FINITE GROUPS WHOSE CHARACTER DEGREE GRAPHS COINCIDE WITH THEIR PRIME GRAPHS

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Received October 19, 2016. Published online March 2, 2018.

Abstract. In the literature, there are several graphs related to a finite group G. Two of them are the character degree graph, denoted by $\Delta(G)$, and the prime graph, $\Gamma(G)$. In this paper we classify all finite groups whose character degree graphs are disconnected and coincide with their prime graphs. As a corollary, we find all finite groups whose character degree graphs are square and coincide with their prime graphs.

Keywords: finite groups; character degree graph; prime graph

MSC 2010: 20C15

1. Introduction

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. The set of all irreducible complex character degrees of G is denoted by $\mathrm{cd}(G)$ so that $\mathrm{cd}(G) = \{\chi(1)\colon\ \chi \in \mathrm{Irr}(G)\}$. The character degree graph of G, written $\Delta(G)$, is the graph whose set of vertices $\varrho(G)$ is the set of primes that divide degrees in $\mathrm{cd}(G)$ with an edge between distinct primes p and q if and only if pq divides some complex irreducible character degrees of G.

Let $\pi(m)$ denote the set of primes that divide the integer m. The prime graph of G, denoted $\Gamma(G)$, is the graph with vertex set $\pi(G) := \pi(|G|)$. In this graph, two distinct vertices p, q are connected by an edge if and only if there exists an element of order pq in G.

In this paper, we are interested in finite groups G whose character degree graph coincides with its prime graph, namely, $\pi(G) = \varrho(G)$ and G has an element of order pq if and only if there exists an irreducible character degree of G which can be

The work of the second author was supported by Scientific Research Projects Coordination Unit of Istanbul University. The project number is 48029.

DOI: 10.21136/CMJ.2018.0553-16

divisible by pq (for all p and q in $\pi(G)$). In this case we write $\Delta(G) = \Gamma(G)$. In this paper, we classify all finite groups whose character degree graphs are disconnected and coincide with their prime graphs. The main theorems are the following.

Theorem 1.1. *Let G be a finite nonsolvable group. Then $\Delta(G) = \Gamma(G)$ and it is disconnected if and only if G is isomorphic to one of the following groups:*
(a) $\mathrm{PSL}(2, 2^n)$ *for an integer* $n \geqslant 2$,
(b) $\mathrm{PGL}(2, q)$ *where* $5 \leqslant q$ *is odd.*

Theorem 1.2. *Let G be a finite solvable group. Then $\Delta(G) = \Gamma(G)$ and it is disconnected if and only if G belongs to the following family, say $(*)$:*
"G is the semi-direct product of a Frobenius subgroup $H := \langle x \rangle \rtimes \langle y \rangle$ acting on an elementary abelian p-group V for some prime p, $C_H(V) = 1$, $\langle x \rangle$ acts irreducibly on V, $|V| = q^{o(y)}$ where q is a p-power and $o(y)$ is the order of y in G, $p \mid o(y)$ and $(q^{o(y)} - 1)/(q - 1) \mid o(x)$"

2. Disconnected graphs

In an unpublished paper, Gruenberg and Kegel have given the following classification of all finite groups with disconnected prime graph, see [7].

Theorem 2.1 (see [7])**.** *If G is a finite group whose prime graph has more than one component, then G has one of the following structures:*
(a) *Frobenius or 2-Frobenius;*
(b) *simple;*
(c) *an extension of a π_1-group by a simple group;*
(d) *simple by π_1-solvable; or*
(e) *π_1 by simple by π_1 where π_1 is the connected component of $\Gamma(G)$ containing* 2.

As a corollary, a finite solvable group whose prime graph is disconnected is Frobenius or 2-Frobenius.

Disconnected graphs have been studied extensively and the groups having disconnected character degree graph have been classified. The paper [2] contains the classification of the solvable groups G where $\Delta(G)$ is disconnected. Then, Lewis and White have completed the classification of all finite groups having disconnected character degree graph by [4].

