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WEIGHTED GENERALIZATION OF THE RAMADANOV'S THEOREM AND FURTHER CONSIDERATIONS

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Abstract. We study the limit behavior of weighted Bergman kernels on a sequence of domains in a complex space $\mathbb{C}^N$, and show that under some conditions on domains and weights, weighed Bergman kernels converge uniformly on compact sets. Then we give a weighted generalization of the theorem given by M. Skwarczyński (1980), highlighting some special property of the domains, on which the weighted Bergman kernels converge uniformly. Moreover, we show that convergence of weighted Bergman kernels implies this property, which will give a characterization of the domains, for which the inverse of the Ramadanov's theorem holds.

Keywords: weighted Bergman kernel; admissible weight; sequence of domains

MSC 2010: 32A36, 32A25

1. Introduction

The Bergman kernel (see for instance [2], [8], [9], [10], [16], [19]) has become a very important tool in geometric function theory, in both one and several complex variables.

It turned out that not only the classical Bergman kernel, but also the weighted one can be useful. Let $D \subset \mathbb{C}^N$ be a bounded domain. For example (see [5]), if we denote by $\Pi\colon L^2(D) \to L^2_H(D)$ (the Bergman projection), we may define for any $\psi \in L^\infty(D)$ the Toeplitz operator T_ψ as a (bounded linear) operator on $L^2_H(D)$ by $T_\psi f := \Pi(\psi f)$. In particular, for $\psi > 0$ on D we have that T_ψ is positive definite (so injective), so there exists an inverse T_ψ^{-1}. Taking positive continuous weight function $\mu \in L^\infty(D)$, T_μ extends to a bounded operator from $L^2_H(D,\mu)$ into $L^2_H(D)$, and $K_{D,\mu}(\cdot,x) = T_\mu^{-1}K_D(\cdot,x)$, where $K_{D,\mu}(\cdot,x)$ denotes the weighted Bergman kernel (associated to the weighted Bergman space $L^2_H(D,\mu)$) at $x \in D$.

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Another practical application of weighted Bergman kernels may bo found in quantum theory (see [4] and [12], and [13])–we may consider a Kähler manifold Ω as a classical phase space of a physical system (many leading quantized classical systems have such a phase space). The Hilbert space H of quantum states of such a system consists of the holomorphic sections of some Hermitian line bundle E over Ω, which belong to $L^2(\Omega, \mu)$ (for the Liouville measure μ on Ω). One of the most interesting and important objects of this model is the reproducing kernel K of H (that is, the kernel $K_{\Omega,\mu}$). This kernel makes the quantization of classical states possible as follows: one can assign to any classical state $z \in \Omega$ the quantum state

$$v_z := [K(\cdot, z)/\|K(\cdot, z)\|] \in H.$$

Using this embedding, one can calculate the transition probability amplitude from one point to another:

$$a(z, w) := |\langle v_z \mid v_w \rangle|, \quad z, w \in \Omega.$$

Then the calculation of the Feynman path integral for such a system is equivalent to finding the reproducing kernel K (that is, $K_{\Omega,\mu}$).

But in general, it is difficult to say anything about the unweighted (regular) or weighted kernel of a given domain. One of the classic results for unweighted Bergman kernels is the Ramadanov's theorem (see [15]):

Theorem 1 (Ramadanov)**.** *Let $D_1 \Subset D_2 \Subset D_3 \ldots$ be an increasing sequence of domains and set $D := \bigcup_j D_j$. Then $K_{D_j} \to K_D$ uniformly on compact subsets of $D \times D$.*

It is very natural to ask whether similar theorem for weighted Bergman kernels is true. Let us recall the Forelli-Rudin construction (see [6] and [11]): If μ is a continuous weight on D and Ω denotes the Hartogs domain

