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ON LINEAR PRESERVERS OF TWO-SIDED GUT-MAJORIZATION ON $\mathbf{M}_{n,m}$

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Abstract. For $X, Y \in \mathbf{M}_{n,m}$ it is said that X is gut-majorized by Y, and we write $X \prec_{\rm gut} Y$, if there exists an n-by-n upper triangular g-row stochastic matrix R such that $X = RY$. Define the relation $\sim_{\rm gut}$ as follows. $X \sim_{\rm gut} Y$ if X is gut-majorized by Y and Y is gut-majorized by X. The (strong) linear preservers of $\prec_{\rm gut}$ on $\mathbb{R}^n$ and strong linear preservers of this relation on $\mathbf{M}_{n,m}$ have been characterized before. This paper characterizes all (strong) linear preservers and strong linear preservers of $\sim_{\rm gut}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$.

Keywords: g-row stochastic matrix; gut-majorization; linear preserver; strong linear preserver; two-sided gut-majorization

MSC 2010: 15A04, 15A21

1. Introduction

Let $\mathbf{M}_{n,m}$ be the algebra of all n-by-m real matrices, and $\mathbb{R}^n$ be the set of all n-by-1 real column vectors. An n-by-n real matrix (not necessarily nonnegative) A is g-*row stochastic* (generalized row stochastic) if all its row sums are one. Let $X, Y \in \mathbf{M}_{n,m}$. Matrix X is said to be *gut-majorized* by Y and it is denoted by $X \prec_{\rm gut} Y$ if there exists an n-by-n upper triangular g-row stochastic matrix R such that $X = RY$. We also say that $X \sim_{\rm gut} Y$ if and only if $X \prec_{\rm gut} Y \prec_{\rm gut} X$, and call this *two-sided gut-majorization*.

A linear function $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserves an order relation $\prec$ in $\mathbf{M}_{n,m}$ if $TX \prec TY$ whenever $X \prec Y$. Also, T is said to strongly preserve if for all X, $Y \in \mathbf{M}_{n,m}$

$$X \prec Y \Leftrightarrow TX \prec TY.$$

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The (strong) linear preservers and strong preservers of $\prec_{\text{gut}}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$ are fully characterized in [1]. For more information about linear preservers of majorization we refer the reader to [2]–[10].

Some of our notation and symbols are as follows:

$\mathcal{R}_n^{\text{gut}}$: the collection of all n-by-n upper triangular g-row stochastic matrices;

E: the n-by-n matrix with all of the entries of the last column equal to one and the other entries equal to zero;

e: the column real vectors with all of the entries equal to one;

$\{e_1, \dots, e_n\}$: the standard basis of $\mathbb{R}^n$;

$[x_1 \mid \dots \mid x_m]$: the n-by-m matrix with columns $x_1, \dots, x_m \in \mathbb{R}^n$;

$A(n_1, \dots, n_l \mid m_1, \dots, m_k)$: the submatrix of A obtained from A by deleting rows $n_1, \dots, n_l$ and columns $m_1, \dots, m_k$;

$A(n_1, \dots, n_l)$: the abbreviation of $A(n_1, \dots, n_l \mid n_1, \dots, n_l)$;

$\mathbb{N}_k$: the set $\{1, \dots, k\} \subset \mathbb{N}$;

A^t: the transpose of a given matrix A;

$[T]$: the matrix representation of a linear function $T\colon \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis;

r_i: the sum of entries on the ith row of $[T]$.

This paper is organized as follows. In Section 2, we first introduce the relation $\sim_{\text{gut}}$ on $\mathbb{R}^n$ and we express an equivalent condition for this majorization. Finally, we obtain some results characterizing the structure of (strong) linear preservers of this relation on $\mathbb{R}^n$. One of the main results of this paper is to find the structure of linear functions $T\colon \mathbb{R}^n \to \mathbb{R}^n$ preserving (strongly preserving) $\sim_{\text{gut}}$. The last section of this paper studies some facts of this concept that are necessary for studying the strong linear preservers of $\sim_{\text{gut}}$ on $\mathbf{M}_{n,m}$. Also, the strong linear preservers of $\sim_{\text{gut}}$ on $\mathbf{M}_{n,m}$ are obtained.

2. Two-sided gut-majorization on $\mathbb{R}^n$

First, we review some sticking point of $\sim_{\text{gut}}$ on $\mathbb{R}^n$, and then we establish some properties to prove the main theorems. Also, we characterize all linear functions $T\colon \mathbb{R}^n \to \mathbb{R}^n$ preserving (strongly preserving) $\sim_{\text{gut}}$.