Theorem 2.2 (see [4])**.** *Let G be a group. Then $\Delta(G)$ has three connected components if and only if $G = S \times A$ where $S \cong \mathrm{PSL}(2, 2^n)$ for an integer $n \geqslant 2$ and A is an abelian group.*

Theorem 2.3 (see [4]). *Let G be a nonsolvable group. Then $\Delta(G)$ has two connected components if and only if there exist normal subgroups $N \subseteq K$ such that the following conditions hold:*

(i) $K/N \cong \mathrm{PSL}(2,q)$, *where $q \geqslant 4$ is a power of a prime p.*

(ii) *If $C/N = C_{G/N}(K/N)$, then $C/N \subseteq Z(G/N)$ and G/K is abelian.*

(iii) *If $q \geqslant 5$, then p does not divide $|G : CK|$.*

(iv) *If $N > 1$, then either $K \cong \mathrm{SL}(2,q)$ or there is a normal subgroup L of G such that $K/L \cong \mathrm{SL}(2,q)$, L is elementary abelian of order q^2, and K/L acts transitively on the nonprincipal characters in $\mathrm{Irr}(L)$.*

(v) *If $p = 2$ or $q = 5$, then either $CK < G$ or $N > 1$.*

(vi) *If $p = 2$ and $N > 1$, then every nonprincipal character in $\mathrm{Irr}(L)$ extends to its stabilizer in G.*

The next theorem will be used often in the proof of Theorem 1.2.

Theorem 2.4 (see [2]). *Let G be a finite solvable group. Then $\Delta(G)$ has two connected components if and only if G belongs to one of the following families:*

(i) *G has a normal nonabelian Sylow p-subgroup P and an abelian p-complement K for some prime p, $P' \leqslant C_P(K)$ and every nonlinear irreducible character of P is fully ramified with respect to $P/C_P(K)$;*

(ii) *G is the semi-direct product of a subgroup H acting on a subgroup P where P is elementary abelian of order 9 and $\mathrm{cd}(H) = \{1,2,3\}$, $C_H(P) \leqslant Z(H)$ and $H/C_H(P) \cong \mathrm{SL}(2,3)$;*

(iii) *G is the semi-direct product of a subgroup H acting on a subgroup P where P is elementary abelian of order 9 and $\mathrm{cd}(H) = \{1,2,3,4\}$, $C_H(P) \leqslant Z(H)$ and $H/C_H(P) \cong \mathrm{GL}(2,3)$;*

(iv) *G is the semi-direct product of a subgroup H acting on an elementary abelian p-group V for some prime p, $|H : F(H)| > 1$, $|V| = q^{|H:F(H)|}$ where q is a p-power, $C_H(V) \leqslant Z(H)$, $F(H)/C_H(V)$ is abelian, $F(H)$ acts irreducibly on V, $(|H : F(H)|, |F(H) : C_H(V)|) = 1$ and $(q^{|H:F(H)|} - 1)/(q-1)$ divides $|F(H) : C_H(V)|$;*

(v) *G has a normal nonabelian 2-subgroup Q and an abelian 2-complement K such that $|G : KQ| = 2$ and G/Q is not abelian, $Q' \leqslant C_Q(K)$ and $C_K(Q)$ is central in G, every nonlinear irreducible character of Q is fully ramified with respect to $Q/C_Q(K)$ which is an elementary abelian 2-group of order 2^{2a} for some positive integer a and is an irreducible K-module, moreover, $K/C_K(Q)$ is abelian of order $2^a + 1$;*

(vi) *G is the semi-direct product of an abelian group D acting coprimely on a group T such that $[T, D]$ is a Frobenius group with nonabelian p-group* *(for a prime p), Frobenius kernel $T' = [T, D]'$ and a Frobenius complement B with $[B, D] \leqslant B$, every character in $\mathrm{Irr}(T \mid T'')$ is D-invariant, T'/T'' is B-irreducible, $|T' : T''| = q^m$ where q is a p-power, $m = |D : C_D(T')|$ and $(q^m - 1)/(q - 1)$ divides $|B|$.*

