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ON REALIZABILITY OF SIGN PATTERNS BY REAL POLYNOMIALS

Vladimir Kostov, Nice

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Abstract. The classical Descartes' rule of signs limits the number of positive roots of a real polynomial in one variable by the number of sign changes in the sequence of its coefficients. One can ask the question which pairs of nonnegative integers (p, n), chosen in accordance with this rule and with some other natural conditions, can be the pairs of numbers of positive and negative roots of a real polynomial with prescribed signs of the coefficients. The paper solves this problem for degree 8 polynomials.

Keywords: real polynomial in one variable; sign pattern; Descartes' rule of signs

MSC 2010: 26C10, 30C15

1. Formulation of the problem and of the results

The classical Descartes' rule of signs states that a real polynomial in one variable does not have more real positive roots than the number of sign changes in the sequence of its coefficients. Any sequence of $\pm$-signs $\overline{\sigma} := (\sigma_0, \sigma_1, \dots, \sigma_d)$ is called a *sign pattern*. In the present paper we consider sign patterns defined by the signs of the coefficients of degree d polynomials P, so in particular $\sigma_d = \mathrm{sign}(P(0))$. For a given sign pattern its *Descartes' pair* $(p_{\overline{\sigma}}, n_{\overline{\sigma}})$ is the number of sign changes and sign preservations in the sequence of coefficients. Denote by $(\mathrm{pos}_P, \mathrm{neg}_P)$ the numbers of positive and negative roots of P counted with multiplicity. Hence, the following restrictions must hold true:

$$\mathrm{pos}_P \leqslant p_{\overline{\sigma}}, \quad \mathrm{neg}_P \leqslant n_{\overline{\sigma}}, \quad \mathrm{pos}_P \equiv p_{\overline{\sigma}} \pmod 2, \quad \mathrm{neg}_P \equiv n_{\overline{\sigma}} \pmod 2. \tag{1.1}$$

(The inequality $\mathrm{neg}_P \leqslant n_{\overline{\sigma}}$ follows from Descartes' rule applied to the polynomial $P(-x)$.) Pairs $(\mathrm{pos}, \mathrm{neg})$ satisfying conditions (1.1) are called *admissible* for the sign pattern $\overline{\sigma}$ (and the latter is *admitting* them).

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The present paper finishes the study which was begun in [3] of sign patterns and their admissible pairs for polynomials of degree up to 8. The present introduction reproduces with some small modifications the one of [3] and the results obtained in that paper, see Theorems 1.1, 1.2 and 1.3. The new results are given in Theorem 1.4 and then presented in another way (suitable to be compared to the previously obtained ones) at the end of this section.

Clearly conditions (1.1) are only necessary ones, i.e. for a given sign pattern $\overline{\sigma}$ and an admissible pair (p, n) it is not a priori clear whether there exists a degree d polynomial with this sign pattern and with exactly p distinct positive and exactly n distinct negative roots. If such a polynomial exists, then we say that the given combination of sign pattern and admissible pair is *realizable*.

Notation 1.1. For a given sign pattern $\overline{\sigma}$ we define its corresponding *reverted* sign pattern $\overline{\sigma}^r$ as $\overline{\sigma}$ read from the back and by $\overline{\sigma}_m$ the sign pattern obtained from the given one by changing the signs in second, fourth, etc. position while keeping the other signs the same. If $\overline{\sigma}$ is defined by a degree d polynomial $P(x)$, then $\overline{\sigma}^r$ is the sign pattern of $x^d P(1/x)$ and $\overline{\sigma}_m$ is the one of $(-1)^d P(-x)$.

Example 1.1. For $d = 4$ the sign pattern $(+, -, -, -, +)$ is equal to $(+, -, -, -, +)^r$ and one has $(+, -, -, -, +)_m = (+, +, -, +, +) = (+, -, -, -, +)^r_m$. For $d = 8$ the sign pattern $(+, +, -, +, -, -, -, -, +)$ is equal to $(+, +, -, +, -, -, -, -, +)^r_m$.

Remark 1.1.

(1) In what follows we assume that the leading coefficients of the polynomials are positive, so sign patterns (except in some places of the proofs) begin with $+$.

(2) It is clear that $(\overline{\sigma}^r)^r = \overline{\sigma}$, $(\overline{\sigma}_m)_m = \overline{\sigma}$ and $(\overline{\sigma}^r)_m = (\overline{\sigma}_m)^r$ (so we write simply $\overline{\sigma}^r_m$).

(3) The sign patterns and admissible pairs $(\overline{\sigma}, (p, n))$, $(\overline{\sigma}^r, (p, n))$, $(\overline{\sigma}_m, (n, p))$ and $(\overline{\sigma}^r_m, (n, p))$ are either all realizable or none of them. Therefore it makes sense to consider the question of realizability of given sign patterns with given admissible pairs modulo the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action defined by $\overline{\sigma} \mapsto \overline{\sigma}^r$ and $\overline{\sigma} \mapsto \overline{\sigma}_m$.

It seems that for the first time the question of realizability of sign patterns with admissible pairs has been asked in [2]. In [4] Grabiner has obtained the first example of nonrealizability. Namely, he has shown that for $d = 4$ the sign pattern $(+, -, -, -, +)$ is not realizable with the admissible pair $(0, 2)$ (Descartes' pair of the pattern equals $(2, 2)$). In [1] Albouy and Fu have given the exhaustive answer to this question of realizability for degrees not greater than 6. In Theorems 1.1, 1.2 and 1.3 we change at some places (with respect to the original formulations in [1] or [3]) a sign pattern σ to σ_m and the corresponding pair (p, n) to (n, p) in order to have mostly pairs of the form $(0, n)$ in the formulations:

Theorem 1.1.

(1) *For degree* 1, 2 *and* 3 *any sign pattern is realizable with any of its admissible pairs.*

(2) *For degree* 4 *the only case of nonrealizability (up to the standard* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-action) is the one of Grabiner's example.*

(3) *For degree* 5 *the only such case is given by the sign pattern* $(+,-,-,-,-,+)$ *with the pair* $(0,3)$.

(4) *For degree* 6 *the only such cases are:* $(+,-,-,-,-,-,+)$ *with* $(0,2)$ *or* $(0,4)$*;* $(+,-,+,-,-,-,+)$ *with* $(0,2)$*;* $(+,+,-,-,-,-,+)$ *with* $(0,4)$.

The cases $d = 7$ and $d = 8$ have been considered in [3]. The exhaustive answer to the question of realizability for $d = 7$ is as follows:

Theorem 1.2. *For* $d = 7$ *there are* 1472 *cases (modulo the standard* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-action) of sign pattern and admissible pair. Of these exactly* 6 *are not realizable:* $(+,+,-,-,-,-,-,+)$, $(+,+,-,-,-,-,+,+)$ *and* $(+,+,+,-,-,-,-,+)$ *with* $(0,5)$*;* $(+,-,-,-,-,+,-,+)$ *with* $(0,3)$*;* $(+,-,-,-,-,-,-,+)$ *with* $(0,3)$ *and* $(0,5)$.

For $d = 8$ the partial answer from [3] can be summarized by the following theorem. In [3] this result is formulated differently, but equivalently. In particular, the authors of [3] have not noticed that the number of cases for which the answer still remained unknown can be decreased by one due to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action.

Theorem 1.3. (1) *For* $d = 8$ *there are* 3648 *possible combinations of sign pattern and admissible pair (up to the standard* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-action). Of these exactly* 13 *are known to be nonrealizable:*

$(+,+,-,-,-,-,-,+,+)$, $(+,-,-,-,-,-,-,+,+)$, $(+,+,+,+,-,-,-,-,+)$ *and* $(+,+,+,-,-,-,-,-,+)$ *with* $(0,6)$*;*

$(+,-,+,-,-,-,+,-,+)$ *and* $(+,-,+,-,+,-,-,-,+)$ *with* $(0,2)$*;*

$(+,-,+,-,-,-,-,-,+)$ *and* $(+,-,-,-,+,-,-,-,+)$ *with* $(0,2)$ *and* $(0,4)$*;*

$(+,-,-,-,-,-,-,-,+)$ *with* $(0,2)$, $(0,4)$ *and* $(0,6)$.