Definition 2.1. Let $x, y \in \mathbb{R}^n$. Then x is said to be *two-sided gut-majorized* by y (in symbol $x \sim_{\text{gut}} y$) if $x \prec_{\text{gut}} y \prec_{\text{gut}} x$.

The following proposition gives an equivalent condition for this relation on $\mathbb{R}^n$. We state the result without proof.

Proposition 2.1. *Let* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. *Then* $x \sim_{\text{gut}} y$ *if and only if*

$$\min\{i\colon\ x_i = x_{i+1} = \ldots = x_n\} = \min\{i\colon\ y_i = y_{i+1} = \ldots = y_n\},$$

and $x_n = y_n$.

The following lemmas are useful for finding the structure of (strong) linear preservers of two-sided gut-majorization on $\mathbb{R}^n$.

Lemma 2.1. *Let* $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ *be a linear preserver of* $\sim_{\text{gut}}$. *Assume* $S\colon\ \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ *is a linear function such that* $[S] = [T](1, \ldots, k)$. *Then* S *preserves* $\sim_{\text{gut}}$ *on* $\mathbb{R}^{n-k}$.

Proof. Let $x' = (x_{k+1}, \ldots, x_n)^t, y' = (y_{k+1}, \ldots, y_n)^t \in \mathbb{R}^{n-k}$ and let $x' \sim_{\text{gut}} y'$. Define $x := (0, \ldots, 0, x_{k+1}, \ldots, x_n)^t$, $y := (0, \ldots, 0, y_{k+1}, \ldots, y_n)^t \in \mathbb{R}^n$. Then, by Propositon 2.1, $x \sim_{\text{gut}} y$ and hence $Tx \sim_{\text{gut}} Ty$. It implies that $Sx' \sim_{\text{gut}} Sy'$. Therefore S preserves $\sim_{\text{gut}}$ on $\mathbb{R}^{n-k}$. □

Lemma 2.2. *Let* $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ *be a linear preserver of* $\sim_{\text{gut}}$, *and let* $[T] = [a_{ij}]$. *Then* $a_{n1} = a_{n2} = \ldots = a_{nn-1} = 0$.

Proof. We proceed by induction. The result is clear for $n = 1$. For $n = 2$ we should prove $a_{21} = 0$. Set $x = 2e_1 + e_2$ and $y = e_2$. As $x \sim_{\text{gut}} y$, it follows that $Tx \sim_{\text{gut}} Ty$. Thus, $2a_{21} + a_{22} = a_{22}$, and hence $a_{21} = 0$. Suppose that $n > 2$ and that the assertion has been established for all linear preservers of $\sim_{\text{gut}}$ on $\mathbb{R}^{n-1}$. Let $S\colon\ \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a linear function with $[S] = [T](1)$. Lemma 2.1 states that S preserves $\sim_{\text{gut}}$ on $\mathbb{R}^{n-1}$. The induction hypothesis ensures that $a_{n2} = \ldots = a_{nn-1} = 0$. So it is enough to show that $a_{n1} = 0$. Consider $x = e_1 + e_n$ and $y = e_n$. Observe that $x \sim_{\text{gut}} y$, and then $Tx \sim_{\text{gut}} Ty$. It implies that $a_{n1} = 0$ as well. □

Lemma 2.3. *Let* $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ *be a linear function such that* $a_{kt} \neq 0$ *for some* $k, t \in \mathbb{N}_{n-1}$, *where* $[T] = [a_{ij}]$. *Suppose that* $a_{k+1t} = \ldots = a_{nt} = 0$ *and there exists some* j $(t + 1 \leqslant j \leqslant n - 1)$ *such that* $a_{k+1j} = \ldots = a_{nj} = 0$. *Then* T *does not preserve* $\sim_{\text{gut}}$.

Proof. Set $x = -(a_{kj}/a_{kt})e_t + e_j$ and $y = y_t e_t + e_j$ where $y_t \in \mathbb{R}\setminus\{-a_{kj}/a_{kt}\}$. It is easy to see that $x \sim_{\text{gut}} y$ but $Tx \not\sim_{\text{gut}} Ty$. Therefore T does not preserve $\sim_{\text{gut}}$. □

Lemma 2.4. *Let* $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ *be a linear preserver of* $\sim_{\text{gut}}$. *Then* $[T]$ *is an upper triangular matrix.*

