Czechoslovak Mathematical Journal, 68 (143) (2018), 771–789

# EQUIVALENT CONDITIONS FOR THE VALIDITY OF THE HELMHOLTZ DECOMPOSITION OF MUCKENHOUPT $A_p$-WEIGHTED $L^p$-SPACES

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Received December 23, 2016. Published online April 16, 2018.

*Abstract.* We discuss the validity of the Helmholtz decomposition of the Muckenhoupt $A_p$-weighted $L^p$-space $(L^p_w(\Omega))^n$ for any domain $\Omega$ in $\mathbb{R}^n$, $n \in \mathbb{Z}$, $n \geqslant 2$, $1 < p < \infty$ and Muckenhoupt $A_p$-weight $w \in A_p$. Set $p' := p/(p-1)$ and $w' := w^{-1/(p-1)}$. Then the Helmholtz decomposition of $(L^p_w(\Omega))^n$ and $(L^{p'}_{w'}(\Omega))^n$ and the variational estimate of $L^p_{w,\pi}(\Omega)$ and $L^{p'}_{w',\pi}(\Omega)$ are equivalent. Furthermore, we can replace $L^p_{w,\pi}(\Omega)$ and $L^{p'}_{w',\pi}(\Omega)$ by $L^p_{w,\sigma}(\Omega)$ and $L^{p'}_{w',\sigma}(\Omega)$, respectively. The proof is based on the reflexivity and orthogonality of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$ and the Hahn-Banach theorem. As a corollary of our main result, we obtain the extrapolation theorem with the aid of the Helmholtz projection of $(L^p_w(\Omega))^n$.

*Keywords*: Helmholtz decomposition; Muckenhoupt $A_p$-weighted $L^p$-spaces; variational estimate

*MSC 2010*: 35Q30, 46E30, 76D05

## 1. Introduction

Let $n \in \mathbb{Z}$ and $n \geqslant 2$, and consider the Muckenhoupt $A_p$-weighted $L^p$-space $(L^p_w(\Omega))^n$ for any domain $\Omega$ in $\mathbb{R}^n$, $1 < p < \infty$ and Muckenhoupt $A_p$-weight $w \in A_p$. From the viewpoint of the Stokes and Navier-Stokes equations as abstract evolution equations in $(L^p_w(\Omega))^n$, it is crucial to discuss the Helmholtz decomposition

$$(L^p_w(\Omega))^n = L^p_{w,\sigma}(\Omega) \oplus L^p_{w,\pi}(\Omega) \tag{1.1}$$

of $(L^p_w(\Omega))^n$, where $L^p_{w,\sigma}(\Omega)$ and $L^p_{w,\pi}(\Omega)$ are closed subspaces of solenoidal vector fields and gradient vector fields of $(L^p_w(\Omega))^n$, respectively. Indeed, the Helmholtz

DOI: 10.21136/CMJ.2018.0646-16

decomposition of $(L^p_w(\Omega))^n$ constructs the Helmholtz projection $P_{p,w}$ from $(L^p_w(\Omega))^n$ onto $L^p_{w,\sigma}(\Omega)$, and the theory of the Stokes operator $A_{p,w}$ in $L^p_{w,\sigma}(\Omega)$ defined as

$$A_{p,w} = -P_{p,w}\Delta, \quad \mathrm{Dom}(A_{p,w}) = (W^{2,p}_w(\Omega))^n \cap (W^{1,p}_{w,0}(\Omega))^n \cap L^p_{w,\sigma}(\Omega) \tag{1.2}$$

was established by Farwig and Sohr, see [6], and Fröhlich, see [9], [10]. In the case of $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ or a bounded domain $\Omega$ in $\mathbb{R}^n$ with $C^{1,1}$-boundary $\partial\Omega$, the resolvent estimate with an $A_p$-consistency increasing constant holds for any $1 < p < \infty$ and $w \in A_p$. Consequently, $A_{p,w}$ has the property of maximal $L^q$ regularity for any $1 < q < \infty$, i.e., for any $f \in L^q(\mathbb{R}_+; L^p_{w,\sigma}(\Omega))$, the mild solution $u$ of the Stokes initial value problem

$$\begin{cases} d_t u + A_{p,w} u = f & \text{in } \mathbb{R}_+, \\ u(0) = 0 \end{cases} \tag{1.3}$$

satisfies the estimate

$$\begin{aligned} &\|d_t u\|_{L^q(\mathbb{R}_+; L^p_{w,\sigma}(\Omega))} + \|A_{p,w} u\|_{L^q(\mathbb{R}_+; L^p_{w,\sigma}(\Omega))} \\ &\qquad \leqslant C(p, A_p(w), q)\|f\|_{L^q(\mathbb{R}_+; L^p_{w,\sigma}(\Omega))}. \end{aligned} \tag{1.4}$$

The Helmholtz decomposition of the $L^p$-space $(L^p(\Omega))^n$ is well known for $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ or any domain $\Omega$ in $\mathbb{R}^n$ with compact $C^1$-boundary $\partial\Omega$ and $1 < p < \infty$ (see Simader, Sohr and Varnhorn [17]). As for any domain $\Omega$ in $\mathbb{R}^n$ with uniform $C^1$-boundary $\partial\Omega$, the function space

$$\widetilde{L}^p(\Omega) = \begin{cases} L^2(\Omega) + L^p(\Omega), & 1 < p < 2, \\ L^2(\Omega) \cap L^p(\Omega), & 2 \leqslant p < \infty, \end{cases}$$

which constructs the Helmholtz decomposition, is introduced by Farwig, Kozono and Sohr in [4], [5]. In the case of a domain $\Omega$ in $\mathbb{R}^n$ with compact $C^{0,1}$-boundary $\partial\Omega$, conditions on $\Omega$ or $p$ are essentially required. More precisely, $3/2 \leqslant p \leqslant 3$ is optimal in the validity of the Helmholtz decomposition of $(L^p(\Omega))^n$, which is due to Fabes, Mendez and Mitrea in [2] and Lang and Méndez in [15]. If $\Omega$ is bounded convex, it follows from Geng and Shen in [12] and Kim and Shen in [13] that the Helmholtz decomposition of $(L^p(\Omega))^n$ holds for any $1 < p < \infty$. Concerning any domain in $\mathbb{R}^n$ with boundary given as a graph $x_n = \eta(x')$ of a uniform $C^{0,1}$-function in $\mathbb{R}^{n-1}$, we can refer to Maekawa and Miura in [16].

Analogously to $(L^p(\Omega))^n$, it follows from [6] and Fröhlich in [7] that the Helmholtz decomposition of $(L^p_w(\Omega))^n$ holds for $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ or any domain $\Omega$ in $\mathbb{R}^n$

with compact $C^1$-boundary $\partial\Omega$, $1 < p < \infty$ and $w \in A_p$. Moreover, any aperture domain $\Omega$ in $\mathbb{R}^n$ with $C^1$-boundary $\partial\Omega$ admits the Helmholtz decomposition of $(L^p_w(\Omega))^n$, but the restriction of $A_p$ is imposed for any perturbed half space $\Omega$ in $\mathbb{R}^n$ with $C^1$-boundary $\partial\Omega$. See Fröhlich [8] and Kobayashi and Kubo [14]. In the case of an unbounded domain in $\mathbb{R}^n$, which is bounded with respect to $x' = (x_1, \ldots, x_{n-k})$, $1 \leqslant k \leqslant n-1$, we can refer to Farwig, see [3].

Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. The aim of this paper is to provide equivalent conditions for the validity of the Helmholtz decomposition of $(L^p_w(\Omega))^n$. More precisely, we consider the weak Neumann problem of $(L^p_w(\Omega))^n$, the Helmholtz projection of $(L^p_w(\Omega))^n$ and the variational estimates of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$. See Definitions 2.2, 2.3 and 2.4 below, which are Muckenhoupt $A_p$-weighted cases of [17]. Proofs of our main results are based on the reflexivity and orthogonality of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$ and the Hahn-Banach theorem. Concerning the variational estimates of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$, our proofs seem to be simpler than that of [17]. Furthermore, as corollaries of our main results, we obtain the extrapolation theorems with the aid of the Helmholtz projection of $(L^p_w(\Omega))^n$.

This paper is organized as follows: In Subsections 2.1 and 2.2, we define Muckenhoupt $A_p$-weights and basic notation used in this paper. Subsection 2.3 provides the notion of the Helmholtz decomposition of $(L^p_w(\Omega))^n$, the weak Neumann problem of $(L^p_w(\Omega))^n$, the Helmholtz projection of $(L^p_w(\Omega))^n$ and the variational estimates of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$. In Subsection 2.4, we state our main results. Subsection 3.1 deals with the reflexivity and orthogonality of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$. Finally, our main results are proved in Subsections 3.2, 3.3 and 3.4.

## 2. Preliminaries and main results

**2.1. Muckenhoupt $A_p$-weights.** This subsection provides the notion of Muckenhoupt $A_p$-weights and Muckenhoupt $A_p$-weighted $L^p$-spaces. First, the Muckenhoupt class $A_p$ of weights is defined as follows:

**Definition 2.1.** Let $1 < p < \infty$, $w \in L^1_{\rm loc}(\mathbb{R}^n)$ and $w \geqslant 0$. Then $w$ is called an $A_p$-weight if

$$A_p(w) := \sup\left\{ \frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x \left( \frac{1}{|Q|}\int_Q w^{-1/(p-1)}(x)\,\mathrm{d}x \right)^{p-1} \colon Q \text{ is a cube in } \mathbb{R}^n \right\} < \infty,$$

where $|Q|$ is the Lebesgue measure of $Q$. Moreover, the set of all $A_p$-weights is denoted by $A_p$.

It is important to indicate some examples of Muckenhoupt $A_p$-weights for any $1 < p < \infty$. Let $x_0 \in \mathbb{R}^n$ and $-n < \alpha < n(p-1)$. A typical example is a radially symmetric weight with respect to $x_0$

$$w(x) := |x - x_0|^{\alpha}, \quad x \in \mathbb{R}^n.$$

More generally, we can give the $m$-dimensional compact manifold case of this weight. Let $m \in \{1, \dots, n-1\}$, $M$ be an $m$-dimensional compact manifold in $\mathbb{R}^n$ with $C^{0,1}$-boundary $\partial M$ and $-(n-m) < \alpha < (n-m)(p-1)$. The following example extends $x_0$ to $M$ in the above:

$$w(x) := d(x, M)^{\alpha}, \quad x \in \mathbb{R}^n.$$

Note that further techniques for construction of Muckenhoupt $A_p$-weights are considered. For more details we refer to [6].

Second, we proceed to the notion of Muckenhoupt $A_p$-weighted $L^p$-spaces. For any Lebesgue measurable set $\Omega$ in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$, the Muckenhoupt $A_p$-weighted $L^p$-space $L^p_w(\Omega)$ is defined as

$$L^p_w(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\overline{\Omega}) \colon \int_{\Omega} |f(x)|^p w(x) \,\mathrm{d}x < \infty \right\}$$

with the norm

$$\|f\|_{L^p_w(\Omega)} := \left( \int_{\Omega} |f(x)|^p w(x) \,\mathrm{d}x \right)^{1/p}.$$

Set $p' := p/(p-1)$ and $w' := w^{-1/(p-1)}$. Analogously to $L^p(\Omega)$, the dual space $(L^p_w(\Omega))^*$ of $L^p_w(\Omega)$ can be regarded as $L^{p'}_{w'}(\Omega)$, i.e., $(L^p_w(\Omega))^* = L^{p'}_{w'}(\Omega)$ with equivalent norms. Furthermore, we define the Muckenhoupt $A_p$-weighted homogeneous $L^p$-Sobolev space $\dot{W}^{1,p}_w(\Omega)$ as

$$\dot{W}^{1,p}_w(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\overline{\Omega}) \colon \forall\, \alpha \in \mathbb{Z}^n_{\geqslant 0},\ |\alpha| = 1,\ \int_{\Omega} |\partial^{\alpha} f(x)|^p w(x) \,\mathrm{d}x < \infty \right\}$$

with the seminorm

$$\|f\|_{\dot{W}^{1,p}_w(\Omega)} := \sum_{|\alpha|=1} \|\partial^{\alpha} f\|_{L^p_w(\Omega)}.$$

Note that the quotient space $\dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ forms a separable and reflexive Banach space. Hereafter, the superscript $n$ denotes the space of vector fields, e.g., $(L^p_w(\Omega))^n$. For any $f \in (L^p_w(\Omega))^n$ and $g \in (L^{p'}_{w'}(\Omega))^n$ we denote

$$\langle f, g \rangle_{\Omega} := \int_{\Omega} f(x) g(x) \,\mathrm{d}x,$$

which is well defined as a bilinear map from $(L^p_w(\Omega))^n \times (L^{p'}_{w'}(\Omega))^n$ into $\mathbb{R}$.

Finally, an extrapolation theorem for the Muckenhoupt classes $A_p$ is stated. The following lemma plays an important role in corollaries of our main results.

**Lemma 2.1.** *Let $\Omega$ be a Lebesgue measurable set in $\mathbb{R}^n$, $P$ be a sublinear map from a linear space of measurable vector fields in $\Omega$ to the space of all measurable vector fields in $\Omega$ and $1 < q < \infty$. Assume that*

$$\|Pf\|_{(L^q_w(\Omega))^n} \leqslant C(A_q(w))\|f\|_{(L^q_w(\Omega))^n} \tag{2.1}$$

*holds for any $w \in A_q$ and $f \in (L^q_w(\Omega))^n$. Then*

$$\|Pf\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n} \tag{2.2}$$

*holds for any $1 < p < \infty$, $w \in A_p$ and $f \in (L^p_w(\Omega))^n$.*

Proof. See [11], Theorem IV.5.19. □

**2.2. Function spaces.** Function spaces and basic notation which we use throughout this paper are introduced as follows: Let $X$ be a normed linear space and has the norm $\|\cdot\|_X$. The dual space of $X$ is denoted by $X^*$ and has the norm

$$\|f\|_{X^*} := \sup\Big\{\frac{|f(x)|}{\|x\|_X} \colon\ x \in X,\ x \neq 0\Big\}.$$

Let $Y$ be a normed linear space with the norm $\|\cdot\|_Y$, and consider a linear map $A$ from $X$ into $Y$. We denote by $\operatorname{Im}(A) := \{Ax \colon x \in X\}$ and $\operatorname{Ker}(A) := \{x \in X \colon Ax = 0\}$ the image and the kernel of $A$, respectively. Moreover, the adjoint map of $A$ is denoted by $A^*$.