$$\Omega = \{(z, w) \in D \times \mathbb{C}^n \colon \|w\|^{2n} < \mu(z)\}$$

in $\mathbb{C}^{N+n}$, then

$$K_{D,\mu}(z, p) = \frac{\pi^n}{n!} K_\Omega((z, 0), (p, 0))$$

(that is, the weighted Bergman kernel $K_{D,\mu}(z, p)$ of D is the restriction of the unweighted Bergman kernel $K_\Omega((z, w), (p, s))$ of Ω to the hyperplane $w = s = 0$). Thus, using Ramadanov's theorem for the kernels $K_{\Omega_j}((z, 0), (p, 0))$ we can derive (under some conditions on weights–monotonicity for instance) the weighted analogue of this theorem. And in fact, we may find some versions in literature (in [7], Proposition 3.17, Theorem 3.18 for instance), but the considered weights are in a special form, as moduli of holomorphic functions or C^2 functions, or as a product of one of those with the given weight ψ. Additionally, some relevant considerations on convergence of a sequence of reproducing kernels were given in [1] (particularly Section 9 of Part I). We can easily see that continuity of the weight μ in the Forelli-Rudin construction provides basically that Ω is an open set. In this paper we derive a weighted version of Ramadanov's theorem for the so called 'admissible weights' μ (we do not require μ to be continuous) without using Forelli-Rudin construction. It is very natural to consider such kind of weights, just by their definition (see below). We will prove the inverse of this theorem as well (see also [17], page 37, for an unweighted situation). In the second part of the paper we show that density of holomorphic functions on a considered domain is very related to the convergence of the weighted Bergman kernels. In fact, we will get an equivalence in the unweighted case. This will provide us with a characterization of the domains, for which the inverse of Ramadanov's theorem holds. We shall start from the definitions and basic facts used in this paper.

2. Definitions and notation

Let $D \subset \mathbb{C}^N$ be a domain, and let $W(D)$ be the set of weights on D, i.e. $W(D)$ is the set of all Lebesgue measurable, real-valued, positive functions on D (we consider two weights as equivalent if they are equal almost everywhere with respect to the Lebesgue measure on D). If $\mu \in W(D)$, we denote by $L^2(D,\mu)$ the space of all Lebesgue measurable, complex-valued, μ-square integrable functions on D, equipped with the norm $\|\cdot\|_{D,\mu} := \|\cdot\|_\mu$ given by the scalar product

$$\langle f \mid g\rangle_\mu := \int_D f(z)\overline{g(z)}\mu(z)\,\mathrm{d}V, \quad f,g \in L^2(D,\mu).$$

The space $L^2_H(D,\mu) = H(D) \cap L^2(D,\mu)$ is called the *weighted Bergman space*, where $H(D)$ stands for the space of all holomorphic functions on the domain D. For any $z \in D$ we define the evaluation functional E_z on $L^2_H(D,\mu)$ by the formula

$$E_z f := f(z), \quad f \in L^2_H(D,\mu).$$

Let us recall Definition 2.1 of admissible weight given in [14].

Definition 2 (Admissible weight). A weight $\mu \in W(D)$ is called an *admissible weight*, an a-weight for short, if $L^2_H(D,\mu)$ is a closed subspace of $L^2(D,\mu)$ and for any $z \in D$ the evaluation functional E_z is continuous on $L^2_H(D,\mu)$. The set of all a-weights on D will be denoted by $AW(D)$.

The definition of admissible weight provides us basically with existence and uniqueness of the related Bergman kernel and completeness of the space $L^2_H(D,\mu)$. The concept of a-weight was introduced in [13], and in [14] several theorems concerning admissible weights are proved. An illustrative one is:

Theorem 3 ([14], Corolarry 3.1)**.** *Let $\mu \in W(D)$. If the function μ^{-a} is locally integrable on D for some $a > 0$, then $\mu \in AW(D)$.*

Now, let us fix a point $t \in D$ and minimize the norm $\|f\|_\mu$ in the class $E_t = \{f \in L^2_H(D,\mu)\ f(t) = 1\}$. It can be proved in a similar way as in the classical case that if μ is an admissible weight, then there exists exactly one function minimizing the norm. Let us denote it by $\phi_\mu(z,t)$. *Weighted Bergman kernel function* $K_{D,\mu}$ is defined as

$$K_{D,\mu}(z,t) = \frac{\phi_\mu(z,t)}{\|\phi_\mu\|^2_\mu}.$$

3. Variations on the Ramadanov's theorem and domain dependence

In this section we study the limit behavior of weighted Bergman kernels for admissible weights. Moreover, we give a weighted characterization of the Bergman kernel (see also [17], page 36) by means of which we prove a kind of converse of the Ramadanov's theorem. We show that density of holomorphic functions is very related to the convergence of sequences of weighted Bergman kernels, and in the case of $\mu_n \equiv 1$ we even have an equivalence (see also [18]).

3.1. Weighted generalization of the Ramadanov's theorem

Main Theorem 4 (Weighted generalization of the Ramadanov's theorem)**.** *Let $\{D_i\}_{i=1}^{\infty}$ be a sequence of domains in $\mathbb{C}^N$ and set $D := \bigcup_j D_j$. Let $\mu \in AW(D)$, $\mu_k \in AW(D_k)$ (extend μ_k by μ on D). Assume moreover that*

a) *for any $n \in \mathbb{N}$ there is $N = N(n)$ s.t. $D_n \subset D_m$ and $\mu_n(z) \leqslant \mu_m(z) \leqslant \mu(z)$ for $m \geqslant N(n)$, $z \in D_n$;*

b) *$\mu_k \underset{k\to\infty}{\longrightarrow} \mu$ pointwise a.e. on D.*

Then

$$\lim_{k\to\infty} K_{D_k,\mu_k} = K_{D,\mu}$$

locally uniformly on $D \times D$.