3. Subgroups of $\mathrm{Aut}(\mathrm{PSL}(2, q))$

Let us consider the special linear group $\mathrm{SL}(2, q)$ where $q = p^f$ for some prime p. Let δ and φ be automorphisms of $\mathrm{SL}(2, q)$ whose actions on the elements of the group are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\delta} = \begin{bmatrix} a & \nu^{-1}b \\ \nu c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\varphi} = \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix}$$

where ν is a fixed generator of F_q^*, the multiplicative group of the field F_q of q elements. Since the center of $\mathrm{SL}(2, q)$ is invariant under the actions of δ and φ, these maps induce the automorphisms $\overline{\delta}$ and $\overline{\varphi}$ on $\mathrm{PSL}(2, q) = \mathrm{SL}(2, q)/Z$ by $(gZ)^{\overline{\delta}} = g^{\delta}Z$ and $(gZ)^{\overline{\varphi}} = g^{\varphi}Z$, as usual. We will denote these induced automorphisms on $\mathrm{PSL}(2, q)$ by δ, φ as well. In this case, the outer automorphism group of $\mathrm{PSL}(2, q)$ is generated by the diagonal automorphism δ and the field automorphism φ and has order df, where $d = (2, q - 1)$. It is also well known that $\mathrm{PSL}(2, q)\langle\delta\rangle \cong \mathrm{PGL}(2, q)$ and $\mathrm{Aut}(\mathrm{PSL}(2, q)) = \mathrm{PS}L(2, q)\langle\delta, \varphi\rangle \cong \mathrm{PGL}(2, q)\langle\varphi\rangle$.

If q is even, then δ is an inner automorphism and $\mathrm{Aut}(\mathrm{PSL}(2, q)) = \mathrm{PSL}(2, q)\langle\varphi\rangle$, and if q is odd, then δ is an outer automorphism but δ^2 is inner. Hence δ is of order $d = (2, q - 1)$ modulo inner automorphisms. Also φ is of order f. Moreover, if q is odd, then δ and φ commute modulo inner automorphisms, so that $\mathrm{Aut}(\mathrm{PSL}(2, q))/\mathrm{PSL}(2, q) \cong \langle\delta\rangle \times \langle\varphi\rangle$.

Lemma 3.1 (see [6]). *If $\mathrm{PSL}(2, q) < G \leqslant \mathrm{Aut}(\mathrm{PSL}(2, q))$ with $q = p^f$, p an odd prime, then one of the following casses occurs:*

(a) *$\delta \in G$ so that $\mathrm{PGL}(2, q) \leqslant G$ and $G = \mathrm{PGL}(2, q)\langle\varphi^k\rangle$ for some $k \mid f$ with $1 \leqslant k \leqslant f$;*

(b) *$G = \mathrm{PSL}(2, q)\langle\varphi^k\rangle$ for some $k \mid f$ with $1 \leqslant k < f$;*

(c) *$G = \mathrm{PSL}(2, q)\langle\delta\varphi^k\rangle$ for some $k \mid f$ with $1 \leqslant k < f$ and f/k even.*

Theorem 3.2 (see [4]). *Let $N \cong \mathrm{PSL}(2, q)$, where $q = p^n$ for a prime p and $q > 5$. Suppose $N < G \leqslant \mathrm{Aut}(N)$. If p divides $|G : N|$, then $\Delta(G)$ is a connected graph. If p does not divide $|G : N|$, then $\Delta(G)$ has exactly two connected components, $\{p\}$ and $\pi(|G : N|(q^2 - 1))$.*

Lemma 4.1. *If G is a Frobenius group, then $\Delta(G)$ is complete.*

Proof. We can say first that $F(G)$, Frobenius kernel of G, has a complete character degree graph which is a subgraph of $\Delta(G)$ and that $\chi^G \in \mathrm{Irr}(G)$, $\chi^G(1) = \chi(1)[G : F(G)] \in \mathrm{cd}(G)$ for all $\chi \in \mathrm{Irr}(F(G))$. Thus $\Delta(G)$ must be complete since $\varrho(G) = \varrho(F(G)) \cup \pi([G : F(G)])$. □