(2) *For exactly* 6 *another cases it is not known whether they are realizable or not (for Cases* 2–6 *the formulation in* [3] *uses the pair* $(4,0)$*, we add the corresponding sign pattern* $(\sigma_j)_m$ *with the pair* $(0,4)$ *for the reader's convenience):*

$$\begin{aligned}
&\textit{Case } 1: && \sigma_1 := (+,+,+,-,-,-,-,+,+) && \textit{with } (0,6);\\
&\textit{Case } 2: && \sigma_2 := (+,+,-,+,-,-,-,+,+) && \textit{with } (4,0),\\
& && (\sigma_2)_m := (+,-,-,-,-,+,-,-,+) && \textit{with } (0,4);
\end{aligned}$$

$$
\begin{aligned}
&\textit{Case } 3: && \sigma_3 := (+,+,-,+,-,+,-,-,+) && \textit{with } (4,0),\\
& && (\sigma_3)_m := (+,-,-,-,-,-,-,+,+) && \textit{with } (0,4);\\
&\textit{Case } 4: && \sigma_4 := (+,+,+,-,-,+,-,+,+) && \textit{with } (4,0),\\
& && (\sigma_4)_m := (+,-,+,+,-,-,-,-,+) && \textit{with } (0,4);\\
&\textit{Case } 5: && \sigma_5 := (+,+,+,+,-,+,-,-,+) && \textit{with } (4,0),\\
& && (\sigma_5)_m := (+,-,+,-,-,-,-,+,+) && \textit{with } (0,4);\\
&\textit{Case } 6: && \sigma_6 := (+,+,-,+,-,-,-,-,+) && \textit{with } (4,0),\\
& && (\sigma_6)_m := (+,-,-,-,-,+,-,+,+) && \textit{with } (0,4).
\end{aligned}
$$

The aim of the present paper is to definitely settle the case $d = 8$. Namely, we prove the following theorem:

Theorem 1.4. *The* 6 *cases of part* (2) *of Theorem* 1.3 *are not realizable.*

For Case 1 the proof is given in Section 2. Cases 2–6 are considered in Section 3. The proofs of Lemmas 3.1 and 3.2 formulated in Section 3 are given in the Appendix. In the proof of the theorem we sometimes use sign patterns having as components not only $+$ and/or $-$, but also 0 (in the sense that the corresponding coefficient equals 0), and in some cases $\pm$ meaning that we consider the cases with $+$ and $-$ together.

As we see, in all cases of nonrealizability one of the components of the admissible pair equals 0. To finish this section we list the nonrealizable cases for $d = 8$ by their pairs (p, n); the third and the fifth columns contain the corresponding Descartes' pairs. In order to have only the pairs $(0, 2)$, $(0, 4)$ and $(0, 6)$ as defining the classification we use for Cases 2–6 of Theorem 1.3 the corresponding patterns $(\sigma_j)_m$. To find Cases 1–6 in the table more easily we give their numbers as indices to the corresponding sign patterns.

$(0,2)$	$(+,-,+,-,-,-,+,-,+)$	$(6,2)$	$(+,-,+,-,+,-,-,-,+)$	$(6,2)$
	$(+,-,+,-,-,-,-,-,+)$	$(4,4)$	$(+,-,-,-,+,-,-,-,+)$	$(4,4)$
	$(+,-,-,-,-,-,-,-,+)$	$(2,6)$		
$(0,4)$	$(+,-,+,-,-,-,-,-,+)$	$(4,4)$	$(+,-,-,-,+,-,-,-,+)$	$(4,4)$
	$(+,-,-,-,-,-,-,-,+)$	$(2,6)$	$(+,-,-,-,-,+,-,-,+)_2$	$(4,4)$
	$(+,-,-,-,-,-,-,+,+)_3$	$(2,6)$	$(+,-,+,+,-,-,-,-,+)_4$	$(4,4)$
	$(+,-,+,-,-,-,-,+,+)_5$	$(4,4)$	$(+,-,-,-,-,+,-,+,+)_6$	$(4,4)$

$(0,6)$	$(+,-,-,-,-,-,-,-,+)$	$(2,6)$	$(+,+,-,-,-,-,-,+,+)$	$(2,6)$
	$(+,-,-,-,-,-,-,+,+)$	$(2,6)$	$(+,+,+,+,-,-,-,-,+)$	$(2,6)$
	$(+,+,+,-,-,-,-,-,+)$	$(2,6)$	$(+,+,+,-,-,-,-,+,+)_1$	$(2,6)$

Remark 1.2. (1) When the sign pattern consists of a sequence of m_1 pluses followed by a sequence of m_2 minuses and then by a sequence of m_3 pluses, where $m_1+m_2+m_3=d+1$, then for the pair $(0,d-2)$ this sign pattern is not realizable if $\kappa := (d-m_1-1)(d-m_3-1)/m_1m_3 \geqslant 4$ (see Proposition 6 in [3]). For the sign patterns with $(0,6)$ in the above table, the quantity κ equals respectively 36, 25/4, 15, 9/2, 8 and $20/6<4$. The last inequality shows that Proposition 6 of [3] gives only sufficient, but not necessary conditions for nonrealizability of the pair $(0,d-2)$ with the sign patterns containing only two sign changes.

(2) In the problem which we consider, an important role is played, although this is not always explicitly pointed out, by the *discriminant set* of the family of monic polynomials. This is the set of values of the coefficients for which the polynomial has a multiple root. The number of real roots changes, generically by 2, when the tuple of coefficients crosses the discriminant set. The stratification of the discriminant set is explained in [6]. More about discriminants of the general family of univariate polynomials for degree 4 or 5 can be found in [5].

2. Case 1 is not realizable

The proof that the sign pattern σ_1 is not realizable with the pair $(0,6)$ follows from Lemmas 2.2 and 2.3. The following lemma is used in the proof of Lemma 2.2. It is a particular case of Proposition 1 of [1], but we include its proof to make the text self-contained.

Lemma 2.1. *For any $0<u<v$ there exists a polynomial $R=x^8+ax^7+bx^6+cx+d$, where $a>0$, $b>0$, $c>0$, $d>0$ and $R(-u)=R'(-u)=R(-v)=R'(-v)=0$. Hence, by Descartes' rule of signs this polynomial equals $(x+u)^2\times(x+v)^2S(x)$, where the monic degree 4 polynomial S has no real roots.*

Proof. Consider the system of linear equations with unknown variables a, b, c and d and parameters $u>0$ and $v>0$:

$$\begin{aligned} u^8-au^7+bu^6-cu+d&=0, \qquad 8u^7-7au^6+6bu^5-c=0,\\ v^8-av^7+bv^6-cv+d&=0, \qquad 8v^7-7av^6+6bv^5-c=0. \end{aligned}$$

One can solve this system with respect to a, b, c and d (using, say, MAPLE) and express the solutions as functions of u and v. Set

$$g := 35u^4v^4 + 20u^3v^5 + 4u^7v + 10u^2v^6 + u^8 + 4uv^7 + 20u^5v^3 + v^8 + 10u^6v^2.$$

Then

$$\begin{aligned} a &= \frac{2}{g}(u^9 + 4u^8v + 10v^2u^7 + 20v^3u^6 + 35v^4u^5 \\ &\quad + 35v^5u^4 + 20v^6u^3 + 10v^7u^2 + 4v^8u + v^9), \\ b &= \frac{1}{g}(u^{10} + 4vu^9 + 10u^8v^2 + 35u^4v^6 + 20u^7v^3 + 20u^3v^7 \\ &\quad + 35u^6v^4 + 10u^2v^8 + 56u^5v^5 + 4uv^9 + v^{10}), \\ c &= \frac{2u^5v^5}{g}(5vu^4 + 6v^2u^3 + 6v^3u^2 + 3u^5 + 5v^4u + 3v^5), \\ d &= \frac{u^6v^6}{g}(5u^4 + 8u^3v + 9u^2v^2 + 8uv^3 + 5v^4). \end{aligned}$$

All coefficients being positive, if one gives positive values to u and v ($u \neq v$), one obtains positive values of a, b, c and d. □

Lemma 2.2. *If the sign pattern σ_1 is realizable with the pair $(0, 6)$, then there exists a real monic degree 8 polynomial H having three double negative and one double positive root and the sign pattern σ_1.*

Proof. Suppose that the sign pattern σ_1 is realizable with the pair $(0, 6)$ by a real degree 8 polynomial P with six distinct negative roots and a complex conjugate pair. One can suppose that the values of P at its negative critical points are all distinct. One can increase the constant term of P (which does not change the sign pattern) so that two of the negative roots coalesce in a double negative root α which is a local minimum of P.

Denote by $\tau < 0$ and $\kappa < 0$ the other two minima of P on the negative half-axis (one has $P(\tau) < 0$ and $P(\kappa) < 0$).

Denote by R_1 the polynomial of Lemma 2.1 with $u = -\alpha$, $v = -\tau$. Then for $\varepsilon > 0$ small enough the polynomial $T := P + \varepsilon R_1$ has five distinct negative roots (four simple and one double). For some positive value of $\varepsilon = \varepsilon_0$ the polynomial T has a double root at κ as well. As the value of T for each fixed $x > 0$ increases with ε, T has no real positive root.

Consider now the polynomial $T_0 := P + \varepsilon_0 R_1$. Denote by R_2 the polynomial of Lemma 2.1 with $u = -\alpha$, $v = -\kappa$. For some positive value of η the polynomial $T^* := T_0 + \eta R_2$ has double roots at α, κ and τ, no positive root and has the sign pattern σ_1.