The first step in the proof is to show the monotonicity property for the weighted kernels. Then we should check that the limit of a sequence of weighted kernels of the domains D_n, if it exists, is equal to $K_{D,\mu}$.

Lemma 5 (Monotonicity property). *For any $n \in \mathbb{N}$, $t \in D_n$, the inequality $K_{D_n,\mu_n}(t,t) \geqslant K_{D_m,\mu_m}(t,t)$ holds for $m \geqslant N(n)$.*

Proof. Let us fix $n \in \mathbb{N}$, $t \in D_n$. Let $m \geqslant N(n)$. The inequality in the statement of the lemma is true if $K_{D_m,\mu_m}(t,t) = 0$. Then suppose that $K_{D_m,\mu_m}(t,t) > 0$. In the proof we use a simple remark that

$$\frac{1}{K_{D_n,\mu_n}(t,t)} = \int_{D_n} \left| \frac{K_{D_n,\mu_n}(s,t)}{K_{D_n,\mu_n}(t,t)} \right|^2 \mu_n(s)\,\mathrm{d}V$$

since $K_{D_n,\mu_n}(t,t) > 0$ and

$$K_{D_n,\mu_n}(t,t) = \int_{D_n} \overline{K_{D_n,\mu_n}(z,t)} K_{D_n,\mu_n}(z,t) \mu_n(z)\,\mathrm{d}V$$

by the reproducing property ([13]) for $f(\cdot) = K_{D_n,\mu_n}(\cdot,t)$. Moreover, the term $K_{D_n,\mu_n}(\cdot,t)/K_{D_n,\mu_n}(t,t)$ is the only element in the class $\{f \in L^2_H(D_n,\mu_n)\colon\ f(t) = 1\}$ with minimal norm. Thus, for $m \geqslant N(n)$ we have

$$\begin{aligned}
\frac{1}{K_{D_n,\mu_n}(t,t)} &\leqslant \int_{D_n} \left| \frac{K_{D_m,\mu_m}(s,t)}{K_{D_m,\mu_m}(t,t)} \right|^2 \mu_n(s)\,\mathrm{d}V \\
&\leqslant \int_{D_n} \left| \frac{K_{D_m,\mu_m}(s,t)}{K_{D_m,\mu_m}(t,t)} \right|^2 \mu_m(s)\,\mathrm{d}V \\
&\leqslant \int_{D_m} \left| \frac{K_{D_m,\mu_m}(s,t)}{K_{D_m,\mu_m}(t,t)} \right|^2 \mu_m(s)\,\mathrm{d}V = \frac{1}{K_{D_m,\mu_m}(t,t)}.
\end{aligned}$$

□

Remark 6. One can show similarly that $K_{D_n,\mu_n}(t,t) \geqslant K_{D,\mu}(t,t)$ for $n \in \mathbb{N}$.

Lemma 7 (Uniqueness of the limit). *If $\lim\limits_{n\to\infty} K_{D_n,\mu_n} = k$ locally uniformly on $D \times D$, then $k = K_{D,\mu}$.*

Proof. Since the sequence $\{K_{D_n,\mu_n}\}_{n=1}^{\infty}$ converges locally uniformly on $D \times D$ and any function K_{D_n,μ_n} is continuous, we obtain that k is continuous on $D \times D$. Let us recall that

$$\int_{D_m} \overline{K_{D_m,\mu_m}(z,t)} K_{D_m,\mu_m}(z,t) \mu_m(z)\,\mathrm{d}V = K_{D_m,\mu_m}(t,t). \tag{3.1}$$

Fix a compact set $E \subset D$ and $t \in E$. For m large enough, $E \subset D_m$ and $t \in D_m$. By Fatou's lemma

$$\begin{aligned}
\int_E |k(z,t)|^2 \mu(z)\,\mathrm{d}V &\leqslant \liminf_{m\to\infty} \int_E |K_{D_m,\mu_m}(z,t)|^2 \mu_m(z)\,\mathrm{d}V \\
&\leqslant \liminf_{m\to\infty} \int_{D_m} |K_{D_m,\mu_m}(z,t)|^2 \mu_m(z)\,\mathrm{d}V \\
&= \liminf_{m\to\infty} K_{D_m,\mu_m}(t,t) = k(t,t).
\end{aligned}$$

Since E is an arbitrary compact set,

$$\int_D |k(z,t)|^2 \mu(z)\,\mathrm{d}V \leqslant k(t,t). \tag{3.2}$$

By the Weierstrass theorem, $k(\cdot,t) \in H(D)$, so $k(\cdot,t) \in L^2_H(D,\mu)$.

