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RESTRICTED HOMOLOGICAL DIMENSIONS OVER LOCAL HOMOMORPHISMS AND COHEN-MACAULAYNESS

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Abstract. We define and study restricted projective, injective and flat dimensions over local homomorphisms. Some known results are generalized. As applications, we show that (almost) Cohen-Macaulay rings can be characterized by restricted homological dimensions over local homomorphisms.

Keywords: Cohen factorization; restricted homological dimension; Cohen-Macaulay ring

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1. Introduction

Throughout this paper, all rings are commutative and noetherian. A local homomorphism $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is a homomorphism of local rings such that $\varphi(\mathfrak{m})\subseteq\mathfrak{n}$. We define and study notions of restricted homological dimensions for complexes of modules over local homomorphisms.

It is well known that Gorenstein homological dimensions are refinements of the classical homological dimensions. Christensen, Foxby and Frankild introduced restricted homological dimensions in [7], which are refinements of Gorenstein homological dimensions in some sense. Iyengar and Sather-Wagstaff in [10] develop a theory of Gorenstein dimension over local homomorphisms. More precisely, let $\varphi\colon R\to S$ be a local homomorphism and $X\in\mathcal{D}_b^f(S)$, and let $R\xrightarrow{\dot\varphi}R'\xrightarrow{\varphi'}\widehat{S}$ be a Cohen factorization of $\dot\varphi$. The *Gorenstein dimension of X over φ* (see [10], (3.3)) is defined by

$$\text{G-dim}_\varphi X := \text{G-dim}_{R'}\widehat{X} - \operatorname{edim}(\dot\varphi),$$

where $\text{G-dim}_{R'}\widehat{X}$ denotes the Gorenstein dimension of $\widehat{X}$ over R' (see [1]). Recall that the *projective dimension of X over φ*, $\mathrm{pd}_{\varphi}X$ (see [10], (4.2)), is defined by

$$\mathrm{pd}_{\varphi}X = \mathrm{pd}_{R'}\widehat{X} - \mathrm{edim}(\dot{\varphi}).$$

Let $\varphi\colon R \to S$ be a local homomorphism and $X \in \mathcal{D}_b^f(S)$. Recall that the *injective dimension of X over φ* (see [4], (8.2)) is the number

$$\mathrm{id}_{\varphi}X = \sup\{n \in \mathbb{Z}\colon \mu_{\varphi}^n(X) \neq 0\} - \mathrm{edim}(\varphi),$$

where

$$\mu_{\varphi}^n(X) = \mathrm{rank}_l \operatorname{Ext}_R^{n-\mathrm{edim}(\varphi)}(k, K[x;X])$$

is the nth *Bass number of X over φ* (see [4], (4.1)). If $\varphi\colon R \to S$ is a local homomorphism and $X \in \mathcal{D}_b^f(S)$, then it is known that $\mathrm{pd}_{\varphi}X = \mathrm{fd}_R X + \mathrm{depth}_R X - \mathrm{depth}_S X$ (see [10], Proposition 4.5) and $\mathrm{id}_{\varphi}X = \mathrm{id}_R X$ (see [4], Corollary 8.2.2).

Inspired by this, we define and study notions of restricted homological dimensions of complexes over local homomorphisms in this paper. The main goal of this paper is to study the properties of restricted homological dimensions over local homomorphisms and to characterize (almost) Cohen-Macaulay rings by restricted homological dimensions over local homomorphisms.

2. Preliminaries

The derived category is written $\mathcal{D}(R)$. Let M be an R-complex

$$\ldots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \ldots$$

The projective, injective and flat dimensions of M are abbreviated as $\mathrm{pd}_R M$, $\mathrm{id}_R M$ and $\mathrm{fd}_R M$, respectively. The symbols $\sup M$ and $\inf M$ are used for the supremum and infimum of the set $\{i \in \mathbb{Z}\colon \mathrm{H}_i(M) \neq 0\}$, with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. The amplitude of a complex X is defined by $\mathrm{amp}\, X = \sup X - \inf X$. A complex M is called *homologically bounded above* if $\sup M < \infty$, it is called *homologically bounded below* if $\inf M > -\infty$, and it is called *homologically bounded* if it is homologically bounded above and below. The full subcategories $\mathcal{D}_-(R)$ and $\mathcal{D}_+(R)$ consist of complexes X with, respectively, $\sup X < \infty$ and $\inf X > -\infty$. We set $\mathcal{D}_b(R) = \mathcal{D}_-(R) \cap \mathcal{D}_+(R)$. The full subcategories $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ of $\mathcal{D}_b(R)$ consist of complexes of finite, respectively, projective, injective and flat dimension. We use the superscript f to denote finite (finitely generated) homology and the subscript 0 to denote modules. For example, $\mathcal{P}_0^f(R)$ denotes the category of finite R-modules of finite projective dimension.

