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ON THE POINTWISE LIMITS OF SEQUENCES OF ŚWIĄTKOWSKI FUNCTIONS

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Abstract. The characterization of the pointwise limits of the sequences of Świątkowski functions is given. Modifications of Świątkowski property with respect to different topologies finer than the Euclidean topology are discussed.

Keywords: Świątkowski function; cliquish function; pointwise limit; *topology of Hashimoto; $\mathcal{I}$-density topology; density topology

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1. Introduction

In 1977 Mańk and Świątkowski defined a new property of real functions, being a kind of intermediate value property, so similar to the Darboux property: *for all* $a < b$ *with* $f(a) \neq f(b)$ *there is* $x \in (a, b) \cap \mathcal{C}(f)$ *such that* $f(x)$ *is between* $f(a)$ *and* $f(b)$, see [14]. It seems that motivations for study of such property derived from search for the weakest conditions that imply monotonicity of functions.

Mańk and Świątkowski called the mentioned condition "*condition* γ" and asked if the family of Baire one Darboux functions satisfying the condition γ are closed under the uniform limits and sums with continuous functions (both answers are positive). Algebraic properties of the class of all functions possessing this property have been studied by many authors. In particular, Maliszewski and the second author gave the characterization of products of Świątkowski functions, see [13] and [12]. Wódka investigated the level of algebrability of the sets connected with the Świątkowski condition in paper [21]. Other results concerning Świątkowski functions can be found in [1] and [7]. This note is a continuation of [22], where uniform limits of sequences of Świątkowski functions are characterized.

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2. Preliminaries

We shall start with some basic notations and definitions. We use a standard set-theoretic and topological notation. In particular, the letter $\mathbb{R}$ denotes the real line with the Euclidean topology τ_e. Symbols $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the sets of natural numbers, integers and rationals, respectively.

For $A \subset \mathbb{R}$ we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{fr} A$ the interior, the closure and the boundary of A, respectively. For $a, b \in \mathbb{R}$ the symbol $I(a, b)$ denotes the interval with end-points a and b. The set of all continuity points of a function $f\colon \mathbb{R} \to \mathbb{R}$ is denoted by $\mathcal{C}(f)$.

For $x \in \mathbb{R}$ and a nonempty set $A \subset \mathbb{R}$ let $D(x, A)$ denote the distance between x and $A \subset \mathbb{R}$, i.e., $D(x, A) := \inf\{|x - t|\colon\ t \in A\}$.

A set $A \subset \mathbb{R}$ has the *Baire property* if there exist an open set O and a meager set M such that $A = O \triangle M = (O \setminus M) \cup (M \setminus O)$. The algebra of all sets possessing the Baire property is denoted by $\mathcal{B}$. The ideal of meager sets is denoted by $\mathcal{M}$. A set A is residual if $\mathbb{R} \setminus A$ is meager. We say that a set A is nowhere meager in an open set $U \subset \mathbb{R}$ if $A \cap W$ is nonmeager for every nonempty open subset $W \subset U$.

The symbols $\operatorname{osc}(f, x)$ and $\operatorname{osc}_\tau(f, x)$ denote the oscillation and the oscillation with respect to a topology τ of a function f at a point $x \in \mathbb{R}$, respectively.

For a family $\mathscr{F} \subset \mathbb{R}^{\mathbb{R}}$ the symbol $\operatorname{LIM}(\mathscr{F})$ denotes the family of all pointwise limits of the sequences $(f_n)_n$ from $\mathscr{F}$.

We will consider the following classes of functions from $\mathbb{R}$ into $\mathbb{R}$.

- $\mathscr{B}\mathfrak{a}$: the class of all functions with the Baire property. A function $f\colon \mathbb{R} \to \mathbb{R}$ has the Baire property if the preimage of every open set has the Baire property.
- $\mathscr{C}_q$: the class of all cliquish functions. We say that $f\colon \mathbb{R} \to \mathbb{R}$ is *cliquish*, if for all $a < b$ and each $\varepsilon > 0$ there is a nondegenerate interval $I \subset (a, b)$ such that $\operatorname{diam} f[I] < \varepsilon$, see [20]. It is well-known and easy to see that $f\colon \mathbb{R} \to \mathbb{R}$ is cliquish if and only if it is *pointwise discontinuous*, i.e., it has a dense set of points of continuity (cf. [9], [8]).
- $\mathscr{S}$: the class of all *Świątkowski functions*, i.e., all functions possessing the following property: for all $a < b$ with $f(a) \neq f(b)$ there is $x \in (a, b) \cap \mathcal{C}(f)$ such that $f(x) \in I(f(a), f(b))$, see [14] (cf. [17], [18]).
- $\mathscr{S}_s$: the class of all *strong Świątkowski functions*, i.e., all functions with the following property: for all $a < b$ and each y between $f(a)$ and $f(b)$ there is $x \in (a, b) \cap \mathcal{C}(f)$ with $f(x) = y$, see [10].

It is known that the following inclusions hold. (See e.g. [11].)

$$\mathscr{S}_s \subset \mathscr{S} \subset \mathscr{C}_q \subset \mathscr{B}\mathfrak{a}. \tag{1}$$

Moreover, easy examples show that all those inclusions are proper.