Corollary 4.2. *Let G be a finite group. If $\Delta(G) = \Gamma(G)$, then G is not a Frobenius group.*

Proof. Assume that G is a Frobenius group. Then $\Gamma(G)$ is a disconnected graph. So we are done by Lemma 4.1 □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a finite nonsolvable group and let $\Delta(G) = \Gamma(G)$ be disconnected. By a result of Manz, Williams and Wolf the character degree graph for any finite group has at most three connected components. Thus, $\Delta(G)$ has two or three connected components since it is disconnected.

Case 1. $\Delta(G)$ has three connected components:

By Theorem 2.2, we know that $G \cong S \times A$ where $S \cong \mathrm{PSL}(2, 2^n)$ for an integer $n \geqslant 2$ and A is an abelian group. Since $\Gamma(G)$ is disconnected, $Z(G) = 1$ and so $A = 1$. Thus, $G \cong \mathrm{PSL}(2, 2^n)$ for an integer $n \geqslant 2$ as desired. By a result of Dickson ([1], page 213) which gives all subgroups of $\mathrm{PSL}(2, q)$ where $q \geqslant 4$, it follows that $\Gamma(\mathrm{PSL}(2, 2^n))$ has three connected components, $\{2\}$, $\pi(2^n - 1)$ and $\pi(2^n + 1)$ where $n \geqslant 2$, and each component is a complete graph in this graph. Thus we see that $\Delta(\mathrm{PSL}(2, 2^n)) = \Gamma(\mathrm{PSL}(2, 2^n))$ for $n \geqslant 2$ by Theorem 3.1 of [6].

Case 2. $\Delta(G)$ has two connected components:

Then G has normal subgroups N and K that satisfy conditions (i)–(vi) of Theorem 2.3.

(a) If $N = 1$ then $K \cong \mathrm{PSL}(2, q)$, $q \geqslant 4$ and q is a power of a prime p by (i).

First, suppose $p = 2$ so that $K \cong \mathrm{PSL}(2, 2^n)$. Since $\Delta(G)$ has two connected components but $\Delta(K)$ has three connected components, we see that $K < G$. Moreover, $K < G \leqslant \mathrm{Aut}(K)$ since $C_G(K) \leqslant Z(G) = 1$ by (ii). Assume $q > 5$, then 2 does not divide the index $|G : K|$ and 2 is an isolated vertex in $\Delta(G)$ by Theorem 3.2. But this contradicts the fact that G/K is a 2-group by Theorem 2.1 and Corollary 4.2. Thus $q \leqslant 5$ and so $K \cong \mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5) \cong A_5$ where A_5 is the alternating group of degree five and so we find $G = \mathrm{Aut}(\mathrm{PSL}(2, 4)) \cong \mathrm{PGL}(2, 5) \cong S_5$ since $|\mathrm{Aut}(\mathrm{PSL}(2, 4)) : \mathrm{PSL}(2, 4)| = 2$. Indeed, $\Delta(G) = \Gamma(G)$ for $G \cong \mathrm{PGL}(2, 5)$ and these graphs have two connected components.