We use the standard notation $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes_R^{\mathbf{L}} -$ for the derived Hom and derived tensor product of complexes.

Definition 2.1. Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. The *depth* of M is defined as

$$\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k, M).$$

By [8], Theorem 1.5. (3), for every R-complex M one has

$$\operatorname{depth}_R M \geqslant -\sup M. \tag{2.1}$$

If $\sup M = s$ is finite, then equality holds if and only if $\mathfrak{m}$ is an associated prime of the top homology module $\mathrm{H}_s(M)$.

Lemma 2.2 ([10], (2.8))**.** *Let $\varphi : R \to S$ be a local homomorphism and $X \in \mathcal{D}(S)$. If $\mathrm{H}(X)$ is degreewise finite over R, then we have the equality*

$$\operatorname{depth}_S X = \operatorname{depth}_R X.$$

Definition 2.3. Let $(R, \mathfrak{m}, k)$ be a local ring. The *width* of an R-complex M is defined as

$$\operatorname{width}_R M = \inf\, (k \otimes^{\mathbf{L}}_R M).$$

By [8], Theorem 1.5. (1), for every R-complex M we have the inequality

$$\operatorname{width}_R M \geqslant \inf M, \tag{2.2}$$

and equality holds if $\mathrm{H}(M)$ is bounded below and degreewise finite by Nakayama's lemma.

Lemma 2.4. *Let $\varphi\colon R \to S$ be a local homomorphism. If $X \in \mathcal{D}^f_+(S)$, then we have the equality*

$$\inf X = \operatorname{width}_S X = \operatorname{width}_R X.$$

P r o o f. Nakayama's lemma explains the first two equalities in the next display:

$$\operatorname{width}_S X = \inf X = \operatorname{width}_S(\mathfrak{m}_R S; X) = \operatorname{width}_R X.$$

For the third equality, use the Koszul characterization of width: if $\mathrm{x} \in \mathfrak{m}_R$ is a generating sequence of $\mathfrak{m}_R$, then one has $S \otimes^{\mathbf{L}}_R K^R(\mathrm{x}) \simeq K^S(\mathrm{x})$ and so

$$X \otimes^{\mathbf{L}}_S K^S(\mathrm{x}) \simeq X \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R K^R(\mathrm{x})) \simeq X \otimes^{\mathbf{L}}_R K^R(\mathrm{x}).$$

$\square$

We proceed by recalling the definition of Cohen factorizations of local homomorphisms from [3].

Let $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism. The *embedding dimension* of φ is

$$\operatorname{edim}(\varphi) := \operatorname{edim}(S/\mathfrak{m}S).$$

A *regular factorization* of φ is a diagram of local homomorphisms, $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$, where $\varphi=\varphi'\dot\varphi$, with $\dot\varphi$ flat, the closed fibre $R'/\mathfrak{m}R'$ regular and $\varphi'\colon R'\to S$ surjective.

Let $\widehat{S}$ denote the completion of S at its maximal ideal and let $\iota\colon S\to\widehat{S}$ be the canonical inclusion. By [3], (1.1), the composition $\grave{\varphi}=\iota\varphi$ admits a regular factorization $R\to R'\to\widehat{S}$ with R' complete. Such a regular factorization is said to be a *Cohen factorization* of $\grave{\varphi}$.

Note that $\operatorname{edim}(\varphi)\leqslant\operatorname{edim}(\dot\varphi)=\operatorname{edim}(R/\mathfrak{m}R')$. When equality holds the Cohen factorization is said to be *minimal*. It is proved in [3], (1.5), that the homomorphism $\grave{\varphi}$ always has a minimal Cohen factorization.

3. Restricted flat dimensions

Recall that the *small restricted flat dimension* (see [7], Definition 2.9), $\operatorname{rfd}_R X$, of $X\in\mathcal{D}_+(R)$ is

$$\operatorname{rfd}_R X=\sup\{\sup(T\otimes^{\mathbf{L}}_R X) \mid T\in\mathcal{P}^f_0(R)\},$$

and the *large restricted flat dimension* (see [7], Definition 2.1), $\operatorname{Rfd}_R X$, of $X\in\mathcal{D}_+(R)$ is

$$\operatorname{Rfd}_R X=\sup\{\sup(T\otimes^{\mathbf{L}}_R X) \mid T\in\mathcal{F}_0(R)\}.$$

Note that if R is local (more generally, if R has finite Krull dimension) and $X\in\mathcal{D}_b(R)$, then one has $\operatorname{rfd}_R X<\infty$ and $\operatorname{Rfd}_R X<\infty$ by [7], (2.2) and (2.10).

Now we introduce the concepts of restricted flat dimensions over local homomorphisms.