Set $W := (x-\alpha)^2(x-\kappa)^2(x-\tau)^2$. Consider the polynomial $T^* - \mu W$, $\mu > 0$. All coefficients of W are positive. Therefore the sign pattern defined by $T^* - \mu W$ has minuses in the positions in which σ_1 has such. As $T^* - \mu W$ has six negative roots counted with multiplicity, by Descartes' rule of signs the sign pattern defined by it has at most two sign changes.

The polynomial $T^* - \mu W$ for $\mu > 0$ small enough is of the form $(x-\alpha)^2 \times (x-\kappa)^2(x-\tau)^2((x-\delta)^2 + A)$, $\delta > 0$, $A > 0$. Indeed, if $\delta \leqslant 0$, then all coefficients of $T^* - \mu W$ would be positive and it will not define the sign pattern σ_1.

Decrease A. Denote by σ' the sign pattern defined by $T^* - \mu W$ when $A = 0$. When decreasing $A > 0$, the signs of the coefficients of x^j remain negative for $j = 2$, 3, 4 and 5. For $j = 0$, 1 and/or 6 they might change from $+$ to $-$. If σ' has more minuses than σ_1, then it has a sequence of m_1 pluses, $m_1 \leqslant 3$, followed by a sequence of m_2 minuses followed by a sequence of m_3 pluses, $m_3 \leqslant 2$, $m_1 + m_2 + m_3 = 9$ (because $T^* - \mu W$ has 6 negative roots and the sequence of its coefficients must have at least 6 sign preservations, i.e. not more than two sign changes).

One cannot have $m_1 < 3$ or $m_3 < 2$ for $A = 0$. Indeed, in this case one can slightly increase A without changing m_1, m_2 and m_3 and obtain a contradiction with Proposition 6 of [3], see part (1) of Remarks 1.2. Hence $m_1 = 3$, $m_3 = 2$ and for $A = 0$ the polynomial $T^* - \mu W$ defines the sign pattern σ_1, i.e. $\sigma' = \sigma_1$. □

Lemma 2.3. *There exists no real monic degree* 8 *polynomial having three double negative and one double positive root and defining the sign pattern* σ_1.

Proof. Assume that such a polynomial exists. Without loss of generality one can assume that it is the square of the polynomial

$$L := (x^3 + \alpha x^2 + \beta x + \gamma)(x-1) = x^4 + (\alpha-1)x^3 + (\beta-\alpha)x^2 + (\gamma-\beta)x - \gamma$$

in which the first factor has three distinct negative roots. Hence $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. The coefficient of x^s of L^2 is denoted by c_s. Hence

$$\begin{aligned} &c_7 = 2(\alpha-1), && c_6 = 2(\beta-\alpha) + (\alpha-1)^2, \\ &c_5 = 2((\gamma-\beta) + (\alpha-1)(\beta-\alpha)), && c_2 = (\gamma-\beta)^2 - 2(\beta-\alpha)\gamma, \\ &c_1 = -2\gamma(\gamma-\beta), && c_0 = \gamma^2. \end{aligned} \tag{2.1}$$

□

Remark 2.1. (1) As L^2 defines the sign pattern σ_1, one must have $c_7 > 0$ and $c_1 > 0$ from which follows $\alpha > 1$ and $\gamma < \beta$. These two inequalities combined with $c_2 < 0$ yield $\beta > \alpha$.

(2) The condition $\beta > \alpha$ implies that the absolute value of at least one of the roots of the polynomial $x^3 + \alpha x^2 + \beta x + \gamma$ (which are all negative) is greater than 1.

In what follows we denote by $\mathcal{P}$ the set $\{\alpha > 1, \beta > 0, \gamma > 0\}$. For each $\alpha = \alpha_0 > 1$ fixed, the set $\mathcal{P}|_{\alpha=\alpha_0}$ is the positive quadrant $\{\beta > 0, \gamma > 0\}$.

Lemma 2.4. *Suppose that $\alpha = \alpha_0 > 1$ is fixed. Then:*

(1) *The condition $c_5 = 0$ defines a straight line $\mathcal{C}_5$. Its slope $2 - \alpha_0$ is positive for $\alpha_0 \in (1, 2)$, zero for $\alpha_0 = 2$ and negative for $\alpha_0 > 2$. For $\alpha_0 > 2$ the intersection $(\mathcal{P}|_{\alpha=\alpha_0}) \cap \mathcal{C}_5$ is a segment.*

(2) *The condition $c_2 = 0$ defines a hyperbola with centre $(2\alpha_0/3, \alpha_0/3)$ and with asymptotes $\gamma - \alpha_0/3 = (2 \pm \sqrt{3})(\beta - 2\alpha_0/3)$. One of its branches (denoted by $\mathcal{C}_2$) belongs to the set $\mathcal{P}|_{\alpha=\alpha_0}$; the other one is denoted by $\mathcal{C}_2^*$. The point $(0, 0)$ belongs to $\mathcal{C}_2^*$ and the tangent line to $\mathcal{C}_2^*$ at $(0, 0)$ is horizontal. Hence $\mathcal{C}_2^* \cap (\mathcal{P}|_{\alpha=\alpha_0}) = \emptyset$.*

(3) *For $\alpha_0 > \sqrt{3}$ the intersection $\mathcal{C}_5 \cap \mathcal{C}_2$ consists of the two points*

$$I_1 := (\alpha_0, \alpha_0) \quad \text{and} \quad I_2 := (\alpha_0(\alpha_0^2 - 1)/(\alpha_0^2 - 3), \alpha_0(\alpha_0 - 1)^2/(\alpha_0^2 - 3)).$$

For $\alpha_0 \in (1, \sqrt{3}]$ one has $\mathcal{C}_5 \cap \mathcal{C}_2 = I_1$. The tangent line to $\mathcal{C}_2$ at I_1 is vertical, at I_2 its slope is negative for $\alpha_0 > 3$, zero for $\alpha_0 = 3$ and positive for $\alpha_0 \in (1, 3)$. For $\alpha_0 > 3$ this slope is negative for the points of $\mathcal{C}_2$ which are between I_1 and I_2.

(4) *The set of hyperbolic polynomials is defined by the condition*

$$4(\beta - \alpha_0^2/3)^3 + 27(\gamma + 2\alpha_0^3/27 - \alpha_0\beta/3)^2 \leqslant 0. \tag{2.2}$$

The corresponding equality defines a curve $\mathcal{H}$ having as only singular point a cusp at $J := (\alpha_0^2/3, \alpha_0^3/27)$. The set of hyperbolic polynomials is the closure of the interior of $\mathcal{H}$. The slope of the tangent lines to $\mathcal{H}$ at its regular points (and the one of the geometric semi-tangent at its cusp) is positive for $\beta > 0$, $\gamma > 0$. The maximal values of the coordinates of the restriction of $\mathcal{H}$ to $\{\beta > 0, \gamma > 0\}$ are attained, simultaniously for β and γ, at and only at its cusp.

(5) *The curve $\mathcal{H}$ intersects the line $\mathcal{C}_5$ exactly when $\alpha_0 \geqslant u_0 := 3.787042615\ldots$ For $\alpha_0 < u_0$ the cusp point J lies below the line $\mathcal{C}_5$. The point I_2 does not define a hyperbolic polynomial for any $\alpha_0 > 1$.*

Before proving Lemma 2.4 we finish the proof of Lemma 2.3. On Figure 1 we show the sets $\mathcal{C}_2$ (branch of a hyperbola), $\mathcal{C}_5$ (straight line with negative slope), the straight line $\{\beta = \gamma\}$ and $\mathcal{H}$ (curve with a cusp point) for $\alpha_0 = 5$. The set $\{c_2 < 0\}$ is the interior of the branch $\mathcal{C}_2$ and the set $\{c_2 < 0, c_5 < 0\}$ is the lens-shaped domain between $\mathcal{C}_2$ and $\mathcal{C}_5$. The point I_1 is the triple intersection of $\mathcal{C}_2$, $\mathcal{C}_5$ and $\{\beta = \gamma\}$.

Remark 2.2. For $\alpha_0 \in (1, \sqrt{3}]$ the set $\{c_2 < 0, c_5 < 0\}$ is not compact and for $\alpha_0 \in (1, \sqrt{3})$ the point I_2 belongs not to $\mathcal{C}_2$, but to $\mathcal{C}_2^*$; I_2 is at ∞ for $\alpha_0 = \sqrt{3}$. Indeed, the slopes of the asymptotes of the hyperbola $\{c_2 = 0\}$ equal $2 \pm \sqrt{3}$ while the slope of $\mathcal{C}_5$ equals $2 - \alpha_0$, see parts (1) and (2) of Lemma 2.4.