Let us introduce solenoidal function spaces. For any open set $\Omega$ in $\mathbb{R}^n$, $C^\infty(\Omega)$ is the space of all functions in $\Omega$ which are infinitely differentiable in $\Omega$. Moreover, we denote by $C_0^\infty(\Omega)$ the space of all $C^\infty$-functions in $\Omega$ whose support is compact and contained in $\Omega$. Set $C^\infty_{0,\sigma}(\Omega) := \{f \in (C_0^\infty(\Omega))^n \colon \operatorname{div} f = 0\}$. For any $1 < p < \infty$ and $w \in A_p$, the space $L^p_{w,\sigma}(\Omega)$ of solenoidal vector fields is the completion of $C^\infty_{0,\sigma}(\Omega)$ in $(L^p_w(\Omega))^n$ with the norm $\|g\|_{L^p_{w,\sigma}(\Omega)} := \|g\|_{(L^p_w(\Omega))^n}$. The space of gradient vector fields is denoted by $L^p_{w,\pi}(\Omega) := \{\nabla h \colon\ h \in \dot W^{1,p}_w(\Omega)\}$ with the norm $\|\nabla h\|_{L^p_{w,\pi}(\Omega)} := \|\nabla h\|_{(L^p_w(\Omega))^n}$.

Let $(X, Y)$ be one of

$$((L^p_{w,\sigma}(\Omega))^*, L^{p'}_{w',\sigma}(\Omega)),\quad ((L^p_{w,\sigma}(\Omega))^{**}, L^p_{w,\sigma}(\Omega)) \quad\text{and}\quad ((L^{p'}_{w',\sigma}(\Omega))^*, L^p_{w,\sigma}(\Omega)).$$

For any $f \in X$ and $g \in Y$, $\|f\|_X \simeq \|g\|_Y$ means that the equivalent norm inequality

$$\|f\|_X \leqslant \|g\|_Y \leqslant C(A_p(w))\|f\|_X$$

holds. In the case of $((L^p_{w,\pi}(\Omega))^*, L^{p'}_{w',\pi}(\Omega))$, $((L^p_{w,\pi}(\Omega))^{**}, L^p_{w,\pi}(\Omega))$ or $((L^{p'}_{w',\pi}(\Omega))^*, L^p_{w,\pi}(\Omega))$, we denote by $\|f\|_X \simeq \|\nabla h\|_Y$ the same as above.

Concerning constants in all estimates which appear in this paper, simplified notations are given as follows: we denote by $C$ a generic positive constant depending only on $n$ and $\Omega$. Moreover, generic positive constants depending only on the above elements and additive elements (e.g., $p$, $A_p(w)$, a pair of $p$ and $A_p(w)$ and so on) are simply denoted by $C(p)$, $C(A_p(w))$, $C(p, A_p(w))$ and so on, respectively.

**2.3. Helmholtz decomposition.** In this subsection, the Helmholtz decomposition of $(L^p_w(\Omega))^n$ and equivalent conditions for it, which are proved in this paper, are formulated. We begin with the Helmholtz decomposition of $(L^p_w(\Omega))^n$.

**Definition 2.2.** Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then the Helmholtz decomposition of $(L^p_w(\Omega))^n$ holds if for any $f \in (L^p_w(\Omega))^n$ there uniquely exists $(g, h) \in L^p_{w,\sigma}(\Omega) \times (\dot{W}^{1,p}_w(\Omega)/\mathbb{R})$ such that

$$g + \nabla h = f \tag{2.3}$$

and

$$\|g\|_{(L^p_w(\Omega))^n} + \|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}. \tag{2.4}$$

In the next definitions, we introduce four conditions which are proved to be equivalent to the validity of the Helmholtz decomposition of $(L^p_w(\Omega))^n$. The first and second conditions are concerned with the weak Neumann problem and the Helmholtz projection of $(L^p_w(\Omega))^n$.

**Definition 2.3.** Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then

(1) the weak Neumann problem of $(L^p_w(\Omega))^n$ is uniquely solvable if for any $f \in (L^p_w(\Omega))^n$ there uniquely exists $h \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ such that

$$\langle \nabla h, \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega \tag{2.5}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}; \tag{2.6}$$

(2) the Helmholtz projection of $(L^p_w(\Omega))^n$ is uniquely defined if there uniquely exists a linear map $P_{p,w}$ from $(L^p_w(\Omega))^n$ into $(L^p_w(\Omega))^n$ such that

$$\operatorname{Im}(P_{p,w}) = L^p_{w,\sigma}(\Omega), \quad \operatorname{Ker}(P_{p,w}) = L^p_{w,\pi}(\Omega), \quad P^2_{p,w} = P_{p,w} \tag{2.7}$$

and

$$\|P_{p,w}f\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n} \tag{2.8}$$

holds for any $f \in (L^p_w(\Omega))^n$.

From the viewpoint of [7], [17], the variational estimates of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$, i.e., Muckenhoupt $A_p$-weighted cases of [17], are the third and fourth conditions.

**Definition 2.4.** Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then

(1) the variational estimate of $L^p_{w,\pi}(\Omega)$ holds if

$$\|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \sup\Big\{\frac{|\langle \nabla h, \nabla\varphi\rangle_\Omega|}{\|\nabla\varphi\|_{(L^{p'}_{w'}(\Omega))^n}} : \varphi \in \dot{W}^{1,p'}_{w'}(\Omega),\ \nabla\varphi \neq 0\Big\} \tag{2.9}$$

holds for any $h \in \dot{W}^{1,p}_w(\Omega)$;

(2) the variational estimate of $L^p_{w,\sigma}(\Omega)$ holds if

$$\|g\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \sup\Big\{\frac{|\langle g, \varphi\rangle_\Omega|}{\|\varphi\|_{(L^{p'}_{w'}(\Omega))^n}} : \varphi \in L^{p'}_{w',\sigma}(\Omega),\ \varphi \neq 0\Big\} \tag{2.10}$$

holds for any $g \in L^p_{w,\sigma}(\Omega)$.

**2.4. Main results.** This subsection will deal with our main results. The first part of our main results yields the equivalence between the Helmholtz decomposition, the weak Neumann problem and the Helmholtz projection of $(L^p_w(\Omega))^n$.

**Theorem 2.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then* (i), (ii) *and* (iii) *are equivalent.*

(i) *The Helmholtz decomposition of $(L^p_w(\Omega))^n$ holds.*
(ii) *The weak Neumann problem of $(L^p_w(\Omega))^n$ is uniquely solvable.*
(iii) *The Helmholtz projection of $(L^p_w(\Omega))^n$ is uniquely defined.*

**Corollary 2.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and $1 < q < \infty$. Then* (i), (ii) *and* (iii) *are equivalent.*

(i) *The Helmholtz decomposition of $(L^q_w(\Omega))^n$ holds for any $w \in A_q$.*

(ii) *The weak Neumann problem of $(L^q_w(\Omega))^n$ is uniquely solvable for any $w \in A_q$.*

(iii) *The Helmholtz projection of $(L^q_w(\Omega))^n$ is uniquely defined for any $w \in A_q$.*

*Moreover,* (i), (ii) *or* (iii) *implies* (iv), (v) *and* (vi).