Proof. Let $[T] = [a_{ij}]$. Use induction on n. For $n = 1$, the result is clear. If $n = 2$, we should only prove that $a_{21} = 0$. Then Lemma 2.2 ensures the result. For $n > 2$ assume that the matrix representation of every linear preserver of $\sim_{\text{gut}}$ on $\mathbb{R}^{n-1}$ is an upper triangular matrix. Let $S\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. Lemma 2.1 ensures that the linear function S preserves $\sim_{\text{gut}}$ on $\mathbb{R}^{n-1}$. The induction hypothesis ensures that $[S]$ is an $(n-1)$-by-$(n-1)$ upper triangular matrix. Also, Lemma 2.2 states that $a_{n1} = 0$. So it is enough to show that $a_{21} = a_{31} = \ldots = a_{n-11} = 0$. Assume, if possible, that $a_{k1} \neq 0$, where $k = \max\{2 \leqslant i \leqslant n-1\colon a_{i1} \neq 0\}$. By Lemma 2.3, we see that T does not preserve $\sim_{\text{gut}}$, which would be a contradiction. Thus $a_{21} = a_{31} = \ldots = a_{n-11} = 0$, and then the induction argument is completed. Therefore $[T]$ is an upper triangular matrix. □

The following theorem characterizes the structure of all linear functions $T\colon \mathbb{R}^n \to \mathbb{R}^n$, preserving $\sim_{\text{gut}}$.

Theorem 2.1. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T preserves $\sim_{\text{gut}}$ if and only if one of the following assertions holds.*

(i) *$Te_1 = \ldots = Te_{n-1} = 0$. In other words,*

$$[T] = \begin{pmatrix} 0 & \ldots & 0 & a_{1n} \\ 0 & \ldots & 0 & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & a_{nn} \end{pmatrix}.$$

(ii) *There exist $t \in \mathbb{N}_{n-1}$ and $1 \leqslant i_1 < \ldots < i_m \leqslant n-1$ such that $a_{i_1 t}, a_{i_2 t+1}, \ldots, a_{i_m n-1} \neq 0$,*

$$[T] = \begin{pmatrix} 0 & * & & & & & \\ & a_{i_1 t} & * & & & & \\ & & \ddots & & & & \\ & & & a_{i_2 t+1} & & & \\ & & & & \ddots & & \\ & 0 & & & & a_{i_m n-1} & \\ & & & & & 0 & * \end{pmatrix},$$

and

(1) *$r_{i_1} = \ldots = r_n$ or*

(2) *for some $k \in (i_m, n) \cup \bigcup_{j=1}^{i_{m-1}} (i_j, i_{j+1})$, $r_k \neq r_{k+1} = \ldots = r_n$.*

Proof. First, we prove the sufficiency of the conditions. If (i) holds, let $x = (x_1,\dots,x_n)^t$, $y = (y_1,\dots,y_n)^t \in \mathbb{R}^n$ such that $x \sim_{\rm gut} y$. Proposition 2.1 ensures that $x_n = y_n$. So $Tx = Ty$, and then $Tx \sim_{\rm gut} Ty$. Assume that (ii) holds. The proof is by induction on n. If $n = 2$, by the hypothesis we see $[T] = \left(\begin{smallmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{smallmatrix}\right)$, $a_{11} \neq 0$, and $r_1 = r_2$. Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$ such that $x \sim_{\rm gut} y$. So $Tx = (a_{11}x_1 + a_{12}y_2, a_{22}y_2)^t$ and $Ty = (a_{11}y_1 + a_{12}y_2, a_{22}y_2)^t$. Observe that $(Tx)_1 = (Tx)_2$ if and only if $x_1 = y_2$, and also $(Ty)_1 = (Ty)_2$ if and only if $y_1 = y_2$, because $r_1 = r_2$ and $a_{11} \neq 0$. Now, as $x \sim_{\rm gut} y$, we deduce that $(Tx)_1 = (Tx)_2$ is equivalent to $(Ty)_1 = (Ty)_2$. Thus, $Tx \sim_{\rm gut} Ty$. Suppose that $n \geqslant 3$ and the result has been proved for all linear functions on $\mathbb{R}^{n-1}$ with the described conditions in the hypothesis. Let $x = (x_1,\dots,x_n)^t$, $y = (y_1,\dots,y_n)^t \in \mathbb{R}^n$ such that $x \sim_{\rm gut} y$. We have to show that $Tx \sim_{\rm gut} Ty$. For this purpose, let $S\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a linear function with $[S] = [T](1)$. Set $x' = (x_2,\dots,x_n)^t$ and $y' = (y_2,\dots,y_n)^t$. Then $x' \sim_{\rm gut} y'$ and hence, by applying the induction hypothesis for S, $Sx' \sim_{\rm gut} Sy'$. That is, $((Tx)_2,\dots,(Tx)_n)^t \sim_{\rm gut} ((Ty)_2,\dots,(Ty)_n)^t$. If there exists some i $(2 \leqslant i \leqslant n-1)$ such that $(Tx)_i \neq (Tx)_{i+1}$, then the proof is complete. Otherwise, $(Tx)_2 = \ldots = (Tx)_n = (Ty)_2 = \ldots = (Ty)_n$.