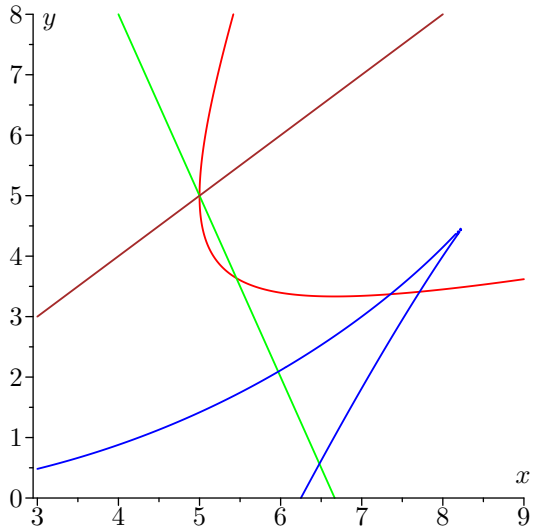

Figure 1. The sets $\mathcal{C}_2$, $\mathcal{C}_5$, $\{\beta = \gamma\}$ and $\mathcal{H}$.

There exists a unique point $Z \in \mathcal{C}_2$ to $\mathcal{C}_2$ at which the tangent is horizontal. Indeed, the branches $\mathcal{C}_2$ and $\mathcal{C}_2^*$ of the hyperbola $\{c_2 = 0\}$ are symmetric with respect to its centre $(2\alpha_0/3, \alpha_0/3)$, see part (2) of Lemma 2.4. The only point of $\mathcal{C}_2^*$ at which the tangent line is horizontal is the origin, see part (2) of Lemma 2.4 (the fact that $(0,0)$ is the only such point follows from the convexity of the hyperbola). Hence $Z = (4\alpha_0/3, 2\alpha_0/3)$.

Compare the γ-coordinates of the points Z and J (see part (4) of Lemma 2.4). For $\alpha_0 < 3\sqrt{2} = 4.2\ldots$ one has $2\alpha_0/3 > \alpha_0^3/27$. The point Z has the least possible γ-coordinate of the points of $\mathcal{C}_2$ whereas J has the largest possible γ-coordinate of the points of $\mathcal{H} \cap \mathcal{P}_{\alpha=\alpha_0}$, see part (4) of Lemma 2.4. Hence, for $\alpha_0 \in (1, 3\sqrt{2})$ one has

$$\mathcal{C}_2 \cap (\mathcal{H} \cap \mathcal{P}_{\alpha=\alpha_0}) = \emptyset \quad \text{and} \quad \{c_2 < 0, c_5 < 0\} \cap (\mathcal{H} \cap \mathcal{P}_{\alpha=\alpha_0}) = \emptyset.$$

Recall that $u_0 < 3\sqrt{2}$, see part (5) of Lemma 2.4. Hence, for $\alpha_0 = u_0$ the cusp J of $\mathcal{H}$ has a smaller γ-coordinate than I_2. As I_2 does not belong to $\mathcal{H}$ (for any $\alpha_0 > 1$, see part (5) of Lemma 2.4), for $\alpha_0 > u_0$ the points I_1 and I_2 are above the two intersection points K_1 and K_2 of $\mathcal{H}$ with $\mathcal{C}_5$ ("above" means "have larger γ-coordinates"); K_1 is presumed to be above K_2. Denote by L^* and L^{**} the vertical straight lines passing through I_2 and K_1. Hence for $a > 3$ the domain $\{c_2 < 0, c_5 < 0\}$ lies to the left of L^* and above I_2, see parts (1) and (3) of Lemma 2.4. At the same time the part of $\mathcal{H} \cap \mathcal{P}_{\alpha=\alpha_0}$ which is to the left of L^* (hence to the left of L^{**} as well) lies below K_1 hence below I_2, so the domain $\{c_2 < 0, c_5 < 0\}$ contains no hyperbolic polynomial. This proves Lemma 2.3 and Theorem 1.4. $\square$

P r o o f of Lemma 2.4. The first two statements of part (1) are to be checked directly. To prove the third statement it suffices to compute the intersection points of the line $\mathcal{C}_5$ with the β- and γ-axes. These points are $(0, \alpha_0(\alpha_0 - 1))$ and $(\alpha_0(\alpha_0 - 1)/(\alpha_0 - 2), 0)$.

Let us prove part (2). The determinants of the matrices $M_1 = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 1 & \alpha_0 \\ 0 & \alpha_0 & 0 \end{pmatrix}$ and $M_2 = \left(\begin{smallmatrix} 1 & -2 \\ -2 & 1 \end{smallmatrix}\right)$ (defined after the quadric $c_2|_{\alpha=\alpha_0}$) are nonzero and M_2 has one positive and one negative eigenvalue. Hence, the equation $c_2 = 0$ defines a hyperbola. To find its centre one sets $\beta \mapsto \beta + \mu$, $\gamma \mapsto \gamma + \nu$ and one looks for (μ, ν) such that the linear terms in the equation $c_2 = 0$ disappear. This yields the system

$$-4\mu + 2\nu + 2\alpha_0 = 0, \quad 2\mu - 4\nu = 0,$$

whose solution is $(\mu, \nu) = (2\alpha_0/3, \alpha_0/3)$. The slopes of the asymptotes are solutions to the equation $\lambda^2 - 4\lambda + 1 = 0$ deduced from the matrix M_2. The branch $\mathcal{S}_2$ occupies the upper right sector defined by the asymptotes.

The equation $c_2 = 0$ is satisfied for $(\beta, \gamma) = (0, 0)$. To compute the equation of the tangent line to the hyperbola $\{c_2 = 0\}$ one writes

$$(-4\beta + 2\gamma + 2\alpha)\,\mathrm{d}\gamma + (2\beta - 4\gamma)\,\mathrm{d}\beta = 0 \tag{2.3}$$

in which the coefficient of $\mathrm{d}\beta$ is 0 for $(\beta, \gamma) = (0, 0)$. The tangent at $(0, 0)$ being horizontal, the branch $\mathcal{S}_2^*$ belongs entirely to the lower half-plane and does not intersect the set $\mathcal{P}|_{\alpha=\alpha_0}$.

Prove part (3). Set $B := \gamma - \beta$, $A := \beta - \alpha_0$. The conditions $c_5 = 0$ and $c_2 = 0$ read (see (2.1)):

$$B = -(\alpha_0 - 1)A, \quad -2(B + A + \alpha_0)A + B^2 = 0,$$

from which one finds that either $A = 0$ (hence $B = 0$ and $\beta = \gamma = \alpha_0$, this defines the point I_1) or $-2(-(\alpha_0 - 1)A + A + \alpha_0) + (\alpha_0 - 1)^2 A = 0$. The last equality implies $A = 2\alpha_0/(\alpha_0^2 - 3)$. Hence

$$\beta = \alpha_0 \frac{\alpha_0^2 - 1}{\alpha_0^2 - 3}, \tag{2.4}$$

so $B = 2\alpha_0(1 - \alpha_0)/(\alpha_0^2 - 3)$ and

$$\gamma = \alpha_0 \frac{(\alpha_0 - 1)^2}{\alpha_0^2 - 3}. \tag{2.5}$$

which gives the point I_2. To show that the tangent line to $\mathcal{C}_2$ at I_1 is vertical it suffices to observe that for $\beta = \gamma = \alpha_0$, equation (2.3) reduces to $\mathrm{d}\beta = 0$. At I_2 the tangent line to $\mathcal{C}_2$ is defined by the equation

$$\frac{2\alpha_0^2}{\alpha_0^2 - 3}\,\mathrm{d}\gamma + \frac{\alpha_0(\alpha_0 - 1)(\alpha_0 - 3)}{\alpha_0^2 - 3}\,\mathrm{d}\beta = 0.$$

Its slope equals $-(\alpha_0 - 1)(\alpha_0 - 3)/2\alpha_0$. The last statement of part (3) follows from the convexity of the hyperbola $\{c_2 = 0\}$.

To prove part (4) one has to recall that the real polynomial x^3+px+q is hyperbolic if and only if $4p^3 + 27q^2 \leqslant 0$ (this means, in particular, that $p \leqslant 0$). As

$$x^3 + \alpha x^2 + \beta x + \gamma = \Big(x + \frac{\alpha}{3}\Big)^3 + \Big(\beta - \frac{\alpha^2}{3}\Big)\Big(x + \frac{\alpha}{3}\Big) + \gamma + \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3},$$

the polynomial $L|_{\alpha=\alpha_0}$ is hyperbolic if and only if condition (2.2) holds true.