Let I and J be nonempty open intervals. We say that $f\colon I \to J$ is *left side surjective* if $f[(\inf I, t)] = J$ for all $t \in I$. Analogously, we say that $f\colon I \to J$ is *right side surjective* if for all $t \in I$ we have $f[(t, \sup I)] = J$. A function $f\colon I \to J$ is a *bi-surjective function* if it is both left and right side surjective. If, additionally, f is continuous, we write $f \in \mathscr{CBS}(I, J)$. A class of continuous bi-surjective functions plays an important role in constructions dealing with Świątkowski functions (see, e.g., [11], [18], [22]).

3. Background informations

It is clear that the class $\mathscr{B}\mathfrak{a}$ is closed with respect to pointwise limits. The classes $\mathrm{LIM}(\mathscr{C}_q)$ and $\mathrm{LIM}(\mathscr{S}_s)$ were characterized by Grande [2] and Maliszewski [11], respectively. The following equalities hold.

- ▷ $\mathrm{LIM}(\mathscr{S}_s) = \mathscr{C}_q$;
- ▷ $\mathrm{LIM}(\mathscr{C}_q) = \mathscr{B}\mathfrak{a}$.

Those facts show that

$$\mathscr{C}_q \subset \mathrm{LIM}(\mathscr{S}) \subset \mathscr{B}\mathfrak{a}.$$

In this section we will show that those inclusions are proper.

Example 1. For $n \in \mathbb{N}$ put $A_n := \{k/2^n\colon\ k \in \mathbb{Z}\}$ and define

$$f_n(x) := \begin{cases} -D(x, A_n) & \text{for } x \notin A_n, \\ 1 & \text{for } x \in A_n. \end{cases}$$

One can see that for each $n \in \mathbb{N}$ the function f_n satisfies the Świątkowski condition. On the other hand $f_n \to f$, where

$$f = \begin{cases} 0 & \text{for } x \notin \bigcup\limits_{n\in\mathbb{N}} A_n, \\ 1 & \text{for } x \in \bigcup\limits_{n\in\mathbb{N}} A_n. \end{cases}$$

Since the function f is discontinuous at any point, it is not cliquish. Therefore $\mathrm{LIM}(\mathscr{S}) \neq \mathscr{C}_q$.

Example 2. Consider a function $f\colon\ \mathbb{R} \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1 & \text{for } x \in \mathbb{Q}_1, \\ 2 & \text{for } x \in \mathbb{Q}_2, \end{cases}$$

where $\mathbb{Q}_1$, $\mathbb{Q}_2$ is a partition of rationals into two dense sets. It is easy to see that $f \in \mathscr{B}\mathfrak{a}$. We shall show that f is not a pointwise limit of any sequence of Świątkowski functions. Indeed, suppose there exists a sequence $(f_n)_n$ of Świątkowski functions such that $f_n \to f$. Let

$$A_n := \left\{x \in \mathbb{R} \setminus \mathbb{Q} \colon \forall_{m \geqslant n}\ |f_m(x)| < \tfrac{1}{2}\right\}.$$

Of course $\mathbb{R} \setminus \mathbb{Q} = \bigcup_n A_n$ and $A_n \subset A_m$ for $n \leqslant m$. Since the set $\mathbb{R} \setminus \mathbb{Q}$ is residual, the Baire category theorem yields that there exists $n_0 \in \mathbb{N}$ for which the set A_{n_0} is nonmeager, and consequently, it is dense in some nonempty open interval I. Fix $x_1 \in I \cap \mathbb{Q}_1$ and $x_2 \in I \cap \mathbb{Q}_2$ with $x_1 < x_2$. There exist numbers n_i, $i = 1, 2$, such that for $n \geqslant n_i$ we have an inequality $|f_n(x_i) - i| < 1/2$. Then for $N := \max\{n_0, n_1, n_2\}$ we have $f_N \notin \mathscr{S}$. In fact, it is easy to observe that $f_N(x) \leqslant 1/2$ for every $x \in I \cap \mathcal{C}(f_N)$, thus there is no $x \in (x_1, x_2) \cap \mathcal{C}(f_N)$ with $f_N(x) \in (f_N(x_1), f_N(x_2))$. Hence $\mathrm{LIM}(\mathscr{S}) \neq \mathscr{B}\mathfrak{a}$.

4. The main theorem

Fix $A \subset \mathbb{R}$, an interval $J \subset \mathbb{R}$ and $\varepsilon \geqslant 0$. We say that a function $f \colon \mathbb{R} \to \mathbb{R}$ satisfies the condition $S(J, A, \varepsilon)$ if for each $a, b \in J$ with $f(a) < f(b)$ there exists $x \in A \cap I(a, b)$ such that $f(x) \in (f(a) - \varepsilon, f(b) + \varepsilon)$.

Note that $f \in \mathscr{S}$ if and only if f satisfies the condition $S(\mathbb{R}, \mathcal{C}(f), 0)$. Thus, the condition $S(\mathbb{R}, A, 0)$ can be treated as a generalization of the Świątkowski property related to a fixed set A. An analogous modification of the strong Świątkowski property has been considered by Marciniak and Szczuka [15], see also [5].

A function f satisfies the condition $S(A, \varepsilon)$ if the union of all open intervals J for which $S(J, A, \varepsilon)$ holds is dense in $\mathbb{R}$.