Now we may suppose that $p > 2$. Since $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong A_5$, we may assume that $q > 5$. Since $\Delta(\mathrm{PSL}(2,q))$ has two connected components by Theorem 3.1 of [6] and $\Gamma(\mathrm{PSL}(2,q))$ has three connected components, $\Delta(K) \neq \Gamma(K)$ and so $\mathrm{PSL}(2,q) \cong K < G \leqslant \mathrm{Aut}(K)$. Thus G is one of the groups (a), (b), (c) of Lemma 3.1. If $G = \mathrm{PGL}(2,q)$ then we know that $\mathrm{cd}(G) = \{1, q, q-1, q+1\}$ and $\mu(G) = \{p, q-1, q+1\}$ where $\mu(G)$ is the subset of elements in the set of orders of elements in G which are maximal under the divisibility relation. Therefore, we see that $\Delta(\mathrm{PGL}(2,q)) = \Gamma(\mathrm{PGL}(2,q))$ and this graph has two connected components. Now assume $G \neq \mathrm{PGL}(2,q)$. If G is one of the groups (a) and (b), then $\varphi^k \in G$ for some $k \mid f$ with $1 \leqslant k < f$ by Lemma 3.1. Since $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\varphi^k} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, φ^k centralizes an element of order p in G. On the other hand, p does not divide the index $|G:K|$ and p is an isolated vertex in $\Delta(G)$, and so $\Gamma(G)$ by Theorem 3.2. Therefore, the order of φ^k is a power of p. But this is a contradiction since $1 \neq \varphi^k K \in G/K$. Now let G be as in (c) of Lemma 3.1. Thus, $G = K\langle \delta\varphi^k \rangle$ for some proper divisor k of f. If $f \neq 2k$, then G has the element $1 \neq \varphi^{2k}$ outside K, and centralizes an element of order p of K, which is a contradiction. Thus, $f = 2k$ and so $|G:K| = 2$. In this case, we find $\Gamma(G) = \Gamma(K)$ since every involution of G lies in K. But this is also a contradiction since $\Gamma(K)$ has three connected components.

(b) If $N > 1$ then by (iv), either $K \cong \mathrm{SL}(2,q)$ or there exists a normal subgroup L of G such that $K/L \cong \mathrm{SL}(2,q)$, L is elementary abelian of order q^2, and K/L acts transitively on the nonprincipal characters in $\mathrm{Irr}(L)$.

Suppose that $K \cong \mathrm{SL}(2,q)$. In this case $p \neq 2$. Otherwise, $K \cong \mathrm{SL}(2,q) \cong \mathrm{PSL}(2,q) \cong K/N$ and so N would be trivial. This contradiction shows that $p \neq 2$. Thus we may assume that $q > 5$. Since $Z(K) \cong Z(\mathrm{SL}(2,q)) > 1$, K is a proper subgroup of G. $K/N \cong \mathrm{PSL}(2,q)$ and $K \cong \mathrm{SL}(2,q)$ yield that the order of N is 2. Then we conclude that $\Gamma(G/N)$ is also disconnected since $\pi(G/N) = \pi(G)$ and $\Gamma(G)$ is disconnected. Thus, the center of G/N is trivial. By using (ii), we obtain that $\mathrm{PSL}(2,q) \cong K/N < G/N \leqslant \mathrm{Aut}(K/N)$. Furthermore, $\Delta(G/N)$ is disconnected by Lemma 3.1 of [4] . So, p does not divide $|G:K|$ and the connected components of $\Delta(G/N)$ are $\{p\}$ and $\pi(|G:K|(q^2-1))$ by Theorem 3.2. Thus, by Corollary 3.2. of [4], p is an isolated vertex in $\Delta(G)$. But $\mathrm{SL}(2,q) \cong K$, a subgroup of G, contains an element with order $2p$ and so p is not an isolated vertex in $\Gamma(G)$ (recall that we consider the case where p is not 2). This contradicts the assumption $\Delta(G) = \Gamma(G)$.

Now we may suppose that G has a normal elemantary abelian subgroup L with order q^2 such that $K/L \cong \mathrm{SL}(2,q)$. Let $1 \neq \upsilon \in \mathrm{Irr}(L)$ and set $T = I_G(\upsilon)$ (the inertia group of υ in G). Since the action of K/L on $\mathrm{Irr}(L) - \{1\}$ is transitive, we have $|K : K \cap T| = q^2 - 1$.