Definition 3.1. Let $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism and $X\in\mathcal{D}_+(S)$. We define the *small restricted flat dimension of X over φ* to be the number

$$\operatorname{rfd}_\varphi X=\operatorname{rfd}_R X+\operatorname{depth}_R X-\operatorname{depth}_S X;$$

the *large restricted flat dimension of X over φ* to be the number

$$\operatorname{Rfd}_\varphi X=\operatorname{Rfd}_R X+\operatorname{depth}_R X-\operatorname{depth}_S X.$$

Remark 3.2. Since $\operatorname{depth}_R X \leqslant \operatorname{depth}_S X$ by [9], (5.2), one has $\operatorname{rfd}_\varphi X \leqslant \operatorname{rfd}_R X$ and $\operatorname{Rfd}_\varphi X \leqslant \operatorname{Rfd}_R X$. If $X \in \mathcal{D}_b(S)$, then it follows from [7], (2.2), (2.12) and (2.10), and [8], (2.5), that

$$\operatorname{depth} R - \operatorname{depth}_S X \leqslant \operatorname{rfd}_\varphi X \leqslant \operatorname{Rfd}_\varphi X < \infty.$$

Remark 3.3. The two restricted flat dimensions defined over R may differ, even for finite modules over local rings. The same also holds for the two restricted flat dimensions over local homomorphisms. Let R be a local ring with $\dim R = 2$ and $\operatorname{depth} R = 0$ (see [7], Example 2.13). Choose $\mathfrak{q} \in \operatorname{Spec} R$ with $\operatorname{depth} R_{\mathfrak{q}} = 1$ and pick $x \in \mathfrak{q}$ such that the fraction $x/1$ is $R_{\mathfrak{q}}$-regular. Let $S = R/(x)$, $\varphi\colon R \to S$ be the natural map and $M = R/(x)$. Then $\operatorname{rfd}_\varphi M = \operatorname{rfd}_R M = 0$ and $\operatorname{Rfd}_\varphi M = \operatorname{Rfd}_R M \geqslant 1$.

Recall that a local ring R is called a *Cohen-Macaulay ring* if $\dim R = \operatorname{depth} R$ (see for example [5]).

Proposition 3.4. *Let R be a Cohen-Macaulay ring, $\varphi\colon R \to S$ a module-finite local homomorphism and $X \in \mathcal{D}_b^f(S)$. If $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\dot\varphi$, then we have the equalities*

$$\operatorname{rfd}_\varphi X = \operatorname{rfd}_{R'} \widehat{X} - \operatorname{edim}(\varphi) \quad \textit{and} \quad \operatorname{Rfd}_\varphi X = \operatorname{Rfd}_{R'} \widehat{X} - \operatorname{edim}(\varphi),$$

where $\widehat{X} = X \otimes_S^{\mathbf{L}} \widehat{S}$.

Proof. Since R is a Cohen-Macaulay ring, by [3], Proposition 2.8, R' is a Cohen-Macaulay ring. Now we have

$$\begin{aligned}
\operatorname{rfd}_\varphi X &= \operatorname{rfd}_R X + \operatorname{depth}_R X - \operatorname{depth}_S X\\
&= \operatorname{depth} R - \operatorname{depth}_S X\\
&= \operatorname{depth} R - \operatorname{depth}_{\widehat{S}} \widehat{X}\\
&= \operatorname{depth} R - \operatorname{depth}_{R'} \widehat{X}\\
&= \operatorname{depth} R' - \operatorname{depth}_{R'} \widehat{X} - \operatorname{depth} R' + \operatorname{depth} R\\
&= \operatorname{rfd}_{R'} \widehat{X} - \operatorname{depth} R' + \operatorname{depth} R\\
&= \operatorname{rfd}_{R'} \widehat{X} - \operatorname{edim}(\varphi).
\end{aligned}$$

The second equality in the computation above follows from [7], Corollary 3.5, as φ is module-finite, the fourth holds by Lemma 2.2, the sixth follows from $\widehat{X} \in \mathcal{D}_b^f(R')$ and [7], Corollary 3.5, and the last equality holds since $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\dot\varphi$.

Similarly, one has $\operatorname{Rfd}_\varphi X = \operatorname{Rfd}_{R'} \widehat{X} - \operatorname{edim}(\varphi)$. □

The next result shows that the large restricted flat dimension is a refinement of the projective dimension over a local homomorphism.

Proposition 3.5. *Let $\varphi\colon R \to S$ be a local homomorphism and $X \in \mathcal{D}_b^f(S)$. Then we have the inequality*

$$\mathrm{Rfd}_{\varphi} X \leqslant \mathrm{pd}_{\varphi} X,$$

and equality holds if $\mathrm{pd}_{\varphi} X < \infty$.