If (1) holds, $(Tx)_{i_m} = (Tx)_n$ implies that $x_{n-1} = y_n$, because $a_{i_m n-1} \neq 0$ and $r_{i_m} = r_n$. Since $x \sim_{\rm gut} y$, we see that $y_{n-1} = y_n$. By continuing this process, we can conclude that $x_t = \ldots = x_n = y_t = \ldots = y_n$. Hence $(Tx)_1 = (Ty)_1$, and then $Tx \sim_{\rm gut} Ty$.

Suppose (2) holds, case (1). If there is some $k \in (i_m, n)$ such that $r_k \neq r_{k+1} = \ldots = r_n$, as $(Tx)_k = (Tx)_n$, we have $a_{kn}y_n = a_{nn}y_n$. The relation $a_{kn} \neq a_{nn}$ ensures that $y_n = 0$, and then $(Ty)_n = 0$. It means that $(Tx)_2 = \ldots = (Tx)_n = (Ty)_2 = \ldots = (Ty)_n = 0$. On the other hand, since $(Tx)_{i_m} = 0$ and $a_{i_m n-1} \neq 0$, we deduce that x_{n-1} and also y_{n-1} are zero. It is a simple matter to see that $x_t = \ldots = x_n = y_t = \ldots = y_n = 0$. So $(Tx)_1 = (Ty)_1 = 0$, which completes the proof.

Case (2). If there exists some $k \in (i_j, i_{j+1})$ for some $j \in \mathbb{N}_{i_m-1}$ such that $r_k \neq r_{k+1} = \ldots = r_n$, as $r_{k+1} = \ldots = r_n$ and $a_{i_{j+1}l}, \ldots, a_{i_m n-1} \neq 0$, we observe that $x_l = \ldots = x_n = y_l = \ldots = y_n$. Now, $(Tx)_k = (Tx)_n$ and $r_k \neq r_n$ imply that $y_n = 0$. So $x_l = \ldots = x_n = y_l = \ldots = y_n = 0$. If $i_1 = 1$, by continuing this procedure, we find that $x_{t+1} = \ldots = x_n = y_{t+1} = \ldots = y_n = 0$. So $(Tx)_2 = \ldots = (Tx)_n = (Ty)_2 = \ldots = (Ty)_n = 0$, $(Tx)_1 = a_{1t}x_t$, and $(Ty)_1 = a_{1t}y_t$. Clearly, $(Tx)_1 \neq 0$ is equivalent to $(Ty)_1 \neq 0$, and then $Tx \sim_{\rm gut} Ty$. If $i_1 > 1$, we can prove that $x_t = \ldots = x_n = y_t = \ldots = y_n = 0$, and thus $(Tx)_1 = (Ty)_1 = 0$, which is the desired conclusion.

For the converse, assume that T preserves $\sim_{\rm gut}$ and (i) does not hold. We show that (ii) holds. We use induction on n. First, consider the case $n = 2$. Lemma 2.4 ensures that T is upper triangular. So $a_{11} \neq 0$. We want to prove $r_1 = r_2$. If $r_1 \neq r_2$, choose $x = ((a_{22} - a_{12})/a_{11}, 1)^t$ and $y = (y_1, 1)^t$, in which $y_1 \in \mathbb{R} \setminus \{1, (a_{22} - a_{12})/a_{11}\}$. Clearly $x \sim_{\text{gut}} y$ and hence $Tx \sim_{\text{gut}} Ty$. It means that $(a_{22}, a_{22})^t \sim_{\text{gut}} (a_{11}y_1 + a_{12}, a_{22})^t$, a contradiction. Thus, $r_1 = r_2$. Now, suppose that $n \geqslant 3$ and the statement holds for linear preservers of $\sim_{\text{gut}}$ on $\mathbb{R}^{n-1}$. Let $S\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. Lemma 2.1 ensures that S preserves $\sim_{\text{gut}}$ on $\mathbb{R}^{n-1}$. Apply the induction hypothesis for S. So the proof will be divided into two steps.