First, assume that $p = 2$. Then we know that $q^2 - 1 \in \mathrm{cd}(K)$ by (vi). Suppose $K = G$, then T is a Sylow 2-subgroup of G and $q^2 - 1$ is an irreducible character degree of G. It implies that $\Delta(G)$ has two complete connected components, $\{2\}$ and $\pi(q^2-1)$ since $\pi(G) = \{2\} \cup \pi(q^2-1)$ and $\Delta(G)$ $(= \Gamma(G))$ is disconnected. Thus there exists an element g in G such that $o(g) = ab$ where $a \in \pi(q-1)$ and $b \in \pi(q+1)$. This implies that $G/L \cong \mathrm{PSL}(2, q)$ has an element with order ab. But this contradicts the fact that $\Gamma(\mathrm{PSL}(2, q))$ has three connected components, $\{2\}$, $\pi(q-1)$ and $\pi(q+1)$. Thus K is proper in G. Since $\pi(G/L) = \pi(G)$ and $\Gamma(G)$ is disconnected, we see that $\Gamma(G/L)$ is also disconnected and so the center of G/L is trivial. Thus, by (ii), $C_{G/L}(K/L) = 1$ and so $K/L < G/L \leqslant \mathrm{Aut}(K/L)$. $\Delta(G/L)$ is also disconnected by Lemma 3.1 of [4]. If $q > 5$, then 2 does not divide the index $|G : K|$ and $\Delta(G/L)$ has exactly two connected components, $\{2\}$ and $\pi(|G : K|(q^2 - 1))$ by Theorem 3.2. Thus, by Corollary 3.2 of [4], 2 is an isolated vertex in $\Delta(G)$ and so in $\Gamma(G)$. Then, by Theorem 2.1 and Corollary 4.2, we see that G/K is a 2-group. This forces that $G = K$ which is a contradiction. So $q = 4$. In this case, $K/L \cong \mathrm{SL}(2, 4) \cong \mathrm{PSL}(2, 4) \cong A_5$. Since $K/L < G/L \leqslant \mathrm{Aut}(K/L) \cong S_5$, we have $G/L \cong S_5$. Since $\pi(G/L) = \pi(G)$ and $\Delta(G)$ is disconnected, we find that $\Delta(G) = \Delta(S_5)$, which is the disconnected graph with two complete connected components, $\{2, 3\}$ and $\{5\}$. But this is a contradiction because we have $15 \in \mathrm{cd}(K)$ by (vi) and so by the normality of K in G, there exists an edge between the primes 3 and 5 in $\Delta(G)$.

Now we consider in the case where $p \neq 2$. We may assume that $K < G$. Otherwise, we find the contradiction that $\Gamma(G)$ is connected since $\pi(G/L) = \pi(G)$ and $\Gamma(G/L)$ is connected. We also know that $|N| = 2|L|$ and so $\pi(G/N) = \pi(G)$. Thus $\Gamma(G/N)$ is disconnected and so $Z(G/N) = 1$. By (ii), $C_{G/N}(K/N) = 1$ and so $K/N < G/N \leqslant \mathrm{Aut}(K/N)$. By Lemma 3.1 of [4], $\Delta(G/N)$ is also disconnected. Therefore, p does not divide the index $|G : K|$ and $\Delta(G/N)$ has exactly two connected components, $\{p\}$ and $\pi(|G : K|(q^2 - 1))$ by Theorem 3.2. Thus, by Corollary 3.2 of [4], p is an isolated vertex in $\Delta(G)$ and so in $\Gamma(G)$. Finally, we see that p is an isolated vertex in $\Gamma(G/L)$. But this is a contradiction since $\mathrm{SL}(2, q) \cong K/L < G/L$ and so G/L has an element of order $2p$. So we are done with the proof of Theorem 1.1. $\square$

Let G be a nonsolvable finite group with $\Delta(G) = \Gamma(G)$. By the proof of Theorem 1.1, we understand that $\Delta(G)$ has two connected components if and only if G is isomorphic to $\mathrm{PGL}(2, q)$ where $5 \leqslant q$ is odd, and $\Delta(G)$ has three connected components if and only if G is isomorphic to $\mathrm{PSL}(2, 2^n)$ for an integer $n \geqslant 2$.

Now we deal with the solvable case which is Theorem 1.2.