P r o o f. It follows from [7], Theorem 2.5, and [10], Proposition 4.5. □

The following result is an extension of the Bass formula for the restricted flat dimensions (see [7], Corollary 3.5), which is a special case by putting $\varphi = \mathrm{id}_R$.

Lemma 3.6. *Let R be a Cohen-Macaulay ring and $\varphi\colon R \to S$ a module-finite local homomorphism and $X \in \mathcal{D}_b^f(S)$. Then*

$$\mathrm{rfd}_{\varphi} X = \mathrm{Rfd}_{\varphi} X = \operatorname{depth} R - \operatorname{depth}_S X.$$

P r o o f. It follows from Lemma 2.2 and [7], Corollary 3.5. □

Recall that a local ring R is called an *almost Cohen-Macaulay ring* if $\operatorname{cmd} R = \dim R - \operatorname{depth} R \leqslant 1$.

Proposition 3.7. *Let R be a local ring. The following conditions are equivalent.*

(i) *R is almost Cohen-Macaulay.*
(ii) *For every local homomorphism $\varphi\colon R \to S$ and for every $X \in \mathcal{D}_+(S)$, $\mathrm{rfd}_{\varphi} X = \mathrm{Rfd}_{\varphi} X$.*
(iii) *For every local homomorphism $\varphi\colon R \to S$ and for every finite S-module M, $\mathrm{rfd}_{\varphi} M = \mathrm{Rfd}_{\varphi} M$.*

P r o o f. It follows from [7], Theorem 3.2. □

Theorem 3.8. *Let R be a local ring. The following conditions are equivalent.*

(i) *R is Cohen-Macaulay.*
(ii) *For every module-finite local homomorphism $\varphi\colon R \to S$ and for every $X \in \mathcal{D}_b^f(S)$, $\mathrm{Rfd}_{\varphi} X = \operatorname{depth} R - \operatorname{depth}_S X$.*
(iii) *For every module-finite local homomorphism $\varphi\colon R \to S$ and for every finite S-module M, $\mathrm{rfd}_{\varphi} M = \operatorname{depth} R - \operatorname{depth}_S M$.*

P r o o f. It follows from Lemma 3.6 and [7], Theorem 3.4. □

Lemma 3.9. *Let* $\varphi\colon R \to S$ *be a local homomorphism and* $X \in \mathcal{D}_+(S)$.

(i) (*See* [10], (8.6)) *We have the inequality* $\mathrm{Rfd}_R X \geqslant \operatorname{depth} R - \operatorname{depth}_S X$; *equality holds if* X *is homologically finite over* R *and* $\text{G-dim}_R X$ *is finite.*

(ii) *We have the inequality* $\mathrm{rfd}_R X \geqslant \operatorname{depth} R - \operatorname{depth}_S X$; *equality holds if* X *is homologically finite over* R *and* $\text{G-dim}_R X$ *is finite.*

P r o o f. (ii) Note that $\operatorname{depth}_R X \leqslant \operatorname{depth}_S X$. Hence the inequality follows by [7], (2.12). If X is homologically finite over R and $\text{G-dim}_R X$ is finite, then equality holds by (i) and [7], (2.10). □

Theorem 3.10. *Let* $\varphi\colon R \to S$ *be a local homomorphism and* $X \in \mathcal{D}_b^f(S)$. *If* $\text{G-dim}_\varphi X$ *is finite, then*

$$\mathrm{rfd}_R X - \operatorname{edim}(\varphi) \leqslant \mathrm{rfd}_\varphi X \leqslant \mathrm{rfd}_R X.$$

P r o o f. By Remark 3.2, one has $\operatorname{depth} R - \operatorname{depth}_S X \leqslant \mathrm{rfd}_\varphi X \leqslant \mathrm{rfd}_R X$. Therefore, we only need to show that

$$(*) \qquad \mathrm{rfd}_R X - \operatorname{edim}(\varphi) \leqslant \operatorname{depth} R - \operatorname{depth}_S X.$$

Let $\widehat{S}$ be the completion of S at its maximal ideal and set $\widehat{X} = \widehat{S} \otimes_S X$. We have

$$\begin{aligned}
\mathrm{rfd}_R X &= \sup\{\sup(T \otimes_R^{\mathbf{L}} X)\colon\ T \in \mathcal{P}_0^f(R)\}\\
&= \sup\{\sup((T \otimes_R^{\mathbf{L}} X) \otimes_S \widehat{S})\colon\ T \in \mathcal{P}_0^f(R)\}\\
&= \sup\{\sup(T \otimes_R^{\mathbf{L}} \widehat{X})\colon\ T \in \mathcal{P}_0^f(R)\}\\
&= \mathrm{rfd}_R \widehat{X}.
\end{aligned}$$