Set $\beta \mapsto \alpha_0^2\beta$ and $\gamma \mapsto \alpha_0^3\gamma$ in the equation of $\mathcal{H}$ (see (2.2)). In the new variables (β, γ) the equation of $\mathcal{H}$ (after division by α_0^3) coincides with its equation for $\alpha_0 = 1$:

$$4\Big(\beta - \frac{1}{3}\Big)^3 + 27\Big(\gamma + \frac{2}{27} - \frac{\beta}{3}\Big)^2 = 0. \tag{2.6}$$

One can parametrize this curve by setting $\beta = 1/3 - 3t^2$, $\gamma = 1/27 + 2t^3 - t^2 = 2(t - 1/3)^2(t + 1/6)$. It has a cusp for $t = 0$, i.e. at $(1/3, 1/27)$. Its tangent vector equals $(-6t, 6t^2 - 2t)$. For $t < 0$ its components are both positive and its slope is also positive. For $t \in (0, 1/3)$ they are both negative and again the slope is positive. One has $\beta > 0$ and $\gamma > 0$ exactly when $t \in (-1/6, 1/3)$ (i.e. only for values of t for which the slope is positive). The coordinate β attains its global maximal value $1/3$ only for $t = 0$. For $t \in (-1/6, 1/3)$ the coordinate γ attains its maximal value $1/27$ only for $t = 0$.

Prove part (5). The equation of $\mathcal{H}$ with $\gamma = \beta - (\alpha_0 - 1)(\beta - \alpha_0)$ reads:

$$\begin{aligned}\mathcal{U}(\alpha_0, \beta) := {} & 4\beta^3 + 44\beta^2\alpha_0^2 - 4\beta\alpha_0^4 - 108\alpha_0\beta + 27\alpha_0^2 - 54\alpha_0^3 \\ & + 108\beta^2 + 180\beta\alpha_0^2 - 144\alpha_0\beta^2 - 64\beta\alpha_0^3 + 23\alpha_0^4 + 4\alpha_0^5 = 0.\end{aligned} \tag{2.7}$$

One has

$$\operatorname{Res}\Big(\mathcal{U}, \frac{\partial\mathcal{U}}{\partial\beta}, \beta\Big) = -64\alpha_0^3(\alpha_0 - 1)(2\alpha_0^2 - 7\alpha_0 + 8)(10\alpha_0^2 - 45\alpha - 0 + 27)^3.$$

The first quadratic factor has no real roots. The roots of the second one equal $0.7129573851\ldots < 1$ and $u_0 := 3.787042615\ldots$ For $\alpha_0 = u_0$ the cusp point of $\mathcal{H}$ is

on $\mathcal{C}_5$. For $\alpha_0 < u_0$ the curve $\mathcal{H} \cap \mathcal{P}|_{\alpha=\alpha_0}$ lies entirely below the line $\mathcal{C}_5$ (this can be deduced from the last statement of part (4) of the lemma and from the fact that for $\alpha_0 > 0$ small enough the cusp point J is close to the origin); for $\alpha_0 > u_0$ it intersects this line at two points.

Remark 2.3. Equation (2.7) is of degree 3 with respect to β. On Figure 1 one sees two of the solutions (the points K_1 and K_2, see the proof of Lemma 2.3). The third solution is an intersection point of $\mathcal{H}$ with $\mathcal{C}_5$, with $\beta < 0$ and $\gamma > 0$. Such an intersection point exists because the γ-coordinate of a point of $\mathcal{C}_5$ grows linearly in $|\beta|$ as $|\beta|$ increases (β being negative) while the γ-coordinate of a point of $\mathcal{H}$ grows as $|\beta|^{3/2}$.

To prove the last statement of part (5) we substitute the right-hand sides of (2.4) and (2.5) for β and γ in (2.2) and we multiply it by $(\alpha_0^2-3)^3/\alpha_0^2(\alpha_0-1)^2 > 0$. This yields the equivalent condition

$$3\alpha_0^6 - 16\alpha_0^5 + 13\alpha_0^4 + 24\alpha_0^3 - 23\alpha_0^2 + 104\alpha_0 - 81 \leqslant 0.$$

However, the left-hand side has no roots greater than 1 and the leading coefficient is positive. Hence, the last inequality fails for $\alpha_0 > 1$. □

3. Cases 2–6 are not realizable

3.1. Preliminaries. The following two lemmas are proved in the Appendix. They allow to simplify the proof of Theorem 1.3 by decreasing the number of parameters.

Lemma 3.1. *Suppose that there exists a monic degree 8 polynomial P realizing Case j, $2 \leqslant j \leqslant 6$. Then there exists a monic degree 8 polynomial U having a quadruple root at 1 and no other real roots, and whose coefficients define the same sign pattern as the one of Case j.*

Remark 3.1. One can observe that roots at 1 remain invariant under reverting of sign patterns.

Lemma 3.2. (1) *Suppose that a monic polynomial $U = (x-1)^4 V$ realizes one of the sign patterns*

$$\sigma_2^r = (+,+,-,-,-,+,-,+,+), \qquad \sigma_4 = (+,+,+,-,-,+,-,+,+),$$
$$\text{or} \qquad \sigma_6^r = (+,-,-,-,-,+,-,+,+),$$

where V is a real monic polynomial with no real root. Then there exists a polynomial of the form $U_t := U - t(x-1)^4$, $t \geqslant 0$, defining the same sign pattern and having one or two negative roots of even multiplicity, hence a polynomial of the form

$$(3.1)\quad W := (x-1)^4\Big(x^2 + Sx + \frac{S^2}{4}\Big)(x^2 + Mx + N), \quad \text{where } S > 0 \text{ and } N \geqslant \frac{M^2}{4}.$$

(2) *If the polynomial U realizes the sign pattern $\sigma_3^r = (+,-,-,+,-,+,-,+,+)$, then in the family of polynomials $U_t^* := U + tx(x-1)^4$, $t > 0$, there exists a polynomial defining the sign pattern σ_3^r and of the form* (3.1).

(3) *If the polynomial U realizes the sign pattern $\sigma_5^r = (+,-,-,+,-,+,+,+,+)$, then in the family of polynomials $U_t^* := U + tx(x-1)^4$, $t > 0$, there exists a polynomial defining one of the sign patterns σ_3^r, σ_5^r or $\sigma^* := (+,-,-,+,-,+,0,+,+)$ and of the form* (3.1).

In what follows we set $W := \sum_{j=0}^{8} w_j x^j$, $w_8 = 1$, and

$$Q := \frac{3S^2}{2} - 4S + 1, \qquad R := S^2 - 6S + 4 \qquad \text{and} \qquad P := \frac{S^2}{4} - 4S + 6.$$

The roots of these three polynomials are real. We denote them by

$$\begin{aligned} 0.27\ldots = (4-\sqrt{10})/3 = q_1 &< q_2 = (4+\sqrt{10})/3 = 2.38\ldots, \\ 0.76\ldots = 3-\sqrt{5} = r_1 &< r_2 = 3+\sqrt{5} = 5.23\ldots, \\ 1.67\ldots = 8-\sqrt{40} = p_1 &< p_2 = 8+\sqrt{40} = 14.32\ldots \end{aligned}$$

The coefficients w_j, $j = 0, \ldots, 7$ are expressed by the following formulae:

$$(3.2)\qquad \begin{aligned} w_0 &= \tfrac{1}{4}S^2N, & w_1 &= \tfrac{1}{4}S(MS + 4N(1-S)), \\ w_2 &= QN + (S-S^2)M + \tfrac{1}{4}S^2, & w_3 &= QM - RN + S - S^2, \\ w_4 &= Q - RM + PN, & w_5 &= -R + PM + (S-4)N, \\ w_6 &= P + (S-4)M + N, & w_7 &= M + S - 4, \end{aligned}$$

3.2. Cases 2 and 4. In Cases 2 and 4 we are using the sign patterns $\sigma_2^r = (+,+,-,-,-,+,-,+,+)$ and $\sigma_4 = (+,+,+,-,-,+,-,+,+)$. They can be united in a single sign pattern $\pi_\pm := (+,+,\pm,-,-,+,-,+,+)$. If the polynomial W (see (3.1)) defines the sign pattern $\pi_\pm$, then one must have $w_j > 0$ for $j = 0$, 1, 3 and 7 and $w_j < 0$ for $j = 2$, 4 and 5.

One has $M > 0$. Indeed, $w_7 = M + S - 4 > 0$, hence $S > 4 - M$. Suppose that $M \leqslant 0$. Then one has $S > 4$, $MS \leqslant 0$ and $4N(1-S) \leqslant 0$, i.e. $w_1 \leqslant 0$, a contradiction.

Suppose that $S > 1$. Then the condition $w_1 > 0$ is equivalent to $N < \frac{1}{4}MS/(S-1)$. On the other hand, as $N \geqslant \frac{1}{4}M^2$, the last two inequalities together imply $M < S/(S-1)$, hence $N < S^2/u$, where $u = 4(S-1)^2$.

For $S \in [p_1, p_2]$ (recall that $p_1 > 1$) one has $P \leqslant 0$, $PN \geqslant PS^2/u$ and $4 - S < M < S/(S-1)$. Therefore

$$w_4 \geqslant \min\Big\{Q(S) - R(S)(4-S) + P(S)\frac{S^2}{u},\ Q(S) - R(S)\frac{S}{S-1} + P(S)\frac{S^2}{u}\Big\}.$$

This minimum is greater than 5, hence greater than 0 (the numerical check of this is easy) and the inequality $w_4 < 0$ fails for $S \in [p_1, p_2]$.

