\begin{aligned} FH + 1 &= \alpha \prod_{a \in \mathcal{A}} (X-a)^{m_a} \prod_{c \in \mathcal{C}} (X-c)^{r_c}, \\ GH + 1 &= \beta \prod_{b \in \mathcal{B}} (X-b)^{n_b} \prod_{c \in \mathcal{C}} (X-c)^{s_c}, \end{aligned} \tag{6}$$

where $\alpha, \beta \in \mathbb{C}^*$ and $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{X}$ are pairwise disjoint sets and $m_a$, $n_b$, $r_c$ and $s_c$ are nonzero exponents.

Let $A$, $B$ and $C$ be the cardinalities of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. In particular $M \geqslant A+B+C$. Consider the parametric line

$$\mathcal{L}\colon\ (\{\gamma_a(t-a)\}_{a\in\mathcal{A}}, \{\gamma_b(t-b)\}_{b\in\mathcal{B}}, \{(t-c)\}_{c\in\mathcal{C}}\colon\ t\in\mathbb{C}) \in \mathbb{C}^M$$

with arbitrarily chosen coefficients $\{\gamma_a\}_{a\in\mathcal{A}}$ and $\{\gamma_b\}_{b\in\mathcal{B}}$, satisfying

$$\prod_{a\in\mathcal{A}} \gamma_a^{m_a} = \alpha \quad \text{and} \quad \prod_{b\in\mathcal{B}} \gamma_b^{n_b} = \beta.$$

We now consider the vectors

$$\vec{m} = (m_a)_{a\in\mathcal{A}}, \quad \vec{n} = (n_b)_{b\in\mathcal{B}}, \quad \vec{r} = (r_c)_{c\in\mathcal{C}}, \quad \vec{s} = (s_c)_{c\in\mathcal{C}}$$

and also the vectors

$$(\vec{m}, \vec{0}, \vec{r}), (\vec{0}, \vec{n}, \vec{s}) \in \mathbb{Z}^A \times \mathbb{Z}^B \times \mathbb{Z}^C. \tag{7}$$

We note that if one of the integers $A$, $B$ or $C$ vanishes, then the corresponding parts of these vectors are not present (for example, if $A=0$ but $B, C \neq 0$, vectors (7) take form $(\vec{0}, \vec{r})$, $(\vec{n}, \vec{s})$).

We remark that vectors (7) are linearly independent as otherwise the rational functions $FH+1$ and $GH+1$ are multiplicatively dependent in $\mathbb{C}(X)/\mathbb{C}^*$, while this is impossible by Lemma 8.

Since vectors (7) are linearly independent, by Lemma 4 applied to the line $\mathcal{L}$, we see that there are at most $C_0(\mathcal{X})$ values of $t\in\mathbb{C}$ such that

$$\prod_{a\in\mathcal{A}} (\gamma_a(t-a))^{m_a} \prod_{c\in\mathcal{C}} (t-c)^{r_c} = \prod_{b\in\mathcal{B}} (\gamma_b(t-b))^{n_b} \prod_{c\in\mathcal{C}} (t-c)^{s_c} = 1$$

for some vectors $\vec{n}$, $\vec{m}$, $\vec{r}$, $\vec{s}$ as above, where $C_0(\mathcal{X})$ depends only on $\mathcal{X}$. In particular, we now see from (6) that the number $Z_0$ of distinct zeros of $H$ is at most

$$Z_0 \leqslant C_0(\mathcal{X}). \tag{8}$$

Furthermore, by Lemma 5, the largest multiplicity of the zeros of the rational function

$$\prod_{a\in\mathcal{A}} (\gamma_a(X-a))^{m_a} \prod_{c\in\mathcal{C}} (X-c)^{r_c} - 1 \in \mathbb{C}(X)$$

is at most $A+C \leqslant M$. In particular, we now see from (6) that the largest multiplicity of zeros of $H$ is at most

$$\mu \leqslant M. \tag{9}$$

Combining (8) and (9), we obtain

$$d = \deg \operatorname{num}(H) \leqslant \mu Z_0 \leqslant M C_0(\mathcal{X}).$$

Thus, $d$ is bounded in terms of $\mathcal{X}$, which concludes the proof. $\square$

## 6. Proof of Theorem 3

The result follows immediately from Lemma 9 applied to polynomials $P = FH+1$, $Q = GH + 1$, $R = 1$, and the identity

$$\max\{\deg P, \deg Q, \deg R\} = \deg H + \max\{\deg F, \deg G\}.$$

$\square$

## 7. Comments

We start with a remark that one can easily extend our results to multivariate rational functions. In fact, using the *Kronecker substitution*, that is, $X_i \to X^{(d+1)^{i-1}}$, $i = 1, 2, \ldots$, where $d$ is the largest degree of all functions involved, one can achieve this without any additional arguments, however a more direct treatment may lead to stronger results.

Most likely, these results can also be extended to rational functions on an algebraic variety.

It is an interesting question to try to relax the condition of quasipolynomiality in Theorems 1 and 2, however, as example (2) shows, it cannot be completely abandoned.

The arguments and results of this paper can probably be extended to more general products $FG+A_1$, $FH+A_2$ and $GH+A_3$ with some fixed nonzero rational functions $A_1, A_2, A_3 \in K[X]$.

Motivated by a conjecture of Győry and Sárközy in [11] on the largest value of $P\left((ab+1)(ac+1)(bc+1)\right)$ taken over several triples $(a, b, c) \in \mathbb{Z}^3$, Bugeaud and Luca in [6], Theorem 1, have proved that for any finite set $\mathcal{T}$ of such triples with $a > b > c > 0$, we have

$$\omega\Bigg(\prod_{(a,b,c)\in\mathcal{T}} (ab+1)(bc+1)(ac+1)\Bigg) \geqslant 10^{-6} \log \#\mathcal{T},$$

where $\omega(k)$ is the number of distinct prime divisors of an integer $k \geqslant 1$. It is also interesting to study function field analogues of this result for finite sets $\mathcal{W}$ of triples $(F, G, H)$ of pairwise distinct rational functions and obtain a lower bound

$$Z\biggl(\prod_{(F,G,H)\in\mathcal{W}} (FG+1)(FH+1)(GH+1)\biggr) \geqslant \psi(\#\mathcal{W})$$

with some explicit (perhaps up to some constant factor) function $\psi$ with $\psi(z) \to \infty$ as $z \to \infty$.

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*Author's address*: Igor E. Shparlinski, Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia, e-mail: `igor.shparlinski@unsw.edu.au`.