Step 1. S satisfies (i). By Lemma 2.3, the first nonzero column of T should be its $(n-1)$st column. Because if the first nonzero column of T is less than its $(n-1)$st column, since $(n-1)$st column of S is zero, T does not preserve $\sim_{\text{gut}}$. If there exists some i $(2 \leqslant i \leqslant n-1)$ such that $a_{in} \neq a_{nn}$, then T satisfies (2). Otherwise we have to just show that $r_1 = \ldots = r_n$. Assume, if possible, that $r_1 \neq r_2 = \ldots = r_n$. Consider $x = (a_{nn} - a_{1n})/(a_{1n-1})e_{n-1} + e_n$ and $y = y_{n-1}e_{n-1} + e_n$, where $y_{n-1} \in \mathbb{R} \setminus \{1, (a_{nn} - a_{1n})/(a_{1n-1})\}$. Thus, $x \sim_{\text{gut}} y$, and so $Tx \sim_{\text{gut}} Ty$, which is a contradiction. Therefore $r_1 = r_n$. We see that (1) holds.

Step 2. S satisfies (ii). If columns $1, \ldots, t-1$ of T are zero, then there is nothing to prove. If not, Lemma 2.3 ensures that the first nonzero column of T should be its $(t-1)$st column, that is,

$$[T] = \begin{pmatrix} a_{1t-1} & & & & * \\ & \ddots & & & \\ & & a_{i_2t} & & \\ & & & \ddots & \\ 0 & & & & a_{i_mn-1} \\ & & & & 0 \quad * \end{pmatrix}.$$

If (2) holds for $[S]$, then there is nothing to prove. Suppose that (1) holds for $[S]$. Then $r_{i_2} = \ldots = r_n$. If $\operatorname{card}\{r_2, \ldots, r_{i_2}\} \geqslant 2$, observe that T satisfies (2), and then the proof is complete. If $r_2 = \ldots = r_{i_2}$, it is enough to prove $r_1 = r_n$. Without loss of generality, we can assume that $a_{1t-1} = 1$. If $r_1 \neq r_n$, by setting $x = x_{t-1}e_{t-1} + \sum_{i=t}^{n} e_i$ and $y = y_{t-1}e_{t-1} + \sum_{i=t}^{n} e_i$, where $x_{t-1} = a_{nn} - \sum_{j=t}^{n} a_{1j}$ and $y_{t-1} \in \mathbb{R} \setminus \left\{1, a_{nn} - \sum_{j=t}^{n} a_{1j}\right\}$, it follows that $x \sim_{\text{gut}} y$, and so $Tx \sim_{\text{gut}} Ty$, which would be a contradiction. Therefore, $r_1 = r_n$, and the desired conclusion holds. □

Now, we focus on finding strong linear preservers of $\sim_{\text{gut}}$ on $\mathbb{R}^n$. We need the following lemma to prove the next theorem.

Lemma 2.5. *Let $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear function that strongly preserves $\sim_{\text{gut}}$. Then T is invertible.*

P r o o f. Suppose that $TX = 0$, where $X \in \mathbf{M}_{n,m}$. Notice that since T is linear, we have $T0 = 0 = TX$. Then it is obvious that $TX \sim_{\rm gut} T0$. Therefore $X \sim_{\rm gut} 0$, because T strongly preserves $\sim_{\rm gut}$. Then $X = 0$, and hence T is invertible. □

We are now ready to prove one of the main theorems of this section. The following theorem characterizes all linear functions $T\colon \mathbb{R}^n \to \mathbb{R}^n$ which strongly preserve $\sim_{\rm gut}$.

Theorem 2.2. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T strongly preserves $\sim_{\rm gut}$ if and only if $[T] = \alpha A$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ and invertible matrix $A \in \mathcal{R}_n^{\rm gut}$.*

P r o o f. First, we prove the necessity of the condition. Assume that T strongly preserves $\sim_{\rm gut}$. It means that T is invertible. Lemma 2.4 ensures that $a_{11} \neq 0$. So, by Theorem 2.1, the desired conclusion is true.

Next, since both T and T^{-1} preserve $\sim_{\rm gut}$ by Theorem 2.1, we have that T strongly preserves $\sim_{\rm gut}$. □

Corollary 2.1. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ preserve $\sim_{\rm gut}$. Then T strongly preserves $\sim_{\rm gut}$ if and only if T is invertible.*

3. Two-sided gut-majorization on $M_{n,m}$

In this section, we discuss some properties of $\sim_{\rm gut}$ on $\mathbf{M}_{n,m}$, and we find the structure of strong linear preservers of this relation on $\mathbf{M}_{n,m}$. First, we state some lemmas.