Proof of Theorem 1.2. Let G be a finite solvable group with disconnected $\Delta(G) = \Gamma(G)$. Since $\Delta(G)$ is a disconnected graph, we know that G belongs to one of the families (i)–(vi) in Theorem 2.4. First assume that G satisfies the hypotheses of (i). Since $1 < P' \leqslant C_P(K)$ and K is abelian, we find that $\Gamma(G)$ is complete. But this contradicts the hypothesis that $\Gamma(G)$ is disconnected.

Let G satisfy the hypotheses of (ii). We know that $\Delta(G)$ has two connected components, $\{2\}$ and $\{3\}$ by Lemma 3.2 of [2], $\pi(|G|) = \{2, 3\}$ since $\Delta(G) = \Gamma(G)$. Thus we find that $\pi(|H|) = \{2, 3\}$ and so $Z(H) = 1$ since $\Delta(G) = \Gamma(G)$ is disconnected and $\mathrm{cd}(H) = \{1, 2, 3\}$. Finally, $H \cong \mathrm{SL}(2, 3)$ since $C_H(P) \leqslant Z(H) = 1$. But this is not possible because there exists an element of order 6 in $\mathrm{SL}(2, 3)$.

If G satisfies the hypotheses of (iii) then $\Delta(G)$ has two connected components, $\{2\}$ and $\{3\}$ by Lemma 3.3 of [2]. Similarly, we find that $H \cong \mathrm{GL}(2, 3)$, but this is also a contradiction. So G cannot be of type (iii).

Now suppose that G satisfies the hypotheses of (v). $\Delta(G)$ has two connected components, $\{2\}$ and $\pi(2^a+1)$ by Lemma 3.5 of [2]. But in this case $\Gamma(G)$ is complete, since $1 < Q' \leqslant C_Q(K)$. So we find that $\Gamma(G)$ does not coincide with $\Delta(G)$.

Finally we will assume that G satisfies the hypotheses of (vi) and look for a contradiction. We know that any solvable group with a disconnected prime graph is a Frobenius group or 2-Frobenius group. Thus G is a 2-Frobenius group by Corollary 4.2. Write F and E/F for the Fitting subgroups of G and G/F, respectively. By Lemma 3.6 of [2], we can see that $F = P$ and $E = T = PQ$ where P is a normal Sylow p-subgroup of G and Q is a p-complement. Thus E is a Frobenius group with the kernel P since G is a 2-Frobenius group. It follows that p is an isolated vertex in $\Gamma(E)$ and so in $\Gamma(G)$ since E is a normal Hall subgroup of G. But this is not possible since $\Delta(G) = \Gamma(G)$ has two connected components, $\pi([E : F]) \cup \{p\}$ and $\pi([D : C_D(T')])$ by Lemma 3.6 of [2].

Now, let G be as in (iv). In this case, G is a 2-Frobenius group. Write F and E/F for the Fitting subgroups of G and G/F respectively. We see that $C_H(V) = Z(G) = 1$, $F = V$ and $E = VF(H)$ by Lemma 3.4 of [2]. Groups $G/V (\cong H)$ and $E = VF(H)$ are Frobenius groups since G is a 2-Frobenius group. Moreover, $G/E (\cong H/F(H))$ and $E/V (\cong F(H))$ are cyclic. Therefore, there exist $x, y \in H$ such that $H = \langle x \rangle \rtimes \langle y \rangle$. Finally, we find that G is the semi-direct product of a Frobenius subgroup $H := \langle x \rangle \rtimes \langle y \rangle$ acting on an elementary abelian p-group V for some prime p, $C_H(V) = 1$, $\langle x \rangle$ acts irreducibly on V, $|V| = q^{o(y)}$ where q is a p-power, $p \mid o(y)$ and $(q^{o(y)} - 1)/(q - 1) \mid o(x)$ as desired. Conversely, for any group G of this type, $\Delta(G)$ coincides with $\Gamma(G)$ and these two graphs have two connected components, $\pi(o(x))$ and $\pi(o(y))$. □