(iv) *The Helmholtz decomposition of $(L^p_w(\Omega))^n$ holds for any $1 < p < \infty$ and $w \in A_p$.*

(v) *The weak Neumann problem of $(L^p_w(\Omega))^n$ is uniquely solvable for any $1 < p < \infty$ and $w \in A_p$.*

(vi) *The Helmholtz projection of $(L^p_w(\Omega))^n$ is uniquely defined for any $1 < p < \infty$ and $w \in A_p$.*

We proceed to the second part of our main results, on the equivalence between the Helmholtz decomposition of $(L^p_w(\Omega))^n$ and the variational estimates of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$. Not only this equivalence but also the duality of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$ are established as follows:

**Theorem 2.2.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then* (i), (ii) *and* (iii) *are equivalent.*

(i) *The Helmholtz decomposition of $(L^p_w(\Omega))^n$ and $(L^{p'}_{w'}(\Omega))^n$ holds.*

(ii) *The variational estimate of $L^p_{w,\pi}(\Omega)$ and $L^{p'}_{w',\pi}(\Omega)$ holds.*

(iii) *The variational estimate of $L^p_{w,\sigma}(\Omega)$ and $L^{p'}_{w',\sigma}(\Omega)$ holds.*

*Moreover,* (i), (ii) *or* (iii) *implies $(L^p_{w,\pi}(\Omega))^* = L^{p'}_{w',\pi}(\Omega)$, $(L^p_{w,\sigma}(\Omega))^* = L^{p'}_{w',\sigma}(\Omega)$ and $(P_{p,w})^* = P_{p',w'}$.*

**Corollary 2.2.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and $1 < q < \infty$. Then* (i), (ii) *and* (iii) *are equivalent.*

(i) *The Helmholtz decomposition of $(L^q_w(\Omega))^n$ and $(L^{q'}_{w'}(\Omega))^n$ holds for any $w \in A_q$.*

(ii) *The variational estimate of $L^q_{w,\pi}(\Omega)$ and $L^{q'}_{w',\pi}(\Omega)$ holds for any $w \in A_q$.*

(iii) *The variational estimate of $L^q_{w,\sigma}(\Omega)$ and $L^{q'}_{w',\sigma}(\Omega)$ holds for any $w \in A_q$.*

*Moreover,* (i), (ii) *or* (iii) *implies* (iv), (v) *and* (vi).

(iv) *The Helmholtz decomposition of $(L^p_w(\Omega))^n$ holds for any $1 < p < \infty$ and $w \in A_p$.*

(v) *The variational estimate of $L^p_{w,\pi}(\Omega)$ holds for any $1 < p < \infty$ and $w \in A_p$.*

(vi) *The variational estimate of $L^p_{w,\sigma}(\Omega)$ holds for any $1 < p < \infty$ and $w \in A_p$.*

## 3. Proofs of Theorems 2.1 and 2.2

**3.1. Auxiliary lemmas.** In this subsection, we will state and prove two auxiliary lemmas which are essentially required for proofs of our main results. The first auxiliary lemma establishes the reflexivity of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$.

**Lemma 3.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then*

(1) *$L^p_{w,\pi}(\Omega)$ is a reflexive Banach space;*
(2) *$L^p_{w,\sigma}(\Omega)$ is a reflexive Banach space.*

Proof. Since $(L^p_w(\Omega))^n$ is a reflexive Banach space and $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$ are closed subspaces of $(L^p_w(\Omega))^n$, $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$ are reflexive Banach spaces. □

We proceed to the second auxiliary lemma, i.e., annihilator properties in $L^p_{w,\pi}(\Omega)$ and in $L^p_{w,\sigma}(\Omega)$. The following lemma yields the orthogonality of $L^p_{w,\pi}(\Omega)$ and $L^p_{w,\sigma}(\Omega)$.

**Lemma 3.2.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$.*

(1) *Define the orthogonal complement $(L^{p'}_{w',\pi}(\Omega))^\perp$ of $L^{p'}_{w',\pi}(\Omega)$ as*

$$(L^{p'}_{w',\pi}(\Omega))^\perp := \{g \in (L^p_w(\Omega))^n \colon \ \forall\, \varphi \in \dot{W}^{1,p'}_{w'}(\Omega),\ \langle g, \nabla\varphi\rangle_\Omega = 0\}.$$

*Then $L^p_{w,\sigma}(\Omega) = (L^{p'}_{w',\pi}(\Omega))^\perp$ holds.*

(2) *Define the orthogonal complement $(L^{p'}_{w',\sigma}(\Omega))^\perp$ of $L^{p'}_{w',\sigma}(\Omega)$ as*

$$(L^{p'}_{w',\sigma}(\Omega))^\perp := \{\psi \in (L^p_w(\Omega))^n \colon \ \forall g \in L^{p'}_{w',\sigma}(\Omega),\ \langle \psi, g\rangle_\Omega = 0\}.$$

*Then $L^p_{w,\pi}(\Omega) = (L^{p'}_{w',\sigma}(\Omega))^\perp$ holds.*

Proof. First, we will prove $L^p_{w,\sigma}(\Omega) \subseteq (L^{p'}_{w',\pi}(\Omega))^\perp$. Let $g \in L^p_{w,\sigma}(\Omega)$. Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^p_{w,\sigma}(\Omega)$, it follows from integration by parts that

$$\langle g, \nabla\varphi\rangle_\Omega = 0$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$. Therefore $g \in (L^{p'}_{w',\pi}(\Omega))^\perp$.

Second, we proceed by contradiction as in [17], Remark 1.4 (d). Assume that $L^p_{w,\sigma}(\Omega) \subsetneq (L^{p'}_{w',\pi}(\Omega))^\perp$. Then there exists $\psi \in (L^{p'}_{w'}(\Omega))^n$ with $\psi \neq 0$ such that

$$\langle g, \psi\rangle_\Omega = 0$$

holds for any $g \in L^p_{w,\sigma}(\Omega)$ and

$$\langle h, \psi \rangle_\Omega \neq 0$$

holds for at least one $h \in (L^{p'}_{w',\pi}(\Omega))^\perp$. By the well known argument of de Rham, see [1], Théoréme 17′, there exists $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ such that $\psi = \nabla\varphi$. Therefore,

$$0 \neq \langle h, \psi \rangle_\Omega = \langle h, \nabla\varphi \rangle_\Omega = 0$$

holds, which is a contradiction and completes the proof of Lemma 3.2 (1).

Lemma 3.2 (2) is an immediate consequence of Lemmas 3.1 (1) and 3.2 (1). Actually, $L^p_{w,\sigma}(\Omega) = (L^{p'}_{w',\pi}(\Omega))^\perp$ implies $(L^{p'}_{w',\sigma}(\Omega))^\perp = (L^p_{w,\pi}(\Omega))^{\perp\perp} = L^p_{w,\pi}(\Omega)$ by classical annihilator properties in $L^p_{w,\pi}(\Omega)$. □

**3.2. Proof of Theorem 2.1:** (i) $\Leftrightarrow$ (ii)**.** In this subsection, we will obtain the equivalence between Theorem 2.1 (i) and (ii). The first part is to prove that (i) implies (ii). Assume that for any $f \in (L^p_w(\Omega))^n$ there uniquely exists $(g, h) \in L^p_{w,\sigma}(\Omega) \times (\dot{W}^{1,p}_w(\Omega)/\mathbb{R})$ such that

$$g + \nabla h = f \tag{3.1}$$

and

$$\|g\|_{(L^p_w(\Omega))^n} + \|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}. \tag{3.2}$$

Since Lemma 3.2 (1) yields $g \in L^p_{w,\sigma}(\Omega) = (L^{p'}_{w',\pi}(\Omega))^\perp$,

$$\langle g, \nabla\varphi \rangle_\Omega = 0$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$. Combining this equality with (3.1) and (3.2),

$$\langle \nabla h, \nabla\varphi \rangle_\Omega = \langle f, \nabla\varphi \rangle_\Omega$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}.$$