Lemma 3.1. *Let $A \in \mathbf{M}_n$. Then the following conditions are equivalent.*

(a) *For each invertible matrix $D \in \mathcal{R}_n^{\rm gut}$, $AD = DA$.*

(b) *For some $\alpha, \beta \in \mathbb{R}$, $A = \alpha I + \beta E$.*

(c) *For each invertible matrix $D \in \mathcal{R}_n^{\rm gut}$ and for all $x, y \in \mathbb{R}^n$,*

$$(Dx + ADy) \sim_{\rm gut} (x + Ay).$$

P r o o f. (a) ⇒ (b): First, by considering

$$D = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ & \frac{1}{2} & \frac{1}{2} & & 0 & \\ & & & \ddots & & \\ & 0 & & & \frac{1}{2} & \frac{1}{2} \\ & & & & & 1 \end{pmatrix},$$

observe that

$$A = \begin{pmatrix} \alpha & \alpha_1 & \alpha_2 & \dots & \alpha_l & \alpha_{l+1} & a_{1n} \\ & \alpha & \alpha_1 & \alpha_2 & \dots & \alpha_l & a_{2n} \\ & & \ddots & \ddots & \ddots & & \\ & & & \alpha & \alpha_1 & \alpha_2 & a_{n-3n} \\ & & 0 & & \alpha & \alpha_1 & a_{n-2n} \\ & & & & & \alpha & \beta \\ & & & & & & \alpha+\beta \end{pmatrix}$$

for some $\alpha, \beta, \alpha_1, \dots, \alpha_{l+1} \in \mathbb{R}$ such that $\alpha_{l+1} + a_{1n} = a_{2n}$, $\alpha_l + a_{2n} = a_{3n}, \dots$, $\alpha_1 + a_{n-2n} = \beta$. Next set

$$D = \begin{pmatrix} 1 & 0 & & \dots & & & 0 \\ & \frac{1}{2} & 0 & & \dots & 0 & \frac{1}{2} \\ & & \frac{1}{2} & 0 & \dots & 0 & \frac{1}{2} \\ & & 0 & & \ddots & & \\ & & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & & & 1 \end{pmatrix}.$$

We deduce that $\alpha_1 = \ldots = \alpha_{l+1} = 0$. Then $a_{1n} = a_{2n} = \ldots = a_{n-2n} = \beta$. Therefore $A = \alpha I + \beta E$.

(b) $\Rightarrow$ (c): Assume that the invertible matrix $D \in \mathcal{R}_n^{\text{gut}}$ and let $x, y \in \mathbb{R}^n$. As $ED = E = DE$, we see that $Dx + ADy = D(x + Ay)$. So $(Dx + ADy) \sim_{\text{gut}} (x + Ay)$.

(c) $\Rightarrow$ (a): Choose $i \in \mathbb{N}_n$ and define $x \colon = e - Ae_i$ and $y := e_i$. Consider the invertible matrix $D \in \mathcal{R}_n^{\text{gut}}$. The hypothesis ensures that $(e - DAe_i + ADe_i) \sim_{\text{gut}} e$. Hence $(-DA + AD)e_i = 0$, and then $AD = DA$. □

For each $i, j \in \mathbb{N}_m$ consider the embedding $E^j \colon \mathbb{R}^n \to \mathbf{M}_{n,m}$ and the projection $E_i \colon \mathbf{M}_{n,m} \to \mathbb{R}^n$, where $E^j(x) = xe_j^t$ and $E_i(A) = Ae_i$. It is easy to show that for every linear function $T \colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$,

$$TX = T[x_1 \mid \ldots \mid x_m] = \left[\sum_{j=1}^m T_1^j x_j \mid \ldots \mid \sum_{j=1}^m T_m^j x_j\right],$$

where $T_i^j = E_i T E^j$.

It is easy to see that if $T \colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is a linear preserver of $\sim_{\text{gut}}$, then T_i^j preserves $\sim_{\text{gut}}$ on $\mathbb{R}^n$ for all $i, j \in \mathbb{N}_m$.

We need the following lemmas to prove the main theorem of this section.

Lemma 3.2 ([1], Lemma 3.3.)**.** *Let $T \colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ satisfy $TX = XR + EXS$ for some $R, S \in \mathbf{M}_m$. Then T is invertible if and only if $R(R+S)$ is invertible.*

Lemma 3.3. *Let $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserve $\sim_{\rm gut}$. If for some $i \in \mathbb{N}_m$ there exists $k \in \mathbb{N}_m$ such that T_i^k is invertible, then*

$$\sum_{j=1}^{m} A_i^j x_j = A_i^k \sum_{j=1}^{m} \alpha_i^j x_j + E \sum_{j=1}^{m} \beta_i^j x_j$$

for some $\alpha_i^j, \beta_i^j \in \mathbb{R}$, where $A_i^j = [T_i^j]$.