Let $\mathfrak{S}$ denote the class of all functions $f \in \mathscr{B}\mathfrak{a}$ which satisfy the condition $S(A, \varepsilon)$ for all residual sets $A \subset \mathbb{R}$ and any $\varepsilon > 0$. Observe that $\mathscr{C}_q \subset \mathfrak{S} \subset \mathscr{B}\mathfrak{a}$. We will show that $\mathrm{LIM}(\mathscr{S}) = \mathfrak{S}$.

The next lemma is probably a part of mathematical folklore.

Lemma 1. *Let $f \in \mathscr{B}\mathfrak{a}$. Then there exists a residual G_δ set A such that $f \restriction A$ is continuous. Moreover, each such set A can be extended to a maximal with respect to inclusion set with the same properties.*

P r o o f. It is well-known that for every function $f \in \mathscr{B}\mathfrak{a}$ there exists a residual G_δ set A such that $f \restriction A$ is continuous. (See e.g. [8].) Now, let

$$B := \{x \in \mathbb{R} \colon \operatorname{osc}(f \restriction A \cup \{x\}, x) = 0\}.$$

It is easy to see that $A \subset B \in G_\delta$, $f \restriction B$ is continuous, and B is maximal set with those properties. $\square$

Lemma 2. *Let $f \in \mathscr{B}\mathfrak{a}$. Then for each open interval I and every $\varepsilon \geqslant 0$ the following conditions are equivalent:*

(i) *f satisfies the condition $S(I, A, \varepsilon)$ for every residual set $A \subset \mathbb{R}$;*

(ii) *f satisfies the condition $S(I, A, \varepsilon)$ for every residual set $A \subset \mathbb{R}$ such that $f \restriction A$ is continuous;*

(iii) *there exists a residual set $A \subset \mathbb{R}$ such that $f \restriction A$ is continuous and f satisfies the condition $S(I, A, \varepsilon)$.*

Proof. Only the implication "(iii)⇒(i)" requires a proof. Let A be a residual set such that $f \restriction A$ is continuous and f satisfies the condition $S(I, A, \varepsilon)$, and let $B \subset \mathbb{R}$ be any residual set. Fix $a, b \in I$ such that $f(a) < f(b)$. By $S(I, A, \varepsilon)$, there exists $x \in A \cap I(a, b)$ such that $f(x) \in (f(a) - \varepsilon, f(b) + \varepsilon)$. Since $A \cap B$ is dense in $\mathbb{R}$ and $f \restriction A$ is continuous, there exists $x_0 \in A \cap B \cap I(a, b)$ with $f(x_0) \in (f(a) - \varepsilon, f(b) + \varepsilon)$. $\square$

Corollary 3. *Assume $f \in \mathscr{C}_q$. Then $f \in \mathscr{S}$ if and only if the condition $S(\mathbb{R}, A, 0)$ holds for every residual set $A \subset \mathbb{R}$.*

Lemma 4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a cliquish function. Then for any $\varepsilon > 0$ and for every nowhere dense set $E \subset \mathbb{R}$ there exists a maximal with respect to inclusion family of pairwise disjoint open intervals $\{I_n\colon n \in \mathbb{N}\}$ such that $\operatorname{diam} f[I_n] < \varepsilon$ for all $n \in \mathbb{N}$ and end-points of any I_n belong to $\mathbb{R} \setminus E$. Moreover, the union of every such family is dense in $\mathbb{R}$.*

Proof. The first part is an easy consequence of the Kuratowski-Zorn lemma. The second part follows easily from cliquishness of f. $\square$

Theorem 5. *Let $f\colon \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent:*

(i) *$f \in \mathrm{LIM}(\mathscr{S})$;*

(ii) *$f \in \mathfrak{S}$.*

Proof. "(i)⇒(ii)" Let $f = \lim_n f_n$ for some sequence $(f_n)_n \subset \mathscr{S}$. Then f and all f_n have the Baire property, thus there exists a residual set A such that $f \restriction A$ and all $f_n \restriction A$ are continuous. By Lemma 2, it is enough to prove that $S(A, \varepsilon)$ holds for f and for any $\varepsilon > 0$. Fix $\varepsilon > 0$ and a nonempty open interval I.

For every $n \in \mathbb{N}$ define

$$A_n = \bigl\{x \in I \cap A\colon \forall_{k \geqslant n}\ |f_k(x) - f(x)| \leqslant \tfrac{1}{3}\varepsilon\bigr\}.$$

Note that sets A_n are closed in A and $A \cap I = \bigcup_n A_n$. Since A is nonmeager, there exists n_0 such that A_{n_0} is dense in some nondegenerate interval $J \subset I$, and consequently, $J \cap A = J \cap A_{n_0}$. We will verify that $S(J, A, \varepsilon)$ holds. Fix $a, b \in J$ such that $f(a) < f(b)$. Since $f_n(a) \to f(a)$ and $f_n(b) \to f(b)$, there is $N \geqslant n_0$ such that $f_N(a) < f_N(b)$ and

$$|f_N(a) - f(a)| < \tfrac{1}{3}\varepsilon \qquad \text{and} \qquad |f_N(b) - f(b)| < \tfrac{1}{3}\varepsilon.$$

By the Świątkowski property of f_N and Corollary 3, there is $x \in A_{n_0} \cap I(a, b)$ with $f_N(x) \in (f_N(a), f_N(b))$ and we have

$$f(a) - \varepsilon < f_N(a) - \tfrac{2}{3}\varepsilon < f_N(x) - \tfrac{2}{3}\varepsilon \leqslant f(x) - \tfrac{1}{3}\varepsilon < f(x),$$

and similarly, $f(x) < f(b) + \varepsilon$. Thus $f(x) \in (f(a) - \varepsilon, f(x) + \varepsilon)$.