Since $\widehat{S}$ is faithfully flat as an S-module, the second equality holds in the computation above. The third equality is by the isomorphism

$$(T \otimes_R^{\mathbf{L}} X) \otimes_S \widehat{S} \cong T \otimes_R^{\mathbf{L}} (\widehat{S} \otimes_S X).$$

Consider the other quantities in the desired inequality $(*)$. Since we have $\operatorname{edim}(\dot{\varphi}) = \operatorname{edim}(\varphi)$ and $\operatorname{depth}_{\widehat{S}} \widehat{X} = \operatorname{depth}_S X$, it suffices to show that

$$\mathrm{rfd}_R \widehat{X} - \operatorname{edim}(\dot{\varphi}) \leqslant \operatorname{depth} R - \operatorname{depth}_{\widehat{S}} \widehat{X}.$$

Hence without loss of generality, we may assume that S is complete. With $R \to R' \to S$ a minimal Cohen factorization of φ, one has

$$\begin{aligned}
\mathrm{rfd}_R X &= \sup\{\sup(T \otimes^{\mathbf{L}}_R X)\colon\ T \in \mathcal{P}^f_0(R)\}\\
&= \sup\{\sup((T \otimes_R R') \otimes^{\mathbf{L}}_{R'} X)\colon\ T \in \mathcal{P}^f_0(R)\}\\
&\leqslant \sup\{\sup((T \otimes_R R') \otimes^{\mathbf{L}}_{R'} X)\colon\ T \otimes_R R' \in \mathcal{P}^f_0(R')\}\\
&= \mathrm{rfd}_{R'} X\\
&= \operatorname{depth} R' - \operatorname{depth}_{R'} X\\
&= \operatorname{depth} R + \operatorname{edim}(\varphi) - \operatorname{depth}_{R'} X\\
&= \operatorname{depth} R + \operatorname{edim}(\varphi) - \operatorname{depth}_S X.
\end{aligned}$$

Since $T \in \mathcal{P}^f_0(R)$, one has $T \otimes_R R' \in \mathcal{P}^f_0(R')$ and so the inequality in the computation above holds. Since $\text{G-dim}_\varphi X$ is finite, the fourth equality follows by Lemma 3.9, the fifth holds as $R \to R'$ is flat and $R'/\mathfrak{m}R'$ is regular and the last follows by Lemma 2.2. This completes the proof. $\square$

Similarly, one has the following result.

Proposition 3.11. *Let $\varphi\colon\ R \to S$ be a local homomorphism and $X \in \mathcal{D}^f_b(S)$. If $\text{G-dim}_\varphi X$ is finite, then*

$$\mathrm{Rfd}_R X - \operatorname{edim}(\varphi) \leqslant \mathrm{Rfd}_\varphi X \leqslant \mathrm{Rfd}_R X.$$

4. Restricted injective dimensions

Recall that the *small restricted injective dimension* $\mathrm{rid}_R Y$ (see [7], Definition 5.1) of $Y \in \mathcal{D}_-(R)$ is

$$\mathrm{rid}_R Y = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(T, Y)\colon\ T \in \mathcal{P}^f_0(R)\},$$

and the *large restricted injective dimension* $\mathrm{Rid}_R Y$ (see [7], Definition 5.10) of $Y \in \mathcal{D}_-(R)$ is

$$\mathrm{Rid}_R Y = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(T, Y)\colon\ T \in \mathcal{P}_0(R)\}.$$

Note that if R is local (more generally, if R has finite Krull dimension), and $X \in \mathcal{D}_b(R)$, then one has $\mathrm{rid}_R X < \infty$ and $\mathrm{Rid}_R X < \infty$ by [7], (5.2) and (5.11).

Next we introduce the concepts of restricted injective dimensions over local homomorphisms.

Definition 4.1. Let $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ be a local homomorphism and $X\in\mathcal{D}_-(S)$. We define the *small restricted injective dimension of* X *over* φ to be the number

$$\operatorname{rid}_\varphi X=\operatorname{rid}_R X+\operatorname{width}_R X-\operatorname{width}_S X;$$

the *large restricted injective dimension of* X *over* φ to be the number

$$\operatorname{Rid}_\varphi X=\operatorname{Rid}_R X+\operatorname{width}_R X-\operatorname{width}_S X.$$

Note that if $X\in\mathcal{D}_b^f(S)$, then $\operatorname{rid}_\varphi X=\operatorname{rid}_R X$ by Lemma 2.4 and so it follows from [7], (5.2) and (5.11) that $-\inf X\leqslant\operatorname{rid}_\varphi X\leqslant\operatorname{Rid}_\varphi X<\infty$.