Corollary 4.3. *Let K be a finite solvable group where $\Delta(K) = \Gamma(K)$ is square. Then $K = A \times B$ where A and B, normal Hall subgroups of K, belong to the following family, say $(**)$:*

"G is the semi-direct product of a Frobenius subgroup $H := \langle x\rangle \rtimes \langle y\rangle$ acting on an elementary abelian p-group V for some prime p, $C_H(V)=1$, $\langle x\rangle$ acts irreducibly on V, $o(x)$ is a power for some prime r, $|V| = q^{o(y)}$ where q and $o(y)$ are a p-power and $(q^{o(y)}-1)/(q-1) \mid o(x)$".

Proof. Let K be a finite solvable group where $\Delta(K) = \Gamma(K)$ is square with vertex set $\varrho(K) = \{p, r, q, s\}$ and edge set $\{pq, ps, rq, rs\}$. By [3], we know that $K = A \times B$ where $\varrho(A) = \{p, r\}$ and $\varrho(B) = \{q, s\}$. A is the normal Hall $\{p, r\}$-subgroup of K and B is the normal Hall $\{q, s\}$-subgroup of K since $\Delta(K) = \Gamma(K)$ is square and $K = A \times B$. It follows that $\Delta(A) = \Gamma(A)$ is the disconnected graph with connected components $\{p\}$, $\{r\}$. Similarly $\Delta(B) = \Gamma(B)$ is the disconnected graph with the connected components $\{q\}$, $\{s\}$. Thus A and B belong to the family $(**)$ by Theorem 1.2. □

Corollary 4.4. *Let K be a finite solvable group with $F = F(K)$ abelian. Suppose that $\Delta(K) = \Gamma(K)$ and there is no complete vertex in $\Delta(K)$. Then $K = D_1 \times \ldots \times D_n$ where D_i, normal Hall subgroups of K, belong to the family $(*)$ of Theorem 1.2 for all i.*

Proof. By [5], we know that for an integer n, $F = M_1 \times \ldots \times M_n \times Z(K)$ with $M_1, \ldots, M_n$ minimal normal subgroups of K and, moreover, $K = D_1 \times \ldots \times D_n$ where $M_i \leqslant D_i$ and $\Delta(D_i)$ is disconnected for all i. Since there is no complete vertex in $\Delta(K)$ $(= \Gamma(K))$, we see that $D_1, \ldots, D_n$ are Hall subgroups of K and $\pi(|K|) = \varrho(K) = \varrho(D_1) \cup \ldots \cup \varrho(D_n)$ so that $\varrho(D_i) \cap \varrho(D_j) = \emptyset$ for every $1 \leqslant i \neq j \leqslant n$. Thus we find that $\varrho(D_i) = \pi(D_i)$ and so $\Delta(D_i) = \Gamma(D_i)$ for all i since $\Delta(K) = \Gamma(K)$. As $\Delta(D_i) = \Gamma(D_i)$ is disconnected, D_i belongs to the family $(*)$ of Theorem 1.2 for all i. □

Corollary 4.5. *Let G be a finite group and $\Delta(G) = \Gamma(G)$. Suppose that $G = D_1 \times \ldots \times D_n$ where D_i is the normal Hall subgroup of G and $\Delta(D_i)$ is disconnected for all i.*

(a) *If G is solvable, then D_i belongs to the family $(*)$ of Theorem 1.2 for all i.*
(b) *If G is nonsolvable, then there exists only one normal Hall subgroup D_j such that $D_j \cong \mathrm{PSL}(2, 2^n)$ (for an integer $n \geqslant 2$) or $D_j \cong \mathrm{PGL}(2, q)$ ($5 \leqslant q$ is odd) and for all $i \neq j$, D_i belongs to the family $(*)$ of Theorem 1.2.*

Proof. It follows from the main theorems. □

We close this paper by asking a question which we are not able to answer. Which finite groups satisfy the property $\Delta(G) = \Gamma(G)$?

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