"(ii)⇒(i)" Let $f \in \mathfrak{S}$. Let A be a maximal with respect to inclusion residual G_δ set such that $f \restriction A$ is continuous. Then $B := \mathbb{R} \setminus A = \bigcup_n B_n$, where $(B_n)_n$ is an increasing sequence of nowhere dense sets.

For every $y \in \mathbb{R}$ define

$$\widetilde{C}_y := f^{-1}[y] \cap \operatorname{int}(\operatorname{cl}(f^{-1}[y]))$$

and set

$$C := \bigcup_{x \in B} \widetilde{C}_{f(x)}.$$

Observe that by the maximality of A for every $x \in B \cap C$ we have

$$x \in \operatorname{int}(\operatorname{cl}(f^{-1}[f(x)] \cap B)).$$

For every $x \in B \cap C$ let I_x denote the connected component of the set

$$\operatorname{int}(\operatorname{cl}(f^{-1}[f(x)] \cap B))$$

containing x. Moreover, set $C_n = B_n \cap C$.

Observe that the set $E = \{x \in \mathbb{R} \colon \limsup_{t \to x, t \in A} f(t) = \pm\infty\}$ is nowhere dense. Let us consider the following function $\tilde{f} \colon \mathbb{R} \to \mathbb{R}$.

$$\tilde{f} = \begin{cases} f(x) & \text{for } x \in (\mathbb{R} \setminus B \cap C) \cup E, \\ \limsup\limits_{t \to x, t \in A} f(t) & \text{for } x \in B \cap C \setminus E. \end{cases}$$

We claim that $\tilde{f}$ is cliquish. Indeed, otherwise there exist $\varepsilon > 0$ and an interval I such that for each $J \subset I$ we have $\operatorname{diam} \tilde{f}[J] \geqslant \varepsilon$. Since $f \restriction A$ is continuous, there exists a nonempty open interval $U \subset I \setminus E$ with $\operatorname{diam} f[U \cap A] < \varepsilon/4$. Since $\tilde{f}[J] \subset \operatorname{cl}(f[J])$, $\operatorname{diam} f[J] \geqslant \varepsilon$ for any $J \subset U$, and consequently, there is a set $\Delta \subset B \setminus C$ dense in U and such that for every $x \in \Delta$ either $f(x) < \inf f[A \cap U] - \varepsilon/4$, or $f(x) > \sup f[A \cap U] + \varepsilon/4$. Let

$$\Delta_- := \{x \in \Delta\colon\ f(x) < \inf f[A \cap U] - \tfrac{1}{4}\varepsilon\};$$
$$\Delta_+ := \{x \in \Delta\colon\ f(x) > \sup f[A \cap U] + \tfrac{1}{4}\varepsilon\}.$$

Fix a nonempty open interval $J \subset U$. Since $\Delta = \Delta_- \cup \Delta_+$, either Δ_- or Δ_+, say Δ_+, is dense in some subinterval $J_0 \subset J$. Observe that f is not constant on $J_0 \cap \Delta_+$, because otherwise $J_0 \cap \Delta_+ \subset C$. Fix $a, b \in J_0 \cap \Delta_+$ with $f(a) < f(b)$ and observe that $f(x) \leqslant f(a) - \varepsilon/4$ for any $x \in A \cap J_0$, hence $f[A \cap J_0] \cap (f(a), f(b)) = \emptyset$. Thus, for any subinterval $J \subset U$ the function f does not satisfy the condition $S(J, A, \varepsilon/8)$, contrary with $f \in \mathfrak{S}$.

Since $\tilde{f} \in \mathscr{C}_q$, there exists a sequence $(\tilde{f}_n)_n$ of strong Świątkowski functions such that $\tilde{f}_n \to \tilde{f}$, see [11], Corollary 6. We will modify functions $\tilde{f}_n$ using a standard trick with bi-surjective functions (see, e.g. [22]). So, for every $n \in \mathbb{N}$ let $\{I^n_m\colon\ m \in \mathbb{N}\}$ be a maximal family (with respect to inclusion) of pairwise disjoint open intervals contained in the set $\mathbb{R} \setminus B_n$ with end-points in A and $\operatorname{diam} \tilde{f}_n[I^n_m] < 1/(4n)$. Since $\tilde{f}_n$ is cliquish, Lemma 4 yields $\operatorname{cl}\Bigl(\bigcup_{m\in\mathbb{N}} I^n_m\Bigr) = \mathbb{R}$. Next, for every $m \in \mathbb{N}$ with $I^n_m \neq \emptyset$ choose $x^n_m \in I^n_m \cap A$. Finally, choose $\tilde{f}^n_m \in \mathscr{CBS}(I^n_m, (\tilde{f}_n(x^n_m) - 1/(2n), \tilde{f}_n(x^n_m) + 1/(2n)))$ and define a function g_n:

$$g_n(x) = \begin{cases} \tilde{f}^n_m(x) & \text{for } x \in I^n_m,\ m \in \mathbb{N}, \\ f(x) & \text{for } x \in C_n, \\ \tilde{f}_n(x) & \text{in other cases.} \end{cases}$$