By Remark 6 we get

$$K_{D_n,\mu_n}(t,t) \geqslant K_{D,\mu}(t,t)$$

for $n = 1,2,\ldots$, $t \in D$. In the limit $n \to \infty$ we obtain

$$k(t,t) \geqslant K_{D,\mu}(t,t).$$

It suffices to show that $k(z,t) = K_{D,\mu}(z,t)$. We should consider two cases:

1. $K_{D,\mu}(t,t) = 0$ for some $t \in D$.

Then for $z \in D$, $K_{D,\mu}(z,t) = 0$ since $K_{D,\mu}(t,t) = \int_D |K_{D,\mu}(z,t)|^2 \mu(z)\,\mathrm{d}V$, and $K_{D,\mu}$ is continuous with respect to z. Thus, for any $f \in L^2_H(D,\mu)$

$$f(t) = \int_D f(w) K_{D,\mu}(t,w)\mu(w)\,\mathrm{d}V = 0$$

and we have that $k(t,t) = 0$ since $f(\cdot) := k(\cdot,t) \in L^2_H(D,\mu)$. But

$$\int_D |k(z,t)|^2 \mu(z)\,\mathrm{d}V \leqslant k(t,t),$$

so $k(z,t) = 0$ for $z \in D$.

2. $K_{D,\mu}(t,t) > 0$ for some $t \in D$.

Then $k(t,t) > 0$ since $k(t,t) \geqslant K_{D,\mu}(t,t) > 0$. We will use once more the well known fact, that in the set $\{f \in L^2_H(D,\mu)\colon\ f(t) = 1\}$ (for some fixed $t \in D$) the function $K_{D,\mu}(\cdot,t)/K_{D,\mu}(t,t)$ is the only minimal element. It is easy to see that $k(\cdot,t)/k(t,t)$ belongs to this set (since $k(\cdot,t) \in L^2_H(D,\mu)$) and moreover, by (3.2) $\|k(\cdot,t)\|_\mu \leqslant \sqrt{k(t,t)}$. Thus,

$$\Big\|\frac{k(\cdot,t)}{k(t,t)}\Big\|_\mu \leqslant \frac{\sqrt{k(t,t)}}{k(t,t)} = \frac{1}{\sqrt{k(t,t)}} \leqslant \frac{1}{\sqrt{K_{D,\mu}(t,t)}} = \Big\|\frac{K_{D,\mu}(\cdot,t)}{K_{D,\mu}(t,t)}\Big\|_\mu.$$

By the minimality property of $K_{D,\mu}(\cdot,t)/K_{D,\mu}(t,t)$ we get from the above that

$$\Big\|\frac{k(\cdot,t)}{k(t,t)}\Big\|_\mu = \frac{1}{\sqrt{k(t,t)}} = \frac{1}{\sqrt{K_{D,\mu}(t,t)}} = \Big\|\frac{K_{D,\mu}(\cdot,t)}{K_{D,\mu}(t,t)}\Big\|_\mu.$$

So $k(t,t) = K_{D,\mu}(t,t)$ and $k(z,t) = K_{D,\mu}(z,t)$ for $z,t \in D$. $\square$

Proof of the Main Theorem 4. We will show that for $n \in \mathbb{N}$ the sequence $\{K_{D_m,\mu_m}\}_{m \geqslant N(n)}$ is locally bounded on $D_n \times D_n$.

Using well known version of the Schwarz inequality for reproducing kernels and Lemma 5 we obtain for any $z, t \in D_n$:

$$\begin{aligned} |K_{D_m,\mu_m}(z,t)| &\leqslant \sqrt{K_{D_m,\mu_m}(z,z)}\sqrt{K_{D_m,\mu_m}(t,t)} \\ &\leqslant \sqrt{K_{D_n,\mu_n}(z,z)}\sqrt{K_{D_n,\mu_n}(t,t)}, \quad m \geqslant N(n). \end{aligned}$$