Lemma 4.2. *Let $\varphi\colon R\to S$ be a local homomorphism. For $X\in\mathcal{D}_b^f(S)$, we have the equalities*

$$\operatorname{rid}_\varphi X=\operatorname{rid}_R X=\operatorname{depth}R-\inf X;\quad \operatorname{Rid}_\varphi X=\operatorname{Rid}_R X.$$

Proof. We have

$$\begin{aligned}\operatorname{rid}_R X&=\sup\{\operatorname{depth}_R(\mathfrak{p},X)-\operatorname{width}_R(\mathfrak{p},X)|\mathfrak{p}\in\operatorname{Spec}R\}\\&=\sup\{\operatorname{depth}_R(\mathfrak{p},X)-\operatorname{width}_S(\mathfrak{p}S,X)|\mathfrak{p}\in\operatorname{Spec}R\}\\&=\operatorname{depth}R-\inf X.\end{aligned}$$

The first equality in the computation above follows by [7], Proposition 5.3, the second by [11], Proposition 2.5, and the third holds by [7], (4.3.2). Now the result follows from Lemma 2.4. □

Proposition 4.3. *Let $\varphi\colon R\to S$ be a module-finite local homomorphism and $X\in\mathcal{D}_b^f(S)$. If R is Cohen-Macaulay, then we have the equality*

$$\operatorname{Rid}_\varphi X=\operatorname{depth}R-\inf X.$$

Proof. It follows from [12], Theorem 2.2. □

Theorem 4.4. *Let $\varphi\colon R\to S$ be a local homomorphism and $X\in\mathcal{D}_b^f(S)$. If $R\xrightarrow{\dot\varphi}R'\xrightarrow{\varphi'}\widehat{S}$ is a minimal Cohen factorization of $\dot\varphi$, then we have the equality $\operatorname{rid}_\varphi X=\operatorname{rid}_{R'}\widehat{X}-\operatorname{edim}(\varphi)$, where $\widehat{X}=X\otimes^{\mathbf{L}}_S\widehat{S}$.*

P r o o f. We have

$$\begin{aligned}\operatorname{rid}_\varphi X &= \operatorname{depth} R - \operatorname{width}_S X\\ &= \operatorname{depth} R - \operatorname{width}_{\widehat{S}} \widehat{X}\\ &= \operatorname{depth} R - \operatorname{width}_{R'} \widehat{X}\\ &= \operatorname{depth} R' - \operatorname{width}_{R'} \widehat{X} - \operatorname{depth} R' + \operatorname{depth} R\\ &= \operatorname{rid}_{R'} \widehat{X} - \operatorname{depth} R' + \operatorname{depth} R\\ &= \operatorname{rid}_{R'} \widehat{X} - \operatorname{edim}(\varphi).\end{aligned}$$

The first equality in the computation above follows by Lemma 4.2, the second by [11], Proposition 2.3, the third by Lemma 2.4, the fifth follows from $\widehat{X} \in \mathcal{D}_b^f(R')$ and [7], Corollary 3.5, and the last equality holds since $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\dot{\varphi}$. $\square$

By analogy with the proof of Theorem 4.4 and [12], Theorem 2.2, we have the following result.

Proposition 4.5. *Let R be a Cohen-Macaulay ring, $\varphi\colon R \to S$ a module-finite local homomorphism and $X \in \mathcal{D}_b^f(S)$. If $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\dot{\varphi}$, then we have the equality $\operatorname{Rid}_\varphi X = \operatorname{Rid}_{R'} \widehat{X} - \operatorname{edim}(\varphi)$.*

The following result shows that the restricted injective dimensions are refinements of the injective dimension over a local homomorphism, at least over almost Cohen-Macaulay rings.

Proposition 4.6. *Let $\varphi\colon R \to S$ be a local homomorphism. For $X \in \mathcal{D}_b^f(S)$, we have the inequalities*

$$\operatorname{rid}_\varphi X \leqslant \operatorname{id}_\varphi X \quad \textit{and} \quad \operatorname{Rid}_\varphi X \leqslant \operatorname{id}_\varphi X$$

and equalities hold if $\operatorname{id}_\varphi X < \infty$ and $\operatorname{cmd} R \leqslant 1$.

P r o o f. It follows from [4], Corollary 8.2.2, and [7], Propositions 5.8 and 5.13. $\square$

For the small restricted injective dimension over local homomorphisms, we have some stability results.