Proof. There is no loss of generality to assume that $i, k = 1$ and $j = 2$. We show that there exist $\alpha_1^2, \beta_1^2 \in \mathbb{R}$ such that $A_1^2 = \alpha_1^2 A_1^1 + \beta_1^2 E$. Let $D \in \mathcal{R}_n^{\rm gut}$ be invertible and $x, y \in \mathbb{R}^n$. Observe that

$$D[x \mid y \mid 0 \mid \ldots \mid 0] \sim_{\rm gut} [x \mid y \mid 0 \mid \ldots \mid 0],$$

and then

$$T[Dx \mid Dy \mid 0 \mid \ldots \mid 0] \sim_{\rm gut} T[x \mid y \mid 0 \mid \ldots \mid 0].$$

So

$$[A_1^1 Dx + A_1^2 Dy \mid * \mid *] \sim_{\rm gut} [A_1^1 x + A_1^2 y \mid * \mid *],$$

and thus

$$A_1^1 Dx + A_1^2 Dy \sim_{\rm gut} A_1^1 x + A_1^2 y.$$

By Theorem 2.2, A_1^1 is a nonzero multiple of an invertible matrix in $\mathcal{R}_n^{\rm gut}$ and hence

$$Dx + (A_1^1)^{-1} A_1^2 Dy \sim_{\rm gut} x + (A_1^1)^{-1} A_1^2 y.$$

Now, Lemma 3.1 ensures that there exist $\alpha_1^2, \beta_1^2 \in \mathbb{R}$ such that $A_1^2 = \alpha_1^2 A_1^1 + \beta_1^2 E$. □

Lemma 3.4. *If $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ strongly preserves $\sim_{\rm gut}$, then for each $i \in \mathbb{N}_m$ there exists $j \in \mathbb{N}_m$ such that T_i^j is invertible.*

Proof. Let $I = \{i \in \mathbb{N}_m\colon T_i^j e_1 = 0 \text{ for all } j \in \mathbb{N}_m\}$. We prove that I is empty. If I is not empty, we can assume without loss of generality $I = \{1, 2, \ldots, k\}$, where $k \in \mathbb{N}_m$. We consider two cases.

Case 1. $k = m$; let $X = [e_1 \mid 0 \mid \ldots \mid 0] \in \mathbf{M}_{n,m}$. We observe that $X \neq 0$ but $TX = 0$. This yields that T is not invertible, which is a contradiction by Lemma 2.5.

Case 2. $k < m$; by Lemma 3.3, for i ($k+1 \leqslant i \leqslant m$) and $j \in \mathbb{N}_m$ there exist invertible matrices A_i and $\alpha_i^j, \beta_i^j \in \mathbb{R}$ such that $\sum\limits_{j=1}^{m} A_i^j x_j = A_i \sum\limits_{j=1}^{m} \alpha_i^j x_j + E \sum\limits_{j=1}^{m} \beta_i^j x_j$. Consider vectors $(\alpha_{k+1}^1, \ldots, \alpha_m^1)^t, \ldots, (\alpha_{k+1}^m, \ldots, \alpha_m^m)^t \in \mathbb{R}^{m-k}$. Since $m - k < m$, there exist $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$, not all zero, such that $\gamma_1(\alpha_{k+1}^1, \ldots, \alpha_m^1)^t + \ldots +$

$\gamma_m(\alpha^m_{k+1},\ldots,\alpha^m_m)^t = 0$. Let $x_j = \gamma_j e_1$ for each $j \in \mathbb{N}_m$. Since for every i $(k+1 \leqslant i \leqslant m)$, $A_i \in \mathcal{R}^n_{\rm gut}$ is invertible, we have $0 \neq A_i e_1 \in \operatorname{Span}\{e_1\}$. As a multiple of e_1 has no effect on the desired answer, we can assume without loss of generality $A_i e_1 = e_1$. This implies that $A_i \sum\limits_{j=1}^{m} \alpha_i^j x_j + E \sum\limits_{j=1}^{m} \beta_i^j x_j = 0$. By putting $X = [x_1 \mid \ldots \mid x_m] \in \mathbf{M}_{n,m}$ we see that $X \neq 0$, and $TX = 0$, a contradiction. Therefore for each $i \in \mathbb{N}_m$ there exists $j \in \mathbb{N}_m$ such that $T_i^j e_1 \neq 0$ and hence T_i^j is invertible. □

The last theorem of this paper, which is our main result in this section, characterizes the strong linear preservers of $\sim_{\rm gut}$ on $\mathbf{M}_{n,m}$.