For $S > p_2$ one has $P \geqslant 0$, $PN \geqslant \frac{1}{4}PM^2 \geqslant 0$ and $0 < M < S/(S-1)$, so

$$w_4 \geqslant \min\Big\{Q(S) - R(S)\frac{S}{S-1},\ Q(S)\Big\}.$$

This minimum is also positive and again $w_4 < 0$ fails.

Let now $S \in (0, p_1)$. The inequality $w_4 < 0$ can be rewritten as $N < (RM - Q)/P$ which together with $\frac{1}{4}M^2 \leqslant N$ implies $PM^2 - 4RM + 4Q < 0$. This is a quadratic inequality with respect to M, with $P > 0$. The discriminant of the quadratic polynomial $Y(M,S) := P(S)M^2 - 4R(S)M + 4Q(S)$ equals $4(R^2(S) - P(S)Q(S))$. It is positive for all $S \in (0, p_1)$ (this is easy to check). Hence, for $S \in (0, p_1)$ the polynomial Y has two real roots $M' < M''$ which depend continuously on S and one must have $M \in (M', M'')$.

For each $S \in (0, p_1)$ fixed, both these roots are smaller than $4 - S$. Indeed, set $M := 4 - S$. The polynomial $Y(4-S, S)$ is positive on $(0, p_1)$ (easy to check). For $S = 1 \in (0, p_1)$ one has $Q = -3/2 < 0$, i.e. one of the roots is negative and the other is positive. Hence, for $S \in (0, p_1)$ the number $4 - S$ lies outside the interval $[M', M'']$, and as $4 - S > 0$, one has $M' < 4 - S$, $M'' < 4 - S$ and $M \in (M', M'')$. But one must have $M > 4 - S$, so the inequalities $w_7 > 0$ and $w_4 < 0$ cannot hold simultaneously for $S \in (0, p_1)$.

3.3. Cases 3, 5 and 6. In Cases 3, 5 and 6 we use the sign patterns

$$\sigma_3^r = (+,-,-,+,-,+,-,+,+), \qquad \sigma_5^r = (+,-,-,+,-,+,+,+,+)$$
$$\text{and} \qquad \sigma_6^r = (+,-,-,-,-,+,-,+,+).$$

and formulae (3.2). The proof of Theorem 1.4 in these cases results from Lemmas 3.4, 3.5 and 3.6.

Lemma 3.3. *In Cases* 3, 5 *and* 6 *one has* $M > 0$.

Proof. One must have $w_1 > 0$ and $w_6 < 0$. For $S \geqslant 1$ the product $N(1-S)$ is negative (see formulae (3.2)), so for $S \geqslant 1$ the condition $w_1 > 0$ implies that one must have $SM > 0$, i.e. $M > 0$. Consider for $S \in (0,1)$ the condition $w_6 < 0$, (i.e. $P + (S-4)M + N < 0$). One has $P(S) > 0$, $N \geqslant 0$ and $S - 4 < 0$, so the inequality $w_6 < 0$ is possible only for $M > 0$. □

Lemma 3.4. *Cases* 3, 5 *and* 6 *are not realizable for* $S \in (0, r_1]$.

Proof. In Cases 3, 5 and 6 one has $w_3 > 0$, i.e. $QM + S - S^2 > RN$, see (3.2). For $S \in (0, r_1]$ one has $R(S) \geqslant 0$ and $QM + S - S^2 > RN \geqslant \frac{1}{4}RM^2$, hence

$$L(S,M) := R(S)\frac{M^2}{4} - Q(S)M - S + S^2 < 0. \tag{3.3}$$

The inequalities (3.3), $0 < S \leqslant r_1$ and $0 \leqslant M < 4 - S$ have no common solution. Indeed, $L(S, 4-S) = \frac{1}{4}(S-2)^2((S-2)^2 + 8)$. This means that for $S = 2$ the line $M + S = 4$ has an ordinary tangency with the curve $L(S,M) = 0$, and this is their only common point in the domain $\{S > 0, M > 0\}$. For $S = M = 1/2$ one has $L(S,M) = 9/64 > 0$ and $S + M - 4 < 0$. Hence, below the line $M + S = 4$ in the domain $\{S > 0, M > 0\}$ one has $L(S,M) > 0$. □

Remark 3.2. (1) The inequalities $S > 0$, $M > 0$ (see Lemma 3.3) and $S + M < 4$ (this follows from $w_7 < 0$ in Cases 3, 5 and 6) imply $S < 4$.

Convention 3.1. (1) In what follows we interpret an equality of the form $w_j = 0$ (see (3.2)) as the equation of a straight line (denoted by l_j) in the space (M,N) with coefficients depending on S as on a parameter. Most often we need equations of the form $A(S)N + B(S)M + C(S) = 0$, and we care to have a positive coefficient of N. E.g. we prefer the equation of the line l_1 (see the quantity w_1 in formulae (3.2)) to be of the form $4(1-S)N + SM = 0$ for $S < 1$ and $4(S-1)N - SM = 0$ for $S > 1$.

(2) We denote by l_j^+ and l_j^- the upper and lower, respectively half-plane defined by the line l_j. In the case of l_1 one has $l_1^+\colon\ 4(1-S)N + SM > 0$ for $S < 1$ and $l_1^+\colon\ 4(S-1)N - SM > 0$ for $S > 1$. For $S = 1$ this line is vertical and we do not define the half-planes $l_1^\pm$. By $s(l_j)$ we denote the *slope* of the line l_j, i.e. the quantity $-B(S)/A(S)$ for $A(S) \neq 0$. For l_1 it equals $S/4(S-1)$.

(3) When in the proofs of the lemmas rational functions appear, it is presumed that the factors of degree 2 have no real roots (so their sign coincides with the one of their leading coefficient). Factorizations are performed by means of MAPLE.

Lemma 3.5. *Cases* 3, 5 *and* 6 *are not realizable for* $S \in [p_1, 4)$.

P r o o f. Consider the four conditions $M > 0$, $w_1 > 0$, $w_3 > 0$ and $w_4 < 0$. The second of them defines the half-plane l_1^- (recall that $l_1\colon\ 4(S-1)N - SM = 0$). The last two of them read

$$(-R(S))N + Q(S)M + S - S^2 > 0 \quad \text{and} \quad (-P(S))N + R(S)M - Q(S) > 0.$$

The straight line $l_3\colon\ (-R(S))N + Q(S)M + S - S^2 = 0$ intersects the N-axis at the point $A := (0, N_A)$ with $N_A := S(S-1)/(-R(S)) > 0$. The lines l_3 and $l_4\colon\ (-P(S))N + R(S)M - Q(S) = 0$ intersect at the point B with coordinates

$$\begin{aligned} M_B &:= \frac{2}{5}\,\frac{5S^4 - 35S^3 + 84S^2 - 64S + 16}{K(S)}, \\ N_B &:= \frac{2}{5}\,\frac{5S^4 - 20S^3 + 36S^2 - 16S + 4}{K(S)}, \quad \text{where} \\ K(S) &:= S^4 - 8S^3 + 30S^2 - 32S + 16, \end{aligned}$$

and both numerators and the denominator K have no real roots. This point lies above the straight line l_1. Indeed, the coefficient of N in the equation of l_1 is positive. Substituting (M_B, N_B) for (M, N) on the left-hand side of this equation yields the expression

$$\mu := \frac{6(S^2 - 2.5\ldots S + 3.8\ldots)(S^2 - 0.5\ldots S + 0.2\ldots)(S - 1.2\ldots)}{(S^2 - 6.6\ldots S + 20.4\ldots)(S^2 - 1.3\ldots S + 0.7\ldots)},$$

which is positive, see Convention 3.1.

For the slopes $s(l_4)$ and $s(l_1)$ one has $s(l_4) > s(l_1) > 0$. The first inequality follows from $R(S)/P(S) - \frac{1}{4}S/(S-1) > 0$ which is equivalent to

$$\frac{15(S^2 - 1.7\ldots S + 0.9\ldots)(S - 4.6\ldots)}{4P(S)(S-1)} > 0$$

and this results from $S - 4.6\ldots < 0$, $S - 1 > 0$ and $P(S) < 0$.

Hence, the set defined by the conditions $M > 0$, $w_3 > 0$ and $w_4 < 0$ is the domain of $\mathbb{R}^2 \simeq (M, N)$ to the right of the N-axis, to the above of the segment AB and to the above of the half-line starting at B, which is part of the line l_4 and which goes to the right and upward. This domain does not intersect the half-plane l_1^- and the four conditions $M > 0$, $w_1 > 0$, $w_3 > 0$ and $w_4 < 0$ cannot hold true simultaneously. $\square$

Lemma 3.6. *Cases* 3, 5 *and* 6 *are not realizable for* $S \in (r_1, p_1)$.