We claim that if $f(x) \neq f(y) \neq f(z) \neq f(x)$ for some $x, y, z \in B \cap C$, then $I_x \cap I_y \cap I_z = \emptyset$. Indeed, suppose that $I_x \cap I_y \cap I_z \neq \emptyset$. First observe that there exists $a \in A \cap I_x \cap I_y \cap I_z$ such that $f(a) \notin \{f(x), f(y), f(z)\}$. In fact, suppose that $f[A \cap I_x \cap I_y \cap I_z] \subset \{f(x), f(y), f(z)\}$. Then there is an interval $J \subset I_x \cap I_y \cap I_z$ with f being constant on $J \cap A$, say $f(a) = f(x)$ for $a \in J \cap A$. Choose $s \in J \cap f^{-1}(f(x)) \cap B$. Then $f \restriction (A \cup \{s\})$ is continuous, contrary to the maximality of A. Hence $f(a) \notin \{f(x), f(y), f(z)\}$. Take

$$\varepsilon := \tfrac{1}{2} \min\{|f(a) - f(x)|, |f(a) - f(y)|, |f(a) - f(z)|\}.$$

Let $U \subset I_x \cap I_y \cap I_z$ be a neighborhood of a such that $\operatorname{diam} f[A \cap U] < \varepsilon$ and let J be any subinterval of U. Choose $\bar{x}, \overline{y}, \overline{z} \in U$ such that $f(\bar{t}) = f(t)$ for $t \in \{x, y, z\}$. Changing, possibly, the names we can assume that $f(a) < f(\bar{x}) < f(\overline{y})$. Then

$$(f(\bar{x}) - \varepsilon, f(\overline{y}) + \varepsilon) \cap f[I(\bar{x}, \overline{y}) \cap A] = \emptyset.$$

Thus, f satisfies the condition $S(J, A, \varepsilon)$ for no subinterval $J \subset U$, contrary to $f \in \mathfrak{S}$.

In the next step we claim that the set

$$D := \{\inf I_x \colon\ x \in B \cap C\} \cap \mathbb{R}$$

is nowhere dense. Indeed, take an arbitrary open set U and $x \in U \cap D$. If there is no $y \in D \cap U \cap I_x$, then $I_x \cap U$ is a nonempty open set disjoint with D. Otherwise, by the previous claim, D is disjoint with $I_x \cap I_y$.

Let $\mathcal{I}$ denote the family of connected components of the set $\mathbb{R} \setminus \operatorname{cl}(D)$. For $I \in \mathcal{I}$ and $n \in \mathbb{N}$ fix a function $f_n^I \in \mathscr{CBS}((\inf I, \inf I + |I|/n), \mathbb{R})$. Finally, for every $n \in \mathbb{N}$ define a function $f_n \colon\ \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} f_n^I(x) & \text{for } x \in (\inf I, \inf I + |I|/n),\ I \in \mathcal{I}, \\ g_n & \text{in the oposite case.} \end{cases}$$

Fix $n \in \mathbb{N}$. We shall show that $f_n \in \mathscr{S}$. Take $a < b$ and assume that $f_n(a) < f_n(b)$ (the second case is analogous). We have to consider a few cases.

Case I. $(a, b) \cap \operatorname{cl}(D) \neq \emptyset$. Let I be a component of $\mathbb{R} \setminus \operatorname{cl}(D)$ with $\inf I \in (a, b)$. Then for every $y \in \mathbb{R}$ there is $t \in (a, b) \cap I$ such that $f_n(t) = f_n^I(t) = y$, thus the Świątkowski condition is satisfied.

Case II. $(a, b) \cap \operatorname{cl}(D) = \emptyset$ and $(a, b) \cap (\inf I, \inf I + |I|/n) \neq \emptyset$ for some component I of $\mathbb{R} \setminus \operatorname{cl}(D)$. Then we have two subcases.

II.1. $(a, b) \subset (\inf I, \inf I + |I|/n)$. Then f_n agrees with f_n^I on (a, b), thus it is continuous on (a, b), so the Świątkowski condition holds.

II.2. $\inf I + |I|/n \in (a, b)$. Then for every $y \in \mathbb{R}$ there is $t \in (a, b) \cap I$ such that $f_n(t) = f_n^I(t) = y$.

Case III. $(a, b) \subset I \setminus (\inf I, \inf I + |I|/n)$ for some component I of $\mathbb{R} \setminus \operatorname{cl}(D)$. Then $f_n(x) = g_n(x)$ for $x \in (a, b)$. Two subcases may occur.

III.1. $\{a, b\} \not\subset C_n$. We may assume that $a \notin C_n$ (the other case is analogous). If $a \in I_n^m$ for some $m \in \mathbb{N}$, then the Świątkowski condition easily holds. If not, $f_n(a) = g_n(a) = \tilde{f}_n(a)$, and by the strong Świątkowski property of $\tilde{f}_n$ there exists $t \in (a, b)$ such that $\tilde{f}_n(t) \in (f_n(a) - 1/(8n), f_n(b) + 1/(8n))$. By the construction of the family $(f_n^m)_m$ there exists $m \in \mathbb{N}$ and $s \in I_n^m$ such that $I_n^m \subset (a, b)$ and

$\tilde{f}_n^m(s) \in (f_n(a), f_n(b))$. Since $\tilde{f}_n^m(s) = f_n(s)$ and $I_n^m \subset \mathcal{C}(f_n)$, the Świątkowski condition holds.