Let $\tilde{h} \in \dot{W}^{1,p}_w(\Omega)$, and assume that

$$\langle \nabla\tilde{h}, \nabla\varphi \rangle_\Omega = \langle f, \nabla\varphi \rangle_\Omega \tag{3.3}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla \tilde{h}\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}. \tag{3.4}$$

Then $f - \nabla \tilde{h} \in (L^{p'}_{w',\pi}(\Omega))^{\perp}$. Indeed, it follows from (3.3) that

$$\langle f - \nabla \tilde{h}, \nabla \varphi\rangle_\Omega = \langle f, \nabla \varphi\rangle_\Omega - \langle \nabla \tilde{h}, \nabla \varphi\rangle_\Omega = 0$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$. Moreover, Lemma 3.2 (1) yields $f - \nabla \tilde{h} \in L^p_{w,\sigma}(\Omega)$. By $f - \nabla \tilde{h} \in L^p_{w,\sigma}(\Omega)$ and (3.4), $f$ is decomposed into

$$f = (f - \nabla \tilde{h}) + \nabla \tilde{h}$$

and

$$\|f - \nabla \tilde{h}\|_{(L^p_w(\Omega))^n} + \|\nabla \tilde{h}\|_{(L^p_w(\Omega))^n} \leqslant (1 + 2C(A_p(w)))\|f\|_{(L^p_w(\Omega))^n}.$$

Therefore, (i) (uniqueness) implies $g = f - \nabla \tilde{h}$ and $\nabla h = \nabla \tilde{h}$.

Second, we conversely prove that (i) is derived from (ii). Suppose that for any $f \in (L^p_w(\Omega))^n$ there uniquely exists $h \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ such that

$$\langle \nabla h, \nabla \varphi\rangle_\Omega = \langle f, \nabla \varphi\rangle_\Omega \tag{3.5}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}. \tag{3.6}$$

Then $f - \nabla h \in L^p_{w,\sigma}(\Omega)$ follows from (3.5) and Lemma 3.2 (1). Furthermore, (3.6) implies

$$(f - \nabla h) + \nabla h = f$$

and

$$\|f - \nabla h\|_{(L^p_w(\Omega))^n} + \|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant (1 + 2C(A_p(w)))\|f\|_{(L^p_w(\Omega))^n}.$$

It remains to obtain the uniqueness of the Helmholtz decomposition of $f$. Set $g := f - \nabla h$, and assume that there exists $(\tilde{g}, \tilde{h}) \in L^p_{w,\sigma}(\Omega) \times (\dot{W}^{1,p}_w(\Omega)/\mathbb{R})$ such that

$$\tilde{g} + \nabla \tilde{h} = f \tag{3.7}$$

and

$$\|\tilde{g}\|_{(L^p_w(\Omega))^n} + \|\nabla \tilde{h}\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}.$$

Then Lemma 3.2 (1) yields $\tilde{g} - g \in L^p_{w,\sigma}(\Omega) = (L^{p'}_{w',\pi}(\Omega))^\perp$. Moreover, it follows from (3.7) that

$$\langle \nabla h - \nabla \tilde{h}, \nabla \varphi \rangle_\Omega = \langle \tilde{g} - g, \nabla \varphi \rangle_\Omega = 0 \tag{3.8}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$. Hence, (ii) (uniqueness) implies $g = \tilde{g}$ and $\nabla h = \nabla \tilde{h}$ by (3.7) and (3.8). This completes the proof of the equivalence between Theorem 2.1 (i) and (ii).

**3.3. Proof of Theorem 2.1:** (ii) $\Leftrightarrow$ (iii)**.** The equivalence between Theorem 2.1 (ii) and (iii) will be proved in this subsection. Assume first (ii). Then for any $f \in (L^p_w(\Omega))^n$ there uniquely exists $h \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ such that

$$\langle \nabla h, \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega \tag{3.9}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \|f\|_{(L^p_w(\Omega))^n}. \tag{3.10}$$

Define a map $P_{p,w}$ from $(L^p_w(\Omega))^n$ into $(L^p_w(\Omega))^n$ as

$$P_{p,w} f := f - \nabla h, \quad f \in (L^p_w(\Omega))^n,$$

where $h \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ is taken as in (3.9) and (3.10). Then (3.9) clearly yields the linearlity of $P_{p,w}$. Moreover, it follows from (3.9) and (3.10) that

$$\mathrm{Im}(P_{p,w}) = (L^{p'}_{w',\pi}(\Omega))^\perp, \quad \mathrm{Ker}(P_{p,w}) = L^p_{w,\pi}(\Omega)$$

and

$$\|P_{p,w} f\|_{(L^p_w(\Omega))^n} \leqslant (1 + C(A_p(w))) \|f\|_{(L^p_w(\Omega))^n}$$

holds for any $f \in (L^p_w(\Omega))^n$. Note that Lemma 3.2 (1) implies $\mathrm{Im}(P_{p,w}) = (L^{p'}_{w',\pi}(\Omega))^\perp = L^p_{w,\sigma}(\Omega)$. It remains to obtain the idempotence of $P_{p,w}$, i.e., $P^2_{p,w} = P_{p,w}$. For any $f \in (L^p_w(\Omega))^n$ there uniquely exists $\tilde{h} \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ such that

$$\langle \nabla \tilde{h}, \nabla \varphi \rangle_\Omega = \langle P_{p,w} f, \nabla \varphi \rangle_\Omega \tag{3.11}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla \tilde{h}\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \|P_{p,w} f\|_{(L^p_w(\Omega))^n}. \tag{3.12}$$

By definition, we have

$$P_{p,w}^2 f = P_{p,w} f - \nabla \tilde{h}.$$

Since $\mathrm{Im}(P_{p,w}) = (L^{p'}_{w',\pi}(\Omega))^{\perp}$, $\nabla \tilde{h} = 0$ follows from (3.11) and (ii) (uniqueness). Therefore $P_{p,w}^2 = P_{p,w}$.

Suppose conversely (iii). Then there uniquely exists a linear map $P_{p,w}$ from $(L^p_w(\Omega))^n$ into $(L^p_w(\Omega))^n$ such that

$$\mathrm{Im}(P_{p,w}) = L^p_{w,\sigma}(\Omega), \quad \mathrm{Ker}(P_{p,w}) = L^p_{w,\pi}(\Omega), \quad P_{p,w}^2 = P_{p,w} \tag{3.13}$$

and

$$\|P_{p,w} f\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \|f\|_{(L^p_w(\Omega))^n} \tag{3.14}$$

holds for any $f \in (L^p_w(\Omega))^n$. Since (3.13) yields that

$$P_{p,w}(f - P_{p,w} f) = P_{p,w} f - P_{p,w}^2 f = 0$$

holds for any $f \in (L^p_w(\Omega))^n$ and $\mathrm{Ker}(P_{p,w}) = L^p_{w,\pi}(\Omega)$, there exists $h \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ such that $f - P_{p,w} f = \nabla h$. Furthermore, $h$ solves the weak Neumann problem for $f$. Indeed, it follows from Lemma 3.2 (1), (3.13) and (3.14) that

$$\langle \nabla h, \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega - \langle P_{p,w} f, \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega \tag{3.15}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla h\|_{(L^p_w(\Omega))^n} \leqslant (1 + C(A_p(w))) \|f\|_{(L^p_w(\Omega))^n}.$$

We proceed to the uniqueness of the weak Neumann problem for $f$. Let $\tilde{h} \in \dot{W}^{1,p}_w(\Omega)$, and assume that

$$\langle \nabla \tilde{h}, \nabla \varphi \rangle_\Omega = \langle f, \nabla \varphi \rangle_\Omega \tag{3.16}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\nabla \tilde{h}\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \|f\|_{(L^p_w(\Omega))^n}.$$

Then $\nabla h - \nabla \tilde{h} \in L^p_{w,\sigma}(\Omega)$ follows from (3.15), (3.16) and Lemma 3.2 (1). Therefore, (3.13) implies

$$\nabla h - \nabla \tilde{h} = P_{p,w}(\nabla h - \nabla \tilde{h}) = 0,$$

i.e., $\nabla h = \nabla \tilde{h}$. This completes the proof of the equivalence between Theorem 2.1 (ii) and (iii). Combining Theorem 2.1 (iii) with Lemma 2.1, we can easily obtain Corollary 2.1.