The term on the right-hand side of the estimation above is locally bounded on $D_n \times D_n$. One can observe that for every compact subset of D there exists a value of n for which the compact set is included in D_n, and therefore the estimation actually implies that the sequence $\{K_{D_m,\mu_m}\}$ is locally bounded on $D \times D$. Thus, by Montel's property, any subsequence of $\{K_{D_m,\mu_m}\}$ has a subsequence convergent locally uniformly on $D \times D$. By Lemma 7 the limit does not depend on a subsequence and is identically equal to $K_{D,\mu}$. Thus

$$\lim_{m\to\infty} K_{D_m,\mu_m}(z,t) = K_{D,\mu}(z,t)$$

locally uniformly on $D \times D$. $\square$

Remark 8. Notice that the case of an increasing sequence of domains is a subcase of the Main Theorem 4 (see [19] for the very interesting considerations and unweighted version of Lemma 7).

3.2. Characterization of the weighted Bergman kernel and further remarks on "decreasing-like" sequence of domains. In [17], page 36, a characterization lemma for unweighted Bergman kernels is given. One can easily conclude similar one for weighted Bergman kernels, as the following Lemma 9 shows. The proof is attached for the convenience of the reader.

Lemma 9. *Denote by $S_{\mu,t} \subset L^2_H(D,\mu)$ the set of all functions f such that $f(t) \geqslant 0$ and $\|f\|_\mu \leqslant \sqrt{f(t)}$, where $t \in D$ is fixed. Then the weighted Bergman function $\varphi_{\mu,t}(\cdot) := K_{D,\mu}(\cdot,t)$ is uniquely characterized by the properties:*

(i) $\varphi_{\mu,t} \in S_{\mu,t}$;

(ii) *if $f \in S_{\mu,t}$ and $f(t) \geqslant \varphi_{\mu,t}(t)$, then $f \equiv \varphi_{\mu,t}$.*

Proof. One can easily see, that there exists at most one element $\varphi_{\mu,t} \in L^2_H(D,\mu)$ which satisfies (i) and (ii) (if φ_1, φ_2 satisfies (i) and (ii), then both $\varphi_1(t)$ and

$\varphi_2(t)$ are nonnegative, and either $\varphi_1(t) \geqslant \varphi_2(t)$ and then $\varphi_1 \equiv \varphi_2$ or $\varphi_2(t) \geqslant \varphi_1(t)$ and then $\varphi_2 \equiv \varphi_1$). We shall show that $\varphi_{\mu,t}(\cdot) = K_{D,\mu}(\cdot, t)$ has both properties.

We have

$$\varphi_{\mu,t}(t) = K_{D,\mu}(t,t) \geqslant 0$$

and

$$\|K_{D,\mu}(\cdot, t)\|_\mu^2 = K_{D,\mu}(t,t)$$

Now let f satisfy the hypothesis of (ii). If $f(t) = 0$, then $\varphi_{\mu,t}(t) = 0$. Hence $\|f\|_\mu = \|\varphi_{\mu,t}\|_\mu = 0$, so $f \equiv 0 \equiv \varphi_{\mu,t}$.

Assume now $f(t) > 0$. By the definition of the weighted Bergman kernel function, $\varphi_{\mu,t}(\cdot)/\varphi_{\mu,t}(t)$ is uniquely characterized as the element in the set $\{h \in L^2_H(D,\mu),\ h(t) = 1\}$ with the minimal norm. But $f(\cdot)/f(t)$ belongs to this set as well, moreover,

$$\Big\|\frac{f(\cdot)}{f(t)}\Big\|_\mu = \frac{\|f\|_\mu}{\sqrt{f(t)}\sqrt{f(t)}} \leqslant \frac{1}{\sqrt{f(t)}} \leqslant \frac{1}{\sqrt{\varphi_{\mu,t}(t)}} = \Big\|\frac{\varphi_{\mu,t}(\cdot)}{\varphi_{\mu,t}(t)}\Big\|_\mu.$$

Thus (by minimality)

$$\frac{1}{\sqrt{f(t)}} = \frac{1}{\sqrt{\varphi_{\mu,t}(t)}}$$

and by uniqueness, for any $z \in D$

$$\frac{f(z)}{f(t)} = \frac{\varphi_{\mu,t}(z)}{\varphi_{\mu,t}(t)}.$$

So $f \equiv \varphi_{\mu,t}$. □

By means of Lemma 9 we can prove the following theorem.