P r o o f. Consider the conditions $w_3 > 0$ and $w_6 < 0$. They read

$$(-R(S))N + Q(S)M + S - S^2 > 0 \qquad \text{and} \qquad N + (S-4)M + P(S) < 0.$$

Consider the point $\Pi := l_3 \cap l_6$. Its coordinates equal

$$\Big(-\frac{S^4 - 22S^3 + 120S^2 - 204S + 96}{2Y(S)},\ -3\frac{S^4 - 16S^3 + 54S^2 - 64S + 16}{4Y(S)}\Big),$$

where $Y(S) := 2S^3 - 17S^2 + 48S - 30$ has a single real root $y_0 := 0.8609094817\ldots$ For $S \in (r_1, y_0)$ and for $S \in (y_0, p_1)$ one has $s(l_3) > s(l_6)$ and $s(l_3) < s(l_6)$, respectively. This follows from

$$\frac{Q(S)}{R(S)} - (4 - S) = (S^2 - 7.6\ldots S + 17.4\ldots)\frac{S - y_0}{R(S)}$$

with $R(S) < 0$. The second coordinate of Π equals

$$\frac{-3(S - 0.3\ldots)(S - 11.9\ldots)(S^2 - 3.7\ldots S + 4.0\ldots)}{4Y(S)}.$$

Hence, it changes sign from $-$ to $+$ when S passes from y_0^- to y_0^+. For $S \in (r_1, y_0)$ one has $\{w_3 > 0\} \cap \{w_6 < 0\} = l_3^+ \cap l_6^-$. For $S = y_0$ the lines l_3 and l_6 are parallel, l_3 is above l_6 and $\{w_3 > 0\} \cap \{w_6 < 0\} = \emptyset$. Thus, for $S \in (r_1, y_0)$ the sector $l_3 \cap l_6$ belongs to the domain $N < 0$ and if some of Cases 3, 5 or 6 is realizable, it can be realizable only for $S \in (y_0, p_1)$.

For $S = y_0^+$ the intersection $\{w_3 > 0\} \cap \{w_6 < 0\}$ is a sector whose vertex has both coordinates positive because the first coordinate of Π equals

$$\frac{-(S - 0.7\ldots)(S - 1.8\ldots)(S - 4.6\ldots)(S - 14.7\ldots)}{2Y(S)} > 0.$$

The point Π lies above the line $l_4\colon\ P(S)N - R(S)M + Q(S) = 0$ for $S \in (y_0, y_1)$, where $y_1 := 1.471576286\ldots$ Indeed, substituting the coordinates of Π for (M, N) on the left-hand side of the equation of l_4 yields

$$\frac{5(S^2 - 5.2\ldots S + 20.3\ldots)(S^2 - 1.2\ldots S + 1.2\ldots)(S - 7.9\ldots)(S - y_1)}{32(S^2 - 7.6\ldots S + 17.4\ldots)(S - y_0)} > 0.$$

Moreover, $s(l_4) < 0 < s(l_3) < s(l_6)$. Hence, for $S \in (y_0, y_1)$ the three conditions $w_3 > 0$, $w_4 < 0$ and $w_6 < 0$ cannot hold true simultaneously.

In order to prove the lemma for $S \in [y_1, p_1)$ we consider the conditions

$$w_1 > 0 \ \text{ i.e. } 4(S-1)N - MS < 0 \ \text{ and } \ w_3 > 0 \ \text{ i.e. } -R(S)N + Q(S)M + S - S^2 > 0.$$

The point $\Gamma := l_1 \cap l_3$ has coordinates (M_Γ, N_Γ) which equal

$$\Big(\frac{4(S-1)^2}{5S^3-16S^2+16S-4},\ \frac{S^2(S-1)}{5S^3-16S^2+16S-4}\Big).$$

Both coordinates are positive for $S \in [y_1, p_1)$ (the only real zero of the denominator equals $0.3\ldots$). The point Γ lies above the straight line l_4. Indeed, substituting (M_Γ, N_Γ) for (M, N) on the left-hand side of the equation of l_4: $P(S)N - R(S)M + Q(S) = 0$ with $P(S) > 0$ yields

$$\frac{3(S^2-2.5\ldots S+3.8\ldots)(S^2-0.5\ldots S+0.2\ldots)(S-1.2\ldots)}{4(S^2-2.8\ldots S+2.1\ldots)(S-0.3\ldots)} > 0.$$

One has $s(l_4) < 0 < s(l_3) < s(l_1)$; the last inequality follows from

$$\frac{S}{4(S-1)} - \frac{Q(S)}{R(S)} = -\frac{5(S^2-2.8\ldots S+2.1\ldots)(S-0.3\ldots)}{4(S-5.2\ldots)(S-1)(S-0.7\ldots)} > 0.$$

Hence, for $S \in [y_1, p_1)$ the sector $\{w_1 > 0\} \cap \{w_3 > 0\}$ does not intersect the half-plane $\{w_4 < 0\} = l_4^-$, i.e. the three conditions $w_1 > 0$, $w_3 > 0$ and $w_4 < 0$ do not hold simultaneously. $\square$

4. Appendix. Proofs of Lemmas 3.1 and 3.2

P r o o f of Lemma 3.1. Denote by $0 < x_1 < x_2 < x_3 < x_4$ the real roots of P. We are looking first for a polynomial $U^0(x)$ of the form $(P(x) + ax^8 - bx^k + c)/(1+a)$ having a quadruple root $x_0 > 0$, where $k = 1$ in Cases 3, 5 and 6, $k = 3$ in Case 2, $k = 5$ in Case 4, and $a > 0$, $b > 0$, $c > 0$. The signs of a, b and c imply that U^0 defines the same sign pattern as P. The polynomial U is obtained from U^0 by suitable rescaling and multiplication by a positive constant which does not change the sign pattern.

For $x = x_0$ the polynomial U^0 satisfies the conditions $(U^0)' = (U^0)'' = (U^0)''' = 0$ which read:

$$\begin{aligned}
(4.1)\qquad & k=1: && P'(x) + 8ax^7 - b = 0, && P'' + 56ax^6 = 0,\\
& && && P''' + 336ax^5 = 0,\\
& k=3: && P'(x) + 8ax^7 - 3bx^2 = 0, && P'' + 56ax^6 - 6bx = 0,\\
& && && P''' + 336ax^5 - 6b = 0,\\
& k=5: && P'(x) + 8ax^7 - 5bx^4 = 0, && P'' + 56ax^6 - 20bx^3 = 0,\\
& && && P''' + 336ax^5 - 60bx^2 = 0.
\end{aligned}$$

Consider first Cases 5 and 6, hence $k = 1$. One eliminates a from the last two equations, which gives $xP'''(x) = 6P''(x)$. The polynomial P' has exactly three positive roots $\mu_1 < \mu_2 < \mu_3$, $\mu_j \in (x_j, x_{j+1})$. Indeed, by Rolle's theorem it has at least three and by Descartes' rule of signs it has at most three of them. So for $x > x_4$ and $x > \mu_3$ the polynomial P and P', respectively, is positive.

The polynomial P'' has at least two real roots $\xi_1 < \xi_2$, $\xi_j \in (\mu_j, \mu_{j+1})$ (again by Rolle's theorem). By Descartes' rule of signs the polynomial P'' has at most three positive roots. The sign of the coefficient of x^2 in P is negative, therefore P'' has exactly three positive roots. The third of them ξ_3 is in $(0, \xi_1)$. Indeed, to the right of ξ_2 the number of positive roots of P'' must be even because for $x > 0$ sufficiently large, P is convex. So $0 < \xi_3 < \xi_1 < \xi_2$ and in fact $\xi_3 \in (0, \mu_1)$.

The polynomial P''' has real roots $\zeta_1 \in (\xi_3, \xi_1)$ and $\zeta_2 \in (\xi_1, \xi_2)$. By Descartes' rule of signs it has at most three positive roots in Case 6 and at most two in Case 5. In Case 6, as P''' must have an even number of roots to the right of ξ_2 (P' is convex for $x > 0$ sufficiently large), the three positive roots $\zeta_3 < \zeta_1 < \zeta_2$ of P''' belong to the intervals $(0, \xi_3)$, (ξ_3, ξ_1) and (ξ_1, ξ_2), respectively.

Hence the signs of $P'''(\xi_1)$ and $P'''(\xi_2)$ are opposite and $xP''' - 6P''$ changes sign at some point $x_0 \in (\xi_1, \xi_2)$.

In Case 3 one has again $k = 1$. The sign patterns σ_3 and σ_5 differ only in their third position. The proof resembles the one in Cases 5 and 6 yet Descartes' rule of signs allows more positive roots for P', P'' and P'''.