**3.4. Proof of Theorem 2.2.** This subsection will deal with three lemmas which we combine to prove Theorem 2.2. First, the variational estimates of $L^{p'}_{w',\pi}(\Omega)$ and $L^{p'}_{w',\sigma}(\Omega)$ are derived from the Helmholtz decomposition of $(L^p_w(\Omega))^n$.

**Lemma 3.3.** *Let* $\Omega$ *be a domain in* $\mathbb{R}^n$, $1 < p < \infty$ *and* $w \in A_p$. *Then* (i) *implies* (ii).

(i) *The Helmholtz decomposition of* $(L^p_w(\Omega))^n$ *holds.*
(ii) *The variational estimates of* $L^{p'}_{w',\pi}(\Omega)$ *and* $L^{p'}_{w',\sigma}(\Omega)$ *hold.*

Proof. We obtain the variational estimate of $L^{p'}_{w',\pi}(\Omega)$ as in [17], Theorem 2.2 b). By the well known duality of $(L^p_w(\Omega))^n$ and $(L^{p'}_{w'}(\Omega))^n$,

$$\begin{aligned}\|\nabla h\|_{(L^{p'}_{w'}(\Omega))^n} &\leqslant \sup\Big\{\frac{|\langle \nabla h, f\rangle_\Omega|}{\|f\|_{(L^p_w(\Omega))^n}}\colon\ f \in (L^p_w(\Omega))^n,\ f \neq 0\Big\}\\ &=: \|\nabla h\|_{((L^p_w(\Omega))^n)'}\end{aligned}$$

holds for any $h \in \dot{W}^{1,p'}_{w'}(\Omega)$. Moreover, (i) implies that there uniquely exists $\varphi \in \dot{W}^{1,p}_w(\Omega)/\mathbb{R}$ such that

$$f - \nabla\varphi \in L^p_{w,\sigma}(\Omega), \quad (f - \nabla\varphi) + \nabla\varphi = f \tag{3.17}$$

and

$$\|\nabla\varphi\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w))\|f\|_{(L^p_w(\Omega))^n}. \tag{3.18}$$

It follows from Lemma 3.2 (1), (3.17) and (3.18) that

$$\begin{aligned}\|\nabla h\|_{((L^p_w(\Omega))^n)'} &= \sup\Big\{\frac{|\langle \nabla h, \nabla\varphi\rangle_\Omega|}{\|f\|_{(L^p_w(\Omega))^n}}\colon\ f \in (L^p_w(\Omega))^n,\ f \neq 0\Big\}\\ &\leqslant C(A_p(w))\sup\Big\{\frac{|\langle \nabla h, \nabla\varphi\rangle_\Omega|}{\|\nabla\varphi\|_{(L^p_w(\Omega))^n}}\colon\ \varphi \in \dot{W}^{1,p}_w(\Omega),\ \nabla\varphi \neq 0\Big\}.\end{aligned}$$

Since the variational estimate of $L^{p'}_{w',\sigma}(\Omega)$ is established as in the above and [17], Theorem 2.3 b), we omit the details. This completes the proof of Lemma 3.3. □

The second lemma is concerned with the weak Neumann problem of $(L^p_w(\Omega))^n$. The unique solvability follows from the variational estimate of $L^p_{w,\pi}(\Omega)$ and $L^{p'}_{w',\pi}(\Omega)$.

**Lemma 3.4.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then* (i) *implies* (ii).

(i) *The variational estimate of $L^p_{w,\pi}(\Omega)$ and $L^{p'}_{w',\pi}(\Omega)$ holds.*

(ii) *The weak Neumann problem of $(L^p_w(\Omega))^n$ and $(L^{p'}_{w'}(\Omega))^n$ is uniquely solvable.*

*Moreover,* (i) *implies $(L^p_{w,\pi}(\Omega))^* = L^{p'}_{w',\pi}(\Omega)$.*

Proof. First, we prove the inclusion relation between $L^{p'}_{w',\pi}(\Omega)$ and $(L^p_{w,\pi}(\Omega))^*$. Let $h \in \dot{W}^{1,p'}_{w'}(\Omega)$, and define a map $f_h$ from $L^p_{w,\pi}(\Omega)$ into $\mathbb{R}$ as

$$f_h(\nabla\varphi) = \langle \nabla h, \nabla\varphi\rangle_\Omega, \quad \varphi \in \dot{W}^{1,p}_w(\Omega).$$

Then it follows from the Hölder inequality and the variational estimate of $L^{p'}_{w',\pi}(\Omega)$ that

$$\|f_h\|_{(L^p_{w,\pi}(\Omega))^*} \leqslant \|\nabla h\|_{(L^{p'}_{w'}(\Omega))^n} \leqslant C(A_p(w))\|f_h\|_{(L^p_{w,\pi}(\Omega))^*}$$

holds, i.e., $\|f_h\|_{(L^p_{w,\pi}(\Omega))^*} \simeq \|\nabla h\|_{(L^{p'}_{w'}(\Omega))^n}$. Therefore, we can regard $L^{p'}_{w',\pi}(\Omega)$ as a closed subspace of $(L^p_{w,\pi}(\Omega))^*$, i.e., $L^{p'}_{w',\pi}(\Omega) \subseteq (L^p_{w,\pi}(\Omega))^*$ with equivalent norms.

The second part is the unique solvability of the weak Neumann problem of $(L^p_w(\Omega))^n$. For any $f \in (L^{p'}_{w',\pi}(\Omega))^*$, the Hahn-Banach theorem admits an extension $\tilde{f} \in (L^p_{w,\pi}(\Omega))^{**}$ of $f$ such that

$$\tilde{f}(\nabla\varphi) = f(\nabla\varphi) \tag{3.19}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|f\|_{(L^{p'}_{w',\pi}(\Omega))^*} = \|\tilde{f}\|_{(L^p_{w,\pi}(\Omega))^{**}}. \tag{3.20}$$

Since Lemma 3.1 (1) yields $\tilde{f} \in (L^p_{w,\pi}(\Omega))^{**} = L^p_{w,\pi}(\Omega)$, there exists $h \in \dot{W}^{1,p}_w(\Omega)$ such that

$$\langle \nabla h, \nabla\varphi\rangle_\Omega = \tilde{f}(\nabla\varphi)$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|\tilde{f}\|_{(L^p_{w,\pi}(\Omega))^{**}} \simeq \|\nabla h\|_{(L^p_w(\Omega))^n}.$$