Proposition 4.7. *Let $\varphi\colon R \to S$ be a local homomorphism and $P \in \mathcal{P}_b^f(S)$. For each $X \in \mathcal{D}_b^f(S)$ we have an equality*

$$\operatorname{rid}_\varphi(P \otimes_S^{\mathbf{L}} X) = \operatorname{rid}_\varphi X - \inf P.$$

P r o o f. Note that $P \otimes_S^{\mathbf{L}} X \in \mathcal{D}_b^f(S)$. We have

$$\begin{aligned}\operatorname{rid}_\varphi(P \otimes_S^{\mathbf{L}} X) &= \operatorname{depth} R - \operatorname{width}_S(P \otimes_S^{\mathbf{L}} X)\\ &= \operatorname{depth} R - \operatorname{width}_S P - \operatorname{width}_S X\\ &= \operatorname{rid}_\varphi X - \inf P,\end{aligned}$$

where the first and third equalities follow by Lemma 4.2 and the second holds by [8], Proposition 4.6. □

Corollary 4.8. *Let $\varphi\colon R \to S$ be a local homomorphism and $P \in \mathcal{P}_b^f(S)$. For each $X \in \mathcal{D}_b^f(S)$ we have the equality*

$$\operatorname{rid}_\varphi \mathbf{R}\operatorname{Hom}_S(P, X) = \operatorname{rid}_\varphi X + \operatorname{pd}_S P.$$

P r o o f. Note that $\mathbf{R}\operatorname{Hom}_S(P, X) \in \mathcal{D}_b^f(S)$ by assumption. Since $\operatorname{pd}_S P$ is finite, one has

$$\mathbf{R}\operatorname{Hom}_S(P, X) \cong \mathbf{R}\operatorname{Hom}_S(P, S \otimes_S^{\mathbf{L}} X) \cong \mathbf{R}\operatorname{Hom}_S(P, S) \otimes_S^{\mathbf{L}} X,$$

where the second isomorphism follows from [2], Lemma 4.4. It follows from [6], Theorem 2.13, that $\operatorname{pd}_S \mathbf{R}\operatorname{Hom}_S(P, S) = -\inf P$ is finite and $\inf \mathbf{R}\operatorname{Hom}_S(P, S) = -\operatorname{pd}_S P$. Thus we have the following equalities.

$$\begin{aligned}\operatorname{rid}_\varphi \mathbf{R}\operatorname{Hom}_S(P, X) &= \operatorname{rid}_\varphi(\mathbf{R}\operatorname{Hom}_S(P, S) \otimes_S^{\mathbf{L}} X)\\ &= \operatorname{rid}_\varphi X - \inf \mathbf{R}\operatorname{Hom}_S(P, S)\\ &= \operatorname{rid}_\varphi X + \operatorname{pd}_S P.\end{aligned}$$

The second equality above follows from Proposition 4.7. □

For any faithfully injective R-module E we use the notation $-^\vee = \mathbf{R}\operatorname{Hom}_R(-, E)$.

Proposition 4.9. *Let $\varphi\colon R \to S$ be a local homomorphism and $X \in \mathcal{D}_b^f(S)$.*

(i) $\operatorname{rid}_\varphi X = \operatorname{rfd}_\varphi X^\vee$.

(ii) *If φ is module-finite, then* $\operatorname{rid}_\varphi X^\vee = \operatorname{rfd}_\varphi X$.

(iii) *If φ is module-finite, then* $\operatorname{Rid}_\varphi X^\vee = \operatorname{Rfd}_\varphi X$.

P r o o f. (i) We have

$$\begin{aligned}\operatorname{rfd}_\varphi X^\vee &= \operatorname{rfd}_R X^\vee + \operatorname{depth}_R X^\vee - \operatorname{depth}_S X^\vee\\ &= \operatorname{rfd}_R X^\vee + \operatorname{width}_R X + \operatorname{depth}_R E - \operatorname{width}_S X - \operatorname{depth}_R E\\ &= \operatorname{rfd}_R X^\vee\\ &= \operatorname{rid}_R X\\ &= \operatorname{rid}_\varphi X.\end{aligned}$$

The second equality in the computation above follows from [8], Proposition 4.6, and [11], Proposition 2.3, the third by Lemma 2.4 and the fourth by [7], Proposition 5.3.

(ii) We have

$$\begin{aligned}\operatorname{rid}_\varphi X^\vee &= \operatorname{rid}_R X^\vee + \operatorname{width}_R X^\vee - \operatorname{width}_S X^\vee\\ &= \operatorname{rfd}_R X + \operatorname{depth}_R X - \operatorname{depth}_S X\\ &= \operatorname{rfd}_\varphi X.\end{aligned}$$

The second equality in the computation above follows from [12], Proposition 2.1, and [7], Proposition 4.8.

(iii) Similarly. □

Remark 4.10. The two restricted injective dimensions over local homomorphisms may also differ. Let R be a local ring with $\dim R = 2$ and $\operatorname{depth} R = 0$ (see [7], Example 2.13). Choose $\mathfrak{q} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{q} = 1$ and pick $x \in \mathfrak{q}$ such that the fraction $x/1$ is $R_\mathfrak{q}$-regular. Let $S = R/(x)$, let $\varphi\colon R \to S$ be the natural map and $M = R/(x)$. Then $\operatorname{rid}_\varphi M^\vee = \operatorname{rfd}_\varphi M = 0$ and $\operatorname{Rid}_\varphi M^\vee = \operatorname{Rfd}_\varphi M = \operatorname{Rfd}_R M \geqslant 1$.