Main Theorem 10. *Let $\{D_n\}_{n=1}^\infty$ be a sequence of domains in $\mathbb{C}^N$ s.t. $D \subset D_n$ for every n and $\mu \in AW(D)$, $\mu_k \in AW(D_k)$. Assume moreover that*

a) $\mu(z) \leqslant \mu_m(z)$ *for $m \in \mathbb{N}$, $z \in D$;*
b) $\mu_k \xrightarrow[k\to\infty]{} \mu$ *pointwise a.e. on D.*

Then $\{K_{D_m,\mu_m}\}_{m=1}^\infty$ converges to $K_{D,\mu}$ locally uniformly on $D \times D$ if and only if for any fixed $t \in D$

$$\lim_{m\to\infty} K_{D_m,\mu_m}(t,t) = K_{D,\mu}(t,t).$$

P r o o f. We shall only make sure that the converse implication is true, since the necessity is obvious. Let $F \subset D$ be a compact set. Then there is a constant

$M = M(F)$ such that $\max_{z\in F} |K_{D,\mu}(z,z)| \leqslant M$. By Schwarz inequality

$$\begin{aligned} |K_{D_m,\mu_m}(z,t)| &\leqslant \sqrt{K_{D_m,\mu_m}(z,z)}\sqrt{K_{D_m,\mu_m}(t,t)} \\ &\leqslant \sqrt{K_{D,\mu}(z,z)}\sqrt{K_{D,\mu}(t,t)} \leqslant M \end{aligned}$$

for any $z,t \in F$. Thus, $\{K_{D_m,\mu_m}\}_{m=1}^{\infty}$ is a Montel family on $D \times D$. It suffices to show that every convergent subsequence of this family converges to $K_{D,\mu}$. Without loss of generality let us consider $\{K_{D_m,\mu_m}\}$ itself and assume that it does converge to some k. For $t \in D$, by Fatou's lemma

$$\begin{aligned} \int_F |k(z,t)|^2\mu(z)\,\mathrm{d}V &\leqslant \liminf_{m\to\infty} \int_F |K_{D_m,\mu_m}(z,t)|^2\mu_m(z)\,\mathrm{d}V \\ &\leqslant \liminf_{m\to\infty} \int_{D_m} |K_{D_m,\mu_m}(z,t)|^2\mu_m(z)\,\mathrm{d}V \\ &= \liminf_{m\to\infty} K_{D_m,\mu_m}(t,t) = K_{D,\mu}(t,t) = k(t,t). \end{aligned}$$

Since $F \subset D$ is an arbitrary compact set,

$$\|k(\cdot,t)\|_\mu^2 \leqslant k(t,t) = K_{D,\mu}(t,t) < \infty.$$

Thus, taking $f(\cdot) = k(\cdot,t)$ in Lemma 9 we obtain $K_{D,\mu}(z,t) = k(z,t)$ for any $z,t \in D$. $\square$

Remark 11. Notice that a decreasing sequence of domains satisfies the assumptions of Main Theorem 10. This theorem for classical Bergman kernels and decreasing case of domains could be found in [17], page 37. Main Theorem 4 could be proved in the same fashion using Lemma 9 (see [20]).

3.3. Domain dependence. In this paragraph, among others, we will give a generalization of [17], page 38, for weighted Bergman kernels. Moreover, we will show that the converse of this theorem holds as well. We shall start with notation used in this paragraph.

Let us assume that $D = \mathrm{int}(\overline{D})$ to exclude slit domains from our considerations (a disc with one radius removed for instance) and consider a 'decreasing-like' version of Ramadanov's theorem. Let us recall the definition of 'approximation from outside' given in [17], Definition V.6; page 38.

Definition 12. We say that a sequence of domains $\{D_n\}_{n=1}^{\infty}$ approximates D from outside if $D \subset D_n$ for all n and for each open G such that $\overline{D} \subset G$ the inclusion $D \subset D_m \Subset G$ holds for all sufficiently large m.

Let $E \subset \mathbb{C}^N$ be Lebesgue measurable, μ be a-weight on E (we say that $\mu \in AW(E)$ if for some open set $W \subset \mathbb{C}^N$ such that $E \subset W$ there is $\nu \in AW(W)$ such that $\nu_{|E} = \mu$) and $L^2_{\text{hol}}(E, \mu)$ be the Hilbert space of all complex-valued functions which are μ-square integrable on the set E and holomorphic in the interior of E. Let moreover $H(E, \mu)$ be the subset of $L^2_{\text{hol}}(E, \mu)$ consisting of all functions possessing holomorphic extension to an open neighborhood of E. We will need the following:

Property 13. *$H(E, \mu)$ is dense in $L^2_{\text{hol}}(E, \mu)$.*

Main Theorem 14. *Suppose D is a domain, $\mu \in AW(\overline{D})$, and the density Property 13 holds when $E = \overline{D}$. Let $\{D_m\}_{m=1}^{\infty}$ be a sequence of domains approximating D from outside, and let μ_m be an admissible weight on D_m. Suppose additionally that*

a) *for every k there exists $N(k)$ such that when $m \geqslant N(k)$, both $D_m \subset D_k$ and $\mu_m(z) \leqslant \mu_k(z)$ when $z \in D_m$;*
b) *$\mu_1 \in L^1(D_1)$;*
c) *for almost every z in $\overline{D}$ both $\mu_m(z) \geqslant \mu(z)$ for every m and $\lim\limits_{m\to\infty} \mu_m(z) = \mu(z)$.*

Then

$$\lim_{m\to\infty} K_{D_m,\mu_m} = K_{D,\mu}$$

locally uniformly on $D \times D$.

Proof. Let $t \in D$ and $f \in L^2_H(D, \mu)$ be fixed. We can extend f by 0 on ∂D to provide $f \in L^2_{\text{hol}}(\overline{D}, \mu) = L^2_H(D, \mu)$ (we have already assumed $D = \text{int}(\overline{D})$). Consider any $h \in H(\overline{D}, \mu)$. Then for m large enough, $h \in L^2_H(D_m, \mu_m)$ (because of b)). We have

$$\begin{aligned}
|h(t)| &= \left| \int_{D_m} h(z) K_{D_m,\mu_m}(z,t) \mu_m(z) \, \mathrm{d}V \right| \\
&= \left| \int_{D_m} h(z) \mu_m(z)^{1/2} K_{D_m,\mu_m}(z,t) \mu_m(z)^{1/2} \, \mathrm{d}V \right| \\
&\leqslant \|h\|_{\mu_m} \left(\int_{D_m} |K_{D_m,\mu_m}(z,t)|^2 \mu_m(z) \, \mathrm{d}V \right)^{1/2} \\
&= \|h\|_{\mu_m} K_{D_m,\mu_m}(t,t)^{1/2}.
\end{aligned}$$

In the limit $m \to \infty$ we get (by the dominated convergence theorem) $|h(t)| \leqslant k(t,t)^{1/2} \|h\|_{\overline{D},\mu}$, where $k(t,t) = \lim\limits_{m\to\infty} K_{D_m,\mu_m}(t,t)$. This limit exists because we can choose a subsequence $\{\mu_{m_k}\}$ of $\{\mu_m\}$ such that $m_1 = 1$, $m_2 > \max\{1, N(1)\}$, $m_3 > \max\{m_2, N(m_2)\}, \ldots, m_{k+1} > \max\{m_k, N(m_k)\}$. Then for any $z \in D$ we have $\mu_{m_k}(z) \geqslant \mu_{m_{k+1}}(z) \geqslant \mu(z)$ and similarly as in the proof of Lemma 5 and Remark 6,

$K_{D,\mu}(t,t) \geqslant K_{D_{m_{k+1}},\mu_{m_{k+1}}}(t,t) \geqslant K_{D_{m_k},\mu_{m_k}}(t,t)$, $k = 1, 2, \dots$ for any $t \in D$. Hence, $k(t,t) = \lim_{k\to\infty} K_{D_{m_k},\mu_{m_k}}(t,t)$ exists. On the other hand, $\lim_{k\to\infty} m_k = \infty$ and then for any $n > N(1)$ there exists $k \in \mathbb{N}$ such that $m_k \leqslant n \leqslant m_{k+1}$. This implies that for any $z \in D$ $\mu_{m_k}(z) \leqslant \mu_n(z) \leqslant \mu_{m_{k-1}}(z)$ and therefore $K_{D_{m_{k-1}},\mu_{m_{k-1}}}(t,t) \leqslant K_{D_n,\mu_n}(t,t) \leqslant K_{D_{m_k},\mu_{m_k}}(t,t)$, $t \in D$. Consequently, $\lim_{n\to\infty} K_{D_n,\mu_n}(t,t)$ exists and is equal to $k(t,t)$. By density Property 13 there is a sequence $\{h_m\}$ of functions in $H(\overline{D},\mu)$ such that $h_m \xrightarrow{L^2_{\mathrm{hol}}(\overline{D},\mu)} f$. So

$$|f(t)| \leqslant k(t,t)^{1/2}\|f\|_{\overline{D},\mu} = k(t,t)^{1/2}\|f\|_{D,\mu}.$$

So for $f(\cdot) = K_{D,\mu}(\cdot,t)$ we have

$$|K_{D,\mu}(t,t)| = K_{D,\mu}(t,t) \leqslant k(t,t)^{1/2}\|K_{D,\mu}\|_\mu = k(t,t)^{1/2}K_{D,\mu}(t,t)^{1/2}.$$