It follows from (3.19) and (3.20) that

$$\langle \nabla h, \nabla\varphi\rangle_\Omega = f(\nabla\varphi) \tag{3.21}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|f\|_{(L^{p'}_{w',\pi}(\Omega))^*} \simeq \|\nabla h\|_{(L^p_w(\Omega))^n}. \tag{3.22}$$

Assume that there exists $\tilde{h} \in \dot{W}^{1,p}_w(\Omega)$ such that

$$\langle \nabla \tilde{h}, \nabla \varphi \rangle_\Omega = f(\nabla \varphi) \tag{3.23}$$

holds for any $\varphi \in \dot{W}^{1,p'}_{w'}(\Omega)$ and

$$\|f\|_{(L^{p'}_{w',\pi}(\Omega))^*} \simeq \|\nabla \tilde{h}\|_{(L^p_w(\Omega))^n}.$$

Then it follows from the variational estimate of $L^p_{w,\pi}(\Omega)$, (3.21) and (3.23) that

$$\begin{aligned} &\|\nabla h - \nabla \tilde{h}\|_{(L^p_w(\Omega))^n} \\ &\quad \leqslant C(A_p(w)) \sup\Big\{ \frac{|\langle \nabla h - \nabla \tilde{h}, \nabla \varphi \rangle_\Omega|}{\|\nabla \varphi\|_{(L^{p'}_{w'}(\Omega))^n}} \colon \varphi \in \dot{W}^{1,p'}_{w'}(\Omega),\ \nabla \varphi \neq 0 \Big\} = 0, \end{aligned}$$

i.e., $\nabla h = \nabla \tilde{h}$. According to (3.21), (3.22) and the uniqueness of $h$, $(L^{p'}_{w',\pi}(\Omega))^*$ can be regarded as a closed subspace of $L^p_{w,\pi}(\Omega)$, i.e., $(L^{p'}_{w',\pi}(\Omega))^* \subseteq L^p_{w,\pi}(\Omega)$ with equivalent norms. Let $f \in (L^p_w(\Omega))^n$, and define a map $f$ from $L^{p'}_{w',\pi}(\Omega)$ into $\mathbb{R}$ as

$$f(\nabla \varphi) = \langle f, \nabla \varphi \rangle_\Omega, \quad \varphi \in \dot{W}^{1,p'}_{w'}(\Omega).$$

Then $f \in (L^{p'}_{w',\pi}(\Omega))^*$ immediately follows from the Hölder inequality. Combining $f \in (L^{p'}_{w',\pi}(\Omega))^*$ with (3.21) and (3.22), the weak Neumann problem of $(L^p_w(\Omega))^n$ is uniquely solvable.

Concerning the unique solvability of the weak Neumann problem of $(L^{p'}_{w'}(\Omega))^n$, it is sufficient to replace $p$ and $w$ by $p'$ and $w'$, respectively, in the above. Moreover, we obtain $L^p_{w,\pi}(\Omega) \subseteq (L^{p'}_{w',\pi}(\Omega))^*$ and $(L^p_{w,\pi}(\Omega))^* \subseteq L^{p'}_{w',\pi}(\Omega)$. Therefore, (i) implies $(L^p_{w,\pi}(\Omega))^* = L^{p'}_{w',\pi}(\Omega)$, $(L^{p'}_{w',\pi}(\Omega))^* = L^p_{w,\pi}(\Omega)$ and (ii), which completes the proof of Lemma 3.4. □

Third, we proceed to the Helmholtz projection $P_{p,w}$ of $(L^p_w(\Omega))^n$. Analogously to the above lemma, the variational estimate of $L^p_{w,\sigma}(\Omega)$ and $L^{p'}_{w',\sigma}(\Omega)$ yields not only the unique definedness of $P_{p,w}$ but also adjoint properties of $P_{p,w}$.

**Lemma 3.5.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $1 < p < \infty$ and $w \in A_p$. Then* (i) *implies* (ii).

(i) *The variational estimate of $L^p_{w,\sigma}(\Omega)$ and $L^{p'}_{w',\sigma}(\Omega)$ holds.*

(ii) *The Helmholtz projection of $(L^p_w(\Omega))^n$ and $(L^{p'}_{w'}(\Omega))^n$ is uniquely defined.*

*Moreover,* (i) *implies $(L^p_{w,\sigma}(\Omega))^* = L^{p'}_{w',\sigma}(\Omega)$, and* (ii) *implies $(P_{p,w})^* = P_{p',w'}$.*

Proof. The first part is to obtain the inclusion relation between $L^{p'}_{w',\sigma}(\Omega)$ and $(L^p_{w,\sigma}(\Omega))^*$. Let $g \in L^{p'}_{w',\sigma}(\Omega)$, and define a map $f_g$ from $L^p_{w,\sigma}(\Omega)$ into $\mathbb{R}$ as

$$f_g(\varphi) = \langle g, \varphi \rangle_\Omega, \quad \varphi \in L^p_{w,\sigma}(\Omega).$$

Then $\|f_g\|_{(L^p_{w,\sigma}(\Omega))^*} \simeq \|g\|_{(L^{p'}_{w'}(\Omega))^n}$ follows from the Hölder inequality and the variational estimate of $L^{p'}_{w',\sigma}(\Omega)$. Therefore, $L^{p'}_{w',\sigma}(\Omega)$ can be regarded as a closed subspace of $(L^p_{w,\sigma}(\Omega))^*$, i.e., $L^{p'}_{w',\sigma}(\Omega) \subseteq (L^p_{w,\sigma}(\Omega))^*$ with equivalent norms.

Second, we proceed to the unique definedness of the Helmholtz projection of $(L^p_w(\Omega))^n$. For any $f \in (L^{p'}_{w',\sigma}(\Omega))^*$, the Hahn-Banach theorem is applied, and $f$ is extended to $\tilde{f} \in (L^p_{w,\sigma}(\Omega))^{**}$ such that

$$\tilde{f}(\varphi) = f(\varphi) \tag{3.24}$$

holds for any $\varphi \in L^{p'}_{w',\sigma}(\Omega)$ and

$$\|f\|_{(L^{p'}_{w',\sigma}(\Omega))^*} = \|\tilde{f}\|_{(L^p_{w,\sigma}(\Omega))^{**}}. \tag{3.25}$$

Since Lemma 3.1 (2) implies $\tilde{f} \in (L^p_{w,\sigma}(\Omega))^{**} = L^p_{w,\sigma}(\Omega)$, there exists $g \in L^p_{w,\sigma}(\Omega)$ such that

$$\langle g, \varphi \rangle_\Omega = \tilde{f}(\varphi)$$

holds for any $\varphi \in L^{p'}_{w',\sigma}(\Omega)$ and

$$\|\tilde{f}\|_{(L^p_{w,\sigma}(\Omega))^{**}} \simeq \|g\|_{(L^p_w(\Omega))^n}.$$

By (3.24) and (3.25), we have

$$\langle g, \varphi \rangle_\Omega = f(\varphi) \tag{3.26}$$

for any $\varphi \in L^{p'}_{w',\sigma}(\Omega)$ and

$$\|f\|_{(L^{p'}_{w',\sigma}(\Omega))^*} \simeq \|g\|_{(L^p_w(\Omega))^n}. \tag{3.27}$$

Suppose that there exists $\tilde{g} \in L^p_{w,\sigma}(\Omega)$ such that

$$\langle \tilde{g}, \varphi \rangle_\Omega = f(\varphi) \tag{3.28}$$

holds for any $\varphi \in L^{p'}_{w',\sigma}(\Omega)$ and

$$\|f\|_{(L^{p'}_{w',\sigma}(\Omega))^*} \simeq \|\tilde{g}\|_{(L^p_w(\Omega))^n}.$$