III.2. $\{a,b\} \subset C_n$. Then $\inf I_b < a < b$ and $f_n(t) = g_n(t) = f(t)$ for $t \in \{a,b\}$, so $f(a) < f(b)$. For $\varepsilon = 1/(8n)$ there is an open interval $J \subset (a,b) \cap I_a \cap I_b$ such that f satisfies the condition $S(J, A, \varepsilon)$. We may assume that $J \subset I_m^n$ for some m. Fix $x_a \in J \cap f^{-1}[f(a)]$ and $x_b \in J \cap f^{-1}[f(b)]$. By $S(J, A, \varepsilon)$, there exists $x \in J \cap A$ with $f(x) \in (f(a) - \varepsilon, f(b) + \varepsilon)$. But then there exist $m \in \mathbb{N}$ and $t \in I_n^m$ such that $I_n^m \subset J$ and $f_n(t) = g_n(t) = \tilde{f}_m^n(t)$, thus $t \in \mathcal{C}(f_n)$ and $f_n(t) \in (f_n(a), f_n(b))$, so the Świątkowski condition is satisfied.

Finally observe that $f_n \to f$. In fact, fix $x \in \mathbb{R}$. Observe that if $x \notin B \cap C$, then $|g_n(x) - \tilde{f}_n(x)| < 1/n$, thus

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x) = f(x).$$

Moreover, there exists n_0 such that $f_n(x) = g_n(x)$ for $n > n_0$, hence $\lim_n f_n(x) = f(x)$. If $x \in B \cap C$, then there is n_0 such that $x \in C_n$ for $n > n_0$ and then $f_n(x) = g_n(x) = f(x)$. □

Corollary 6. *If $B \subset \mathbb{R}$ is a set with the Baire property, then the characteristic function of B is a pointwise limit of a sequence of Świątkowski functions.*

P r o o f. Let χ_B be the characteristic function of B. Since $B \in \mathscr{B}$, there are an open set U and a meager set M with $B = U \triangle M$. Let $A = \mathbb{R} \setminus (M \cup \mathrm{fr}(U))$. Clearly A is residual and $\chi_B \restriction A$ is continuous. Fix $a, b \in \mathbb{R}$ with $\chi_B(a) < \chi_B(b)$. Then $\chi_B(a) = 0$ and $\chi_B(b) = 1$, so for any $\varepsilon > 0$ we have $\chi_B[\mathbb{R}] = \{0,1\} \subset (\chi_B(a) - \varepsilon, \chi_B(b) + \varepsilon)$, hence χ_B satisfies the condition $S(A, \varepsilon)$. □

5. Baire system generated by the family $\mathscr{S}$

For a family $\mathcal{F}$ of real-valued functions defined on X there is a smallest family $\mathcal{B}(\mathcal{F})$ of all real-valued functions defined on $\mathbb{R}$ which contains $\mathcal{F}$ and which is closed under the process of taking limits of sequences. This family is called the *Baire system* generated by $\mathcal{F}$. For a given $\mathcal{F}$ let us define

- ▷ $\mathcal{B}_0(\mathcal{F}) = \mathcal{F}$;
- ▷ $\mathcal{B}_\alpha(\mathcal{F}) = \mathrm{LIM}\Big(\bigcup_{\beta<\alpha} \mathcal{B}_\beta(\mathcal{F})\Big)$ for $\alpha > 0$.

Then $\mathcal{B}(\mathcal{F}) = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha(\mathcal{F})$. This system was described in 1899 by Baire in the case when $\mathcal{F}$ is the family of all continuous functions. The minimal ordinal $\alpha \leqslant \omega_1$ with $\mathcal{B}_\alpha(\mathcal{F}) = \bigcup_{\beta<\alpha} \mathcal{B}_\beta(\mathcal{F})$ is called *Baire order* of the family $\mathcal{F}$.

Theorem 7. *We have*

(i) $\mathcal{B}_1(\mathscr{S}) = \mathfrak{S}$;

(ii) $\mathcal{B}_\alpha(\mathscr{S}) = \mathscr{B}\mathfrak{a}$ *for* $\alpha > 1$.

Hence, the Baire order of the family of all Świątkowski functions is equal to 2.

P r o o f. The equality (i) follows from Theorem 5. Since $\mathfrak{S} \subset \mathscr{B}\mathfrak{a}$ and the class of all functions possessing the Baire property is closed with respect to pointwise limits, $\mathcal{B}_2(\mathscr{S}) = \mathrm{LIM}(\mathfrak{S}) \subset \mathscr{B}\mathfrak{a}$. Since $\mathscr{B}\mathfrak{a} \subset \mathrm{LIM}(\mathscr{C}_q)$, see [2], and $\mathscr{C}_q \subset \mathfrak{S}$, $\mathscr{B}\mathfrak{a} \subset \mathcal{B}_2(\mathscr{S})$. Again, since $\mathrm{LIM}(\mathscr{B}\mathfrak{a}) = \mathscr{B}\mathfrak{a}$, we have $\mathcal{B}_\alpha(\mathscr{S}) = \mathscr{B}\mathfrak{a}$ for $\alpha \geqslant 1$. □

6. A generalization: τ-Świątkowski functions

In this section we will consider a slight generalization of the Świątkowski property. Let τ be a fixed topology on $\mathbb{R}$. For a function $f\colon \mathbb{R} \to \mathbb{R}$ let $\mathcal{C}_\tau(f)$ denote the set of all points x at which f is continuous as a function from the space $(\mathbb{R}, \tau)$ into $\mathbb{R}$ with the Euclidean topology.