Theorem 3.1. *Let $T\colon \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear function. Then T strongly preserves $\sim_{\rm gut}$ if and only if there exist $R, S \in \mathbf{M}_m$ such that $R(R+S)$ is invertible, and invertible matrix $A \in \mathcal{R}_n^{\rm gut}$ such that $TX = AXR + EXS$.*

Proof. First, we prove the sufficiency of the conditions. Let $X, Y \in \mathbf{M}_{n,m}$ such that $X \sim_{\rm gut} Y$. [1], Theorem 1.3 ensures that T strongly preserves $\prec_{\rm gut}$. So $X \sim_{\rm gut} Y$ if and only if $X \prec_{\rm gut} Y \prec_{\rm gut} X$ if and only if $TX \prec_{\rm gut} TY \prec_{\rm gut} TX$ if and only if $TX \sim_{\rm gut} TY$. This shows that T strongly preserves $\sim_{\rm gut}$.

Next, assume that T strongly preserves $\sim_{\rm gut}$. For $m = 1$ see Theorem 2.2. Suppose that $m > 1$. Lemma 3.4 ensures that for each $i \in \mathbb{N}_m$ there exists some $j \in \mathbb{N}_m$ such that T_i^j is invertible. Lemma 3.3 ensures that there exist invertible matrices $A_1, \ldots, A_m \in \mathbf{M}_n$, vectors $a_1, \ldots, a_m \in \mathbb{R}^m$, and a matrix $S' \in \mathbf{M}_m$ such that $TX = [A_1 X a_1 \mid \ldots \mid A_m X a_m] + EXS'$. One can prove that $\operatorname{rank}\{a_1, \ldots, a_m\} \geqslant 2$. Without loss of generality, assume that $\{a_1, a_2\}$ is a linearly independent set. This implies that for every $x, y \in \mathbb{R}^n$ there exists $B_{x,y} \in \mathbf{M}_{n,m}$ such that $B_{x,y} a_1 = x$ and $B_{x,y} a_2 = y$. Let $X \in \mathbf{M}_{n,m}$ and invertible matrix $D \in \mathcal{R}_n^{\rm gut}$. So $DX \sim_{\rm gut} X$, and then $TDX \sim_{\rm gut} TX$. Thus

$$[A_1 DX a_1 \mid \ldots \mid A_m DX a_m] + EDXS \sim_{\rm gut} [A_1 X a_1 \mid \ldots \mid A_m X a_m] + EXS.$$

Clearly, $A_1 DX a_1 + A_2 DX a_2 \sim_{\rm gut} A_1 X a_1 + A_2 X a_2$. So for each $X \in \mathbf{M}_{n,m}$ and each invertible matrix $D \in \mathcal{R}_n^{\rm gut}$ we have

$$DXa_1 + A_1^{-1} A_2 DXa_2 \sim_{\rm gut} Xa_1 + A_1^{-1} A_2 X a_2. \tag{1}$$

By replacing $X = B_{x,y}$ in (1), $Dx + A_1^{-1} A_2 Dy \sim_{\rm gut} x + A_1^{-1} A_2 y$ for each invertible matrix $D \in \mathcal{R}_n^{\rm gut}$ and for each $x, y \in \mathbb{R}^n$. Lemma 3.1 states that $A_2 = \alpha A_1 + \beta E$ for some $\alpha, \beta \in \mathbb{R}$. For every $i \geqslant 3$ if $a_i = 0$, we can choose $A_i = A_1$. If $a_i \neq 0$, then $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is linearly independent. Similarly to the above, $A_i = \gamma_i A_1 + \delta_i E$

for some $\gamma_i, \delta_i \in \mathbb{R}$. Define $A := A_1$. Then for every $i \geqslant 2$, $A_i = \alpha_i A + \beta_i E$ for some $\alpha_i, \beta_i \in \mathbb{R}$. So

$$TX = [AXa_1 \mid AX(r_2a_2) \mid \ldots \mid AX(r_ma_m)] + EXS = AXR + EXS,$$

where $R = [a_1 \mid r_2a_2 \mid \ldots \mid r_ma_m]$ for some $r_2, \ldots, r_m \in \mathbb{R}$ and $S = S' + [0 \mid \beta_2a_2 \mid \ldots \mid \beta_ma_m]$. □

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