Then it follows from the variational estimate of $L^p_{w,\sigma}(\Omega)$, (3.26) and (3.28) that

$$\|g-\tilde{g}\|_{(L^p_w(\Omega))^n} \leqslant C(A_p(w)) \sup\Big\{\frac{|\langle g-\tilde{g},\varphi\rangle_\Omega|}{\|\varphi\|_{(L^{p'}_{w'}(\Omega))^n}}:\ \varphi \in L^{p'}_{w',\sigma}(\Omega),\ \varphi \neq 0\Big\} = 0,$$

i.e., $g = \tilde{g}$. By (3.26), (3.27) and the uniqueness of $g$, we can regard $(L^{p'}_{w',\sigma}(\Omega))^*$ as a closed subspace of $L^p_{w,\sigma}(\Omega)$, i.e., $(L^{p'}_{w',\sigma}(\Omega))^* \subseteq L^p_{w,\sigma}(\Omega)$ with equivalent norms. Let $f \in (L^p_w(\Omega))^n$, and define a map $f$ from $L^{p'}_{w',\sigma}(\Omega)$ into $\mathbb{R}$ as

$$f(\varphi) = \langle f, \varphi\rangle_\Omega, \quad \varphi \in L^{p'}_{w',\sigma}(\Omega).$$

Then the Hölder inequality obviously yields $f \in (L^{p'}_{w',\sigma}(\Omega))^*$. By this functional, a map $P_{p,w}$ from $(L^p_w(\Omega))^n$ into $L^p_{w,\sigma}(\Omega)$ is defined as

$$P_{p,w} f = g,$$

where $g \in L^p_{w,\sigma}(\Omega)$ is taken as in (3.26) and (3.27). Analogously to the proof of Theorem 2.1, it is not difficult to verify (2.7) and (2.8) with the aid of Lemma 3.2 (2), (3.26) and (3.27). Consequently, the Helmholtz projection of $(L^p_w(\Omega))^n$ is uniquely defined.

Replacing $p$ and $w$ by $p'$ and $w'$, respectively, in the above, the Helmholtz projection $P_{p',w'}$ of $(L^{p'}_{w'}(\Omega))^n$ is uniquely defined. Moreover, we obtain $L^p_{w,\sigma}(\Omega) \subseteq (L^{p'}_{w',\sigma}(\Omega))^*$ and $(L^p_{w,\sigma}(\Omega))^* \subseteq L^{p'}_{w',\sigma}(\Omega)$. Therefore, (i) implies $(L^p_{w,\sigma}(\Omega))^* = L^{p'}_{w',\sigma}(\Omega)$, $(L^{p'}_{w',\sigma}(\Omega))^* = L^p_{w,\sigma}(\Omega)$ and (ii).

It remains to prove that (ii) implies $(P_{p,w})^* = P_{p',w'}$. Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^p_{w,\sigma}(\Omega)$ and in $L^{p'}_{w',\sigma}(\Omega)$, it follows from integration by parts that

$$\langle P_{p,w} f, g\rangle_\Omega = \langle P_{p,w} f, P_{p',w'} g\rangle_\Omega = \langle f, P_{p',w'} g\rangle_\Omega$$

holds for any $f \in (L^p_w(\Omega))^n$ and $g \in (L^{p'}_{w'}(\Omega))^n$. This equality means $(P_{p,w})^* = P_{p',w'}$ and $(P_{p',w'})^* = P_{p,w}$, which completes the proof of Lemma 3.5. □

At this point, we are ready to prove Theorem 2.2 and Corollary 2.2. It is obvious to see that Theorem 2.2 is an immediate consequence of Theorem 2.1, Lemmas 3.3, 3.4 and 3.5. Combining Theorem 2.2 with Corollary 2.1, we can easily obtain Corollary 2.2. Therefore, proofs of our main results are complete.

*References*

[1] *G. de Rham*: Variétés différentiables. Formes, courants, formes harmoniques. Publications de l'Institut de Mathématique de l'Université de Nancago III. Actualités Scientifiques et Industrielles 1222 b, Hermann, Paris, 1973. (In French.) zbl MR
[2] *E. Fabes, O. Mendez, M. Mitrea*: Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. J. Funct. Anal. *159* (1998), 323–368. zbl MR doi
[3] *R. Farwig*: Weighted $L^q$-Helmholtz decompositions in infinite cylinders and in infinite layers. Adv. Differ. Equ. *8* (2003), 357–384. zbl MR
[4] *R. Farwig, H. Kozono, H. Sohr*: An $L^q$-approach to Stokes and Navier-Stokes equations in general domains. Acta Math. *195* (2005), 21–53. zbl MR doi
[5] *R. Farwig, H. Kozono, H. Sohr*: On the Helmholtz decomposition in general unbounded domains. Arch. Math. *88* (2007), 239–248. zbl MR doi
[6] *R. Farwig, H. Sohr*: Weighted $L^q$-theory for the Stokes resolvent in exterior domains. J. Math. Soc. Japan *49* (1997), 251–288. zbl MR doi
[7] *A. Fröhlich*: The Helmholtz decomposition of weighted $L^q$-spaces for Muckenhoupt weights. Ann. Univ. Ferrara, Nuova Ser., Sez. VII *46* (2000), 11–19. zbl MR
[8] *A. Fröhlich*: Maximal regularity for the non-stationary Stokes system in an aperture domain. J. Evol. Equ. *2* (2002), 471–493. zbl MR doi
[9] *A. Fröhlich*: The Stokes operator in weighted $L^q$-spaces I. Weighted estimates for the Stokes resolvent problem in a half space. J. Math. Fluid Mech. *5* (2003), 166–199. zbl MR doi
[10] *A. Fröhlich*: The Stokes operator in weighted $L^q$-spaces II. Weighted resolvent estimates and maximal $L^p$-regularity. Math. Ann. *339* (2007), 287–316. zbl MR doi
[11] *J. García-Cuerva, J. L. Rubio de Francia*: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies 116, North-Holland, Amsterdam, 1985. zbl MR doi
[12] *J. Geng, Z. Shen*: The Neumann problem and Helmholtz decomposition in convex domains. J. Funct. Anal. *259* (2010), 2147–2164. zbl MR doi
[13] *A. S. Kim, Z. Shen*: The Neumann problem in $L^p$ on Lipschitz and convex domains. J. Funct. Anal. *255* (2008), 1817–1830. zbl MR doi
[14] *T. Kobayashi, T. Kubo*: Weighted $L^p$-$L^q$ estimates of the Stokes semigroup in some unbounded domains. Tsukuba J. Math. *37* (2013), 179–205. zbl MR doi
[15] *J. Lang, O. Méndez*: Potential techniques and regularity of boundary value problems in exterior non-smooth domains. Regularity in exterior domains. Potential Anal. *24* (2006), 385–406. zbl MR doi
[16] *Y. Maekawa, H. Miura*: Remark on the Helmholtz decomposition in domains with noncompact boundary. Math. Ann. *359* (2014), 1077–1095. zbl MR doi
[17] *C. G. Simader, H. Sohr, W. Varnhorn*: Necessary and sufficient conditions for the existence of Helmholtz decompositions in general domains. Ann. Univ. Ferrara, Sez. VII, Sci. Mat. *60* (2014), 245–262. zbl MR doi

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