We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ has the Świątkowski property with respect to τ (shortly, f is a τ-Świątkowski function) if $f \in S(\mathcal{C}_\tau(f), 0)$. The class of all τ-Świątkowski functions will be denoted by $\mathscr{S}_\tau$.

Note that an analogous modification of the strong Świątkowski property has been considered in [3] and [6].

Theorem 8. *Let* τ *be a topology on* $\mathbb{R}$ *satisfying the following conditions:*

(i) τ *is finer than the Euclidean topology:* $\tau_e \subset \tau$;

(ii) $\tau \setminus \{\emptyset\} \subset \mathcal{B} \setminus \mathcal{M}$.

Then $\mathrm{LIM}(\mathscr{S}_\tau) = \mathfrak{S}$.

P r o o f. "$\supset$" Since $\tau_e \subset \tau$, so $\mathscr{S} \subset \mathscr{S}_\tau$ and consequently, $\mathfrak{S} \subset \mathrm{LIM}(\mathscr{S}_\tau)$.

"$\subset$" First let us see that $\mathscr{S}_\tau \subset \mathscr{B}\mathfrak{a}$. We will use the following well-known fact.

Fact 1. If $f \notin \mathscr{B}\mathfrak{a}$, then there are reals $\alpha < \beta$ such that the sets $A = f^{-1}[(-\infty, \alpha)]$, $B = f^{-1}[(\beta, \infty)]$ are both nowhere meager in some nonempty open set $U \subset \mathbb{R}$.

Thus, if $f \notin \mathscr{B}\mathfrak{a}$, then the set $\mathcal{C}_\tau(f)$ is not dense in $\mathbb{R}$ and therefore f is not a τ-Świątkowski function.

Now assume that $f \in \mathrm{LIM}(\mathscr{S}_\tau)$, i.e. there exists a sequence $(f_n)_n$ of τ-Świątkowski functions tending to f. Then $f \in \mathscr{B}\mathfrak{a}$, so there is a residual set A such that $f \restriction A$ is continuous. By Lemma 2, it is enough to prove that $S(A, \varepsilon)$ holds for any $\varepsilon > 0$. Fix $\varepsilon > 0$ and a nonempty open interval I. For every $n \in \mathbb{N}$ define

$$A_n = \bigl\{x \in I \cap A\colon\ \forall_{k \geqslant n}\ |f_k(x) - f(x)| < \tfrac{1}{3}\varepsilon\bigr\}.$$

Observe that each A_n has the Baire property, $A \cap I = \bigcup_n A_n$ and $A_n \subset A_{n+1}$. Since A is nonmeager, there exists n_0 such that A_{n_0} is residual in some nondegenerate interval $J \subset I$. We will verify that $S(J, A, \varepsilon)$ holds. Fix $a, b \in J$ such that $f(a) < f(b)$. Since $f_n(a) \to f(a)$ and $f_n(b) \to f(b)$, there is $N \geqslant n_0$ such that $f_N(a) < f_N(b)$ and

$$|f_N(a) - f(a)| < \tfrac{1}{3}\varepsilon \qquad \text{and} \qquad |f_N(b) - f(b)| < \tfrac{1}{3}\varepsilon.$$

By the τ-Świątkowski property of f_N, there is $x \in \mathcal{C}_\tau(f_N) \cap I(a,b)$ with $f_N(x) \in (f_N(a), f_N(b))$. Let $V \subset I(a,b)$ be a τ-neighborhood of x such that $f_N[V] \subset (f_N(a), f_N(b))$. Since $V \cap A_{n_0} \neq \emptyset$, there exists $x_0 \in A_{n_0} \cap I(a,b)$ with $f_N(x_0) \in (f_N(a), f_N(b))$ and we have

$$f(a) - \varepsilon < f_N(a) - \tfrac{2}{3}\varepsilon < f_N(x_0) - \tfrac{2}{3}\varepsilon < f(x_0) - \tfrac{1}{3}\varepsilon < f(x_0),$$

and similarly, $f(x_0) < f(b) + \varepsilon$. Thus $f(x_0) \in (f(a) - \varepsilon, f(b) + \varepsilon)$. □

In particular, the following two topologies satisfy assumptions of Theorem 8:

τ_*: the $*$-topology of Hashimoto with respect to the ideal $\mathcal{M}$, see [4]. Recall that

$$\tau_* = \{U \setminus M\colon\ U \in \tau_e;\ M \in \mathcal{M}\}.$$

$\tau_{\mathcal{I}}$: the $\mathcal{I}$-density topology, the category counterpart of the density topology, see [19].

Recall that $\tau_e \subset \tau_* \subset \tau_{\mathcal{I}}$ and both inclusions are here proper.