Thus $K_{D,\mu}(t,t) \leqslant k(t,t)$. Obviously $K_{D_m,\mu_m}(t,t) \leqslant K_{D,\mu}(t,t)$. In the limit $m \to \infty$ we get $k(t,t) \leqslant K_{D,\mu}(t,t)$. Therefore

$$K_{D,\mu}(t,t) = k(t,t) = \lim_{m\to\infty} K_{D_m,\mu_m}(t,t).$$

The conclusion follows from Main Theorem 10. □

What is interesting, it turns out that some kind of the converse of the Main Theorem 14 holds as well, namely:

Main Theorem 15. *Let $D \subset \mathbb{C}^N$ be a domain such that the Lebesgue measure of the boundary ∂D is equal to* 0*, μ be a weight on the closure $\overline{D}$ of D in $\mathbb{C}^N$ and $\mu_{|D} \in AW(D)$. Suppose that for a sequence $\{D_n\}$ approximating D from outside, and a sequence of admissible weights $\{\mu_n\}$ (where $\mu_n \in AW(D_n)$)*

$$\lim_{n\to\infty} K_{D_n,\mu_n|D} = K_{D,\mu}$$

holds locally uniformly on $D \times D$; for any $t \in D$, $K_{D_n,\mu_n}(\cdot,t) \in L^2_H(D,\mu)$ and

$$\lim_{n\to\infty} \|K_{D_n,\mu_n}(\cdot,t)\|^2_\mu = \|K_{D,\mu}(\cdot,t)\|^2_\mu = K_{D,\mu}(t,t).$$

Then Property 13 *holds.*

P r o o f. For any $t \in D$ we have

$$
\begin{aligned}
&\|K_{D_n,\mu_n|D}(\cdot,t) - K_{D,\mu}(\cdot,t)\|_\mu^2 \\
&\quad= \int_D (K_{D_n,\mu_n}(z,t) - K_{D,\mu}(z,t))\overline{(K_{D_n,\mu_n}(z,t) - K_{D,\mu}(z,t))}\mu(z)\,\mathrm{d}V \\
&\quad= \int_D |K_{D_n,\mu_n}(z,t)|^2\mu(z)\,\mathrm{d}V - \int_D K_{D,\mu}(t,z)K_{D_n,\mu_n}(z,t)\mu(z)\,\mathrm{d}V \\
&\qquad- \overline{\int_D K_{D,\mu}(t,z)K_{D_n,\mu_n}(z,t)\mu(z)\,\mathrm{d}V} + \int_D |K_{D,\mu}(z,t)|^2\mu(z)\,\mathrm{d}V \\
&\quad= \|K_{D_n,\mu_n}(\cdot,t)\|_\mu^2 - K_{D_n,\mu_n}(t,t) - \overline{K_{D_n,\mu_n}(t,t)} + K_{D,\mu}(t,t) \\
&\quad= \|K_{D_n,\mu_n}(\cdot,t)\|_\mu^2 - 2K_{D_n,\mu_n}(t,t) + K_{D,\mu}(t,t).
\end{aligned}
$$

By assumptions

$$\lim_{n\to\infty} \|K_{D_n,\mu_n|D}(\cdot,t) - K_{D,\mu}(\cdot,t)\|_\mu^2 = 0,$$

which means that the closure in L^2-norm

$$\mathrm{cl}\{K_{D,\mu}(\cdot,t), t \in D\} \subset \mathrm{cl}\{K_{D_n,\mu_n}(\cdot,t), t \in D, n \in \mathbb{N}\} \subset L_H^2(\overline{D},\mu).$$

On the other hand, by reproducing property

$$\mathrm{cl}\{K_{D,\mu}(\cdot,t), t \in D\} = L_H^2(D,\mu) = L_{\mathrm{hol}}^2(\overline{D},\mu).$$

Taking into account that

$$\{K_{D_n,\mu_n}(\cdot,t), t \in D, n \in \mathbb{N}\} \subset H(\overline{D},\mu)$$

we obtain the desired result. □

Remark 16. Look also in [18] for some considerations concerning unweighted, decreasing case of Main Theorem 15 and very interesting remarks. Notice that taking for any n, $\mu_n \equiv 1$ we get in fact that Property 13 and the hypothesis of Main Theorem 14 are equivalent, which gives us a description of the domains for which a 'decreasing-like' version of Ramadanov's theorem holds. Moreover, using Main Theorem 4 we can prove a weighted version of the counterexample to the Lu Qi-Keng conjecture given in [3].

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