Denote by $p(P')$ the number of positive roots of P'. Combining Rolle's theorem and Descartes' rule of signs, one understands that it is possible to encounter only one of the following triples $(p(P'), p(P''), p(P'''))$:

$$\text{i) } (3,5,4), \qquad \text{ii) } (3,3,4), \qquad \text{iii) } (3,3,2), \qquad \text{iv) } (5,5,4).$$

In Case iii) the proof is carried out in exactly the same way as for Case 5. In the other cases one performs analogous reasoning with only difference—the two more positive roots of P'' and P''' in Case i), of P''' in Case ii) or of P', P'' and P''' in Case iv). For parity reasons the two more roots of the corresponding derivative $P^{(j)}$ (compared to their number in the proof of Case 5) must belong to one and the same interval of $[0, \infty)$ defined by 0, ∞ and the positive roots of $P^{(j-1)}$.

E.g. in Case i) the polynomial P' has positive roots $0 < \mu_1 < \mu_2 < \mu_3$, and by Rolle's theorem in each of the intervals (μ_1, μ_2) and (μ_2, μ_3) its derivative P'' has a root. To the right of μ_3 there are either 0 or 2 roots of P'' because P' is convex for large positive x. If they are 2, then as in Case 5 one concludes that the fifth positive root of P'' is in $(0, \mu_1)$. If they are 0, then there are exactly three roots of P'' in one of the intervals $(0, \mu_1)$, (μ_1, μ_2) or (μ_2, μ_3) and one root in each of the other two;

this follows from P being convex at its minima and concave at its maxima. In each of the intervals (ξ_j, ξ_{j+1}), $j = 1, \ldots, 5$, there is exactly one root of P'''.

One proves as for Case 5 that the signs of P''' at two consecutive roots of P'' are opposite, hence $xP''' - 6P''$ changes sign at some point x_0 from the interval between these two roots.

Consider Case 2, hence $k = 3$. Eliminating b from equations (4.1) yields:

$$2P' - xP'' = 40ax^7 \quad \text{and} \quad P'' - xP''' = 280ax^6.$$

Eliminating a from the last two equations gives the equation

$$14P' - 8xP'' + x^2P''' = (14P' - 2xP'') - \frac{x}{2}(14P' - 2xP'')' = 0.$$

The polynomial P' has at most four positive roots (by Descartes' rule of signs), and at least three of them (denoted by μ_j) belong to the intervals (x_j, x_{j+1}), $j = 1$, 2 and 3, hence the fourth one μ_0 is in $(0, x_1)$ (because $P'(0) > 0$). The polynomial P'' has positive roots $\xi_\nu \in (\mu_\nu, \mu_{\nu+1})$, $\nu = 1, 2$, and $\xi_0 \in (\mu_0, \mu_1)$. Hence, the polynomial $S := 14P' - 2xP''$ has different signs at μ_ν and $\mu_{\nu+1}$ for $\nu = 1$ and 2, hence it has roots $\delta_\nu \in (\mu_\nu, \mu_{\nu+1})$, its derivative has opposite signs at δ_1 and δ_2, so $S - \frac{1}{2}xS' := 14P' - 8xP'' + x^2P'''$ has a real root $x_0 \in (\mu_1, \mu_3)$.

Consider Case 4, hence $k = 5$. One first eliminates b (see equations (4.1)):

$$4P' - xP'' = 24ax^7 \quad \text{and} \quad 3P'' - xP''' = 168ax^6.$$

Then eliminating a results in

$$28P' - 10xP'' + x^2P''' = (28P' - 4xP'') - \frac{x}{4}(28P' - 4xP'')' = 0.$$

Similarly to the proof in Case 2 one shows that the polynomial $28P' - 10xP'' + x^2P'''$ has a positive root x_0.

After the number x_0 is found, one finds first a and then b from system (4.1). Now we have to justify the positive signs of a and b (and after this, the one of c as well). To this end we set $a = ta_*$, $b = tb_*$, where $t > 0$, and we consider the family of polynomials $R_t(x) := P(x) + t\psi_k(x)$ with $\psi_k := a_*x^8 - b_*x^k$, $k = 1$, 3 or 5. We suppose that for some $t > 0$ the polynomial R_t has a triple critical point at x_0. Hence, for a suitably chosen c the polynomial $R_t + c$ has a quadruple root at x_0.

Consider the function ψ_k for $x > 0$. For $a_* \geqslant 0$, $b_* \leqslant 0$ and $a_* - b_* > 0$ it is increasing and convex, for $a_* \leqslant 0$, $b_* \geqslant 0$ and $a_* - b_* < 0$ it is decreasing and concave (for $a_* = 0$ and $k = 1$ it is linear, i.e. convex and concave at the same time). For $a_* > 0$ and $b_* > 0$ or for $a_* < 0$ and $b_* < 0$ it has a minimum or a maximum

at $\lambda_k := (kb_*/(8a_*))^{1/(8-k)}$ with $\psi_k(x) < 0$ for $x \in (0, \lambda_k]$ or with $\psi_k(x) > 0$ for $x \in (0, \lambda_k]$).

Consider the family of polynomials R_t, where t is supposed to belong to an interval $[0, \alpha)$ such that the sign pattern defined by the coefficients of R_t is the one of P. We keep the same notation for the positive roots of R_t and its derivatives as the one for P. Then:

A) If ψ_k is decreasing on $[\mu_2, \mu_3]$, then as t increases, μ_2 moves to the left and μ_3 to the right;

B) If ψ_k is increasing on $[\mu_1, \mu_2]$, then as t increases, μ_1 moves to the left and μ_2 to the right.

In both cases A) and B) it is impossible to have the three positive roots of R_t' coalescing into a single critical point of R_t. If $a_* \geqslant 0$, $b_* \leqslant 0$ and $a_* - b_* > 0$, then case B) takes place. If $a_* \leqslant 0$, $b_* \geqslant 0$ and $a_* - b_* < 0$, then case A) takes place. If $a_* < 0$ and $b_* < 0$, then at least one of cases A) or B) takes place. Hence only for $a_* > 0$ and $b_* > 0$ one can have a critical point of R_t of multiplicity 3. This implies that $a > 0$ and $b > 0$. Besides, $\lambda_k \in (\mu_1, \mu_3)$. Hence $R_t(\mu_1) < 0$ (because $P(\mu_1) < 0$ and $\psi_k(\mu_1) < 0$) and to have $U^0(x_0) = 0$ one has to choose $c > 0$. $\square$

P r o o f of Lemma 3.2. Prove part (1). Consider the one-parameter family of polynomials $U_t := U - t(x-1)^4$, $t \geqslant 0$. The first four coefficients do not depend on t (they are the same as the ones of U). The signs of the five coefficients of $-(x-1)^4$ are $(-, +, -, +, -)$. Hence, the first 8 components of the sign pattern of U_t do not depend on t and in the family U_t for some $t > 0$, due to the decrease of the value of U_t as t increases, one of the two things takes place first:

a) one has $U_t(0) = 0$ or

b) U_t has one or two negative roots, each of them of even multiplicity.

One can notice that the family U_t contains no polynomial with six positive roots (counted with multiplicity) because there are four or five sign changes in the sign pattern of U_t (the sign pattern of U_t is obtained from σ_2^r, σ_4 or σ_6^r by replacing the last component by $+$, 0 or $-$).

If a) takes place for $t_0 > 0$, then as $U_t'(0) > 0$, the root of U_t at 0 is simple and U_t has one or several negative roots whose total multiplicity is odd. Hence, for some $t_1 \in (0, t_0)$, b) has taken place. Therefore in the family U_t there exists (for some $t > 0$) a polynomial of the form (3.1) which realizes the pattern σ_2^r, σ_4 or σ_6^r.

Prove part (2) of the lemma. Suppose that the polynomial U realizes the sign pattern $\sigma_3^r = (+, -, -, +, -, +, -, +, +)$. Consider the family $U_t^* = U + tx(x-1)^4$, $t > 0$. The signs of the coefficients of $x(x-1)^4$ are $(0, 0, 0, +, -, +, -, +, 0)$, so the sign pattern of U_t^* is σ_3^r for any $t > 0$. The value of U_t^* increases (linearly with t) for each $x > 0$, $x \neq 1$ fixed, and decreases for each $x < 0$ fixed. Hence, for some $t > 0$

the polynomial U_t^* has one or two negative roots each of even multiplicity. For this value of t the polynomial U_t^* has the form (3.1).

The proof of part (3) resembles the one of part (2). Suppose that the polynomial U realizes the sign pattern $\sigma_6^r = (+,-,-,+,-,+,+,+,+)$. The difference between σ_6^r and σ_3^r is in the sign of the coefficient of x^2. Hence, in the family U_t^* there is a polynomial with a quadruple root at 1, with one or two negative roots of even multiplicity and with coefficients defining either one of the sign patterns σ_3^r, σ_6^r or the sign pattern σ^* (the sign of the coefficient of x^2 in U_t^* might change for some value of t). In all three cases this is a polynomial of the form (3.1). $\square$

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Author's address: Vladimir Kostov, Université Côte d'Azur, Laboratoire de Mathématiques, Parc Valrose, 06108 Nice cedex 2, France, e-mail: `vladimir.kostov@unice.fr`.