Example 3. Let $\mathbb{Q}_1$, $\mathbb{Q}_2$ be a partition of rationals onto two dense sets. Define $Q_i := \mathbb{Q}_i \cup \{(\frac{1}{2}(5-2i) + 2k) \cdot \pi\colon\ k \in \mathbb{Z}\}$. Then the function

$$f(x) = \begin{cases} \sin(x) & \text{for } x \in \mathbb{R} \setminus (Q_1 \cup Q_2), \\ (-1)^i 2 & \text{for } x \in Q_i,\ \ i = 1, 2 \end{cases}$$

is a τ_*-Świątkowski function which is not Świątkowski.

Corollary 9. *Although the families $\mathscr{S}$, $\mathscr{S}_{\tau_*}$ are different, their pointwise closures coincide.*

Lemma 10. *Assume τ is a topology on $\mathbb{R}$ which satisfies assumptions of Theorem 8. For any function $f\colon \mathbb{R} \to \mathbb{R}$, if the set $\mathcal{C}_\tau(f)$ is dense, then it is residual.*

P r o o f. For $n \in \mathbb{N}$ define

$$G_n = \big\{x \in \mathbb{R}\colon \operatorname{osc}_\tau(f, x) < 1/n\big\}.$$

Clearly, $G_n \in \tau$, hence $\mathcal{C}_\tau(f) = \bigcap_{n\in\mathbb{N}} G_n$ is a G_δ set in the topology τ, so it has the Baire property, thus it is enough to prove that $\mathcal{C}_\tau(f) \cap U \notin \mathcal{M}$ for every nonempty open set $U \in \tau_e$. Fix $U \in \tau_e$ and $n \in \mathbb{N}$. Since $\mathcal{C}_\tau$ is dense, $G_n \cap U$ is τ-open and nonempty, hence $G_n \cap U \notin \mathcal{M}$. Therefore each G_n is residual, thus $\mathcal{C}_\tau(f)$ is residual too. □

Theorem 11. *Assume τ is a topology on $\mathbb{R}$ which satisfies assumptions of Theorem 8. Then if $\tau_* \subset \tau$, then $\mathscr{S}_{\tau_*} = \mathscr{S}_\tau$.*

P r o o f. "$\subset$" Since $\tau_* \subset \tau$, we have $\mathscr{S}_{\tau_*} \subset \mathscr{S}_\tau$.

"$\supset$" Assume $f\colon \mathbb{R} \to \mathbb{R}$ has the τ-Świątkowski property. In the first part of the proof of Theorem 8 it is shown that $f \in \mathscr{B}\mathfrak{a}$, hence there is a residual set $C \subset \mathbb{R}$ for which $f \restriction C$ is continuous. By Lemma 10, the set $D := C \cap \mathcal{C}_\tau(f)$ is residual. Then $f \restriction D$ is continuous and therefore f is τ_*-continuous at each point $x \in D$. To prove that f has the τ_*-Świątkowski property fix $a, b \in \mathbb{R}$ with $f(a) < f(b)$. Since $f \in \mathscr{S}_\tau$, there is $x \in \mathcal{C}_\tau(f) \cap I(a, b)$ with $f(x) \in (f(a), f(b))$. Let $U \in \tau$ be a τ-neighborhood of x such that $U \subset I(a, b)$ and $f[U] \subset (f(a), f(b))$. Then $U \notin \mathcal{M}$, hence $U \cap D \neq \emptyset$, so there is $x_0 \in D \subset \mathcal{C}_{\tau_*}(f)$ such that $x_0 \in I(a, b)$ and $f(x_0) \in (f(a), f(b))$. □

Corollary 12. *We have $\mathscr{S}_{\tau_\mathcal{I}} = \mathscr{S}_{\tau_*}$.*

Finally we will discuss the τ-Świątkowski property related to some topologies τ connected with the Lebesgue measure on the real line. Interestingly, an analog of Theorem 11 for the measure does not occur. (We obtain a new example of an incomplete duality between the measure and category.) Let us consider the following topologies:

d_*: the $*$-topology of Hashimoto with respect to the ideal $\mathcal{N}$ of Lebesgue nullsets. Recall that

$$d_* = \{U \setminus N\colon U \in \tau_e;\ N \in \mathcal{N}\}.$$

d: the density topology, see e.g. [16].

Recall that $\tau_e \subset d_* \subset d$ and both inclusions are here proper. Thus

$$\mathscr{S} \subset \mathscr{S}_{d_*} \subset \mathscr{S}_d.$$

The function f from Example 3 shows that the inclusion $\mathscr{S} \subset \mathscr{S}_{d_*}$ is proper. The next example shows that the inclusion $\mathscr{S}_{d_*} \subset \mathscr{S}_d$ is proper, too.

Example 4. Let $E_1, E_2 \subset \mathbb{R}$ be disjoint F_σ set such that for each nondegenerate interval $J \subset \mathbb{R}$ the sets $J \cap E_i$, $i = 1, 2$, and $J \setminus (E_1 \cup E_2)$ have positive measure (cf. [16], Section 8). Moreover, assume that $\{(\frac{1}{2}(5-2i)+2k)\cdot\pi\colon\ k \in \mathbb{Z}\} \subset E_i$ for $i = 1, 2$. Then the function

$$f(x) = \begin{cases} \sin(x) & \text{for } x \in \mathbb{R} \setminus (E_1 \cup E_2), \\ (-1)^i 2 & \text{for } x \in \ E_i,\ \ i = 1, 2 \end{cases}$$

is a d-Świątkowski function which is not d_*-Świątkowski.

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