Theorem 4.11. *Let R be a local ring. The following conditions are equivalent.*

(i) *R is almost Cohen-Macaulay.*

(ii) *For every module-finite local homomorphism $\varphi\colon R \to S$ and for every $X \in \mathcal{D}_b^f(S)$, $\operatorname{Rfd}_\varphi X^\vee = \operatorname{depth} R - \inf X$.*

(iii) *For every module-finite local homomorphism $\varphi\colon R \to S$ and for every finite S-module M, $\operatorname{Rfd}_\varphi M = \operatorname{depth} R$.*

P r o o f. (i) ⇒ (ii): We have

$$\operatorname{Rfd}_\varphi X^\vee = \operatorname{rfd}_\varphi X^\vee = \operatorname{rid}_\varphi X = \operatorname{depth} R - \inf X.$$

The first equality in the computation above holds by Proposition 3.7, the second by Proposition 4.9 and the last follows by Lemma 4.2.

(ii) ⇒ (iii): It is clear.

(iii) ⇒ (i): By [12], Theorem 3.1. □

5. Restricted projective dimensions

Recall that the *small restricted projective dimension* $\operatorname{rpd}_R X$ (see [7], Definition 5.19) of $X \in \mathcal{D}_+(R)$ is

$$\operatorname{rpd}_R X = \sup\{\inf U - \inf \mathbf{R}\mathrm{Hom}_R(X,U)\colon\ U \in \mathcal{I}_b^f(R) \wedge \mathrm{H}(U) \neq 0\},$$

and the *large restricted projective dimension* $\operatorname{Rpd}_R X$ (see [7], Definition 5.14) of $X \in \mathcal{D}_+(R)$ is

$$\operatorname{Rpd}_R X = \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(X,T)\colon\ T \in \mathcal{I}_0(R)\}.$$

Note that if R is local (more generally, if R has finite Krull dimension) and $X \in \mathcal{D}_b(R)$, then one has $\operatorname{rpd}_R X < \infty$ and $\operatorname{Rpd}_R X < \infty$.

Next we introduce the concepts of restricted projective dimensions over local homomorphisms.

Definition 5.1. Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a local homomorphism and $X \in \mathcal{D}_+(S)$. We define the *small restricted projective dimension of X over φ* to be the number

$$\operatorname{rpd}_\varphi X = \operatorname{rpd}_R X + \operatorname{depth}_R X - \operatorname{depth}_S X;$$

the *large restricted projective dimension of X over φ* to be the number

$$\operatorname{Rpd}_\varphi X = \operatorname{Rpd}_R X + \operatorname{depth}_R X - \operatorname{depth}_S X.$$

Proposition 5.2. *Let $\varphi\colon R \to S$ be a module-finite local homomorphism and $X \in \mathcal{D}_b^f(S)$. If $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\dot\varphi$, then we have the equality $\operatorname{rpd}_\varphi X = \operatorname{rpd}_{R'} \widehat{X} - \operatorname{edim}(\varphi)$, where $\widehat{X} = X \otimes_S^{\mathbf{L}} \widehat{S}$.*

P r o o f. By analogy with the proof of Proposition 3.4 and [7], Lemma 5.20. $\square$

Similarly, we have the following result by [7], Theorem 5.22.

Proposition 5.3. *Let R be a Cohen-Macaulay ring, $\varphi\colon R \to S$ a module-finite local homomorphism and $X \in \mathcal{D}_b^f(S)$. If $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\dot\varphi$, then we have the equality $\operatorname{Rpd}_\varphi X = \operatorname{Rpd}_{R'} \widehat{X} - \operatorname{edim}(\varphi)$.*

The next result is an extension of the Bass formula to the restricted projective dimension (see [7], Corollary 5.23), which is a special case by putting $\varphi = \mathrm{id}_R$.

Proposition 5.4. *Let $\varphi\colon R \to S$ be a module-finite local homomorphism. If $X \in \mathcal{D}_b^f(S)$, then we have the equality*

$$\operatorname{rpd}_R X = \operatorname{rpd}_\varphi X = \operatorname{depth} R - \operatorname{depth}_S X.$$

Proof. It follows from [7], Lemma 5.20, and Lemma 2.2. □

Proposition 5.5. *Let $\varphi\colon R \to S$ be a module-finite local homomorphism and $X \in \mathcal{D}_b^f(S)$. If R is Cohen-Macaulay, then we have the equality*

$$\operatorname{Rpd}_{\varphi} X = \operatorname{depth} R - \operatorname{depth}_S X.$$

Proof. It follows from [7], Corollary 5.23, and Lemma 2.2. □

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