VALENCY SEVEN SYMMETRIC GRAPHS OF ORDER $2pq$

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Abstract. A graph is said to be symmetric if its automorphism group acts transitively on its arcs. In this paper, all connected valency seven symmetric graphs of order $2pq$ are classified, where p, q are distinct primes. It follows from the classification that there is a unique connected valency seven symmetric graph of order $4p$, and that for odd primes p and q, there is an infinite family of connected valency seven one-regular graphs of order $2pq$ with solvable automorphism groups, and there are four sporadic ones with nonsolvable automorphism groups, which is $1, 2, 3$-arc transitive, respectively. In particular, one of the four sporadic ones is primitive, and the other two of the four sporadic ones are bi-primitive.

Keywords: arc-transitive graph; symmetric graph; s-regular graph

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1. Introduction

For a finite, simple and undirected graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ denote the vertex set, edge set, arc set and full automorphism group of X, respectively. Note that an *arc* is an ordered edge, that is, an ordered pair of adjacent vertices. For $u, v \in V(X)$, $\{u, v\}$ denotes the edge incident to u and v in X. An *s-arc* in a graph X for some nonnegative integer s is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices such that $(v_{i-1}, v_i) \in A(X)$ for $1 \leqslant i \leqslant s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. For a subgroup G of the automorphism group $\mathrm{Aut}(X)$ of a graph X, the graph X is said to be *(G, s)-arc-transitive* or *(G, s)-regular* if G acts transitively or regularly on the set of s-arcs of X, and *(G, s)-transitive* if G acts transitively on the set of s-arcs but not on the set of $(s+1)$-arcs of X. A graph X is said to be *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive,

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$(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph is said to be *primitive* if its automorphism group is primitive on the vertex set, and a graph is said to be *bi-primitive* if it is a bipartite graph with bi-parts Δ_1, Δ_2, and the setwise stabilizer of its automorphism group is primitive on both Δ_1 and Δ_2. Throughout this paper, we will denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathbb{Z}_n^*$ the multiplicative group of units modulo n, by D_{2n} the dihedral group of order $2n$, by F_n the Frobenius group of order n, and by A_n and S_n the alternating group and the symmetric group of degree n, respectively.

It is well known that a graph Γ is G-arc-transitive if and only if G is vertex-transitive and the vertex stabilizer G_v of $v \in V(\Gamma)$ in G is transitive on $N_\Gamma(v)$. Hence the structure of the vertex stabilizer of G_v plays an important role in the study of (G, s)-transitive graphs. For example, benefitted from Djoković and Miller [4] result about the vertex stabilizer of cubic symmetric graphs, lots of works about classifications of cubic symmetric graphs were obtained by many authors (see [7], [8], [9], [23], [24]). Due to the vertex stabilizers given in [27], symmetric tetravalent graphs have also been studied extensively in the literature (see [11], [12], [22], [32], [34]). Simlarly, Guo and Feng [14] determined structure of vertex stabilixers of pentavalent symmetric graphs, some works about classifications of pentavalent symmetric graphs were also obtained (see [6], [14], [17], [18], [26]). Naturally, the next step is to characterize valency seven symmetric graphs. Recently, Guo et al. [15] gave the structure of vertex stabilizers of valency seven symmetric graphs, and this encourages us to consider some work on valency seven symmetric graphs. In [16], Guo et al. classified valency seven symmetric graphs of order $4p$, and in [25], Pan et al. classified prime-valent symmetric graphs of square-free order. But, we obtain this result for valency seven symmetric graphs of order $2pq$ independently. Let p, q be two distinct primes. In this paper, we classify valency seven symmetric graphs of order $2pq$.

2. Preliminaries

Let X be a graph, and N a subgroup of $\mathrm{Aut}(X)$. Denote by X_N the quotient graph corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in X_N if there is an edge in X between those orbits. In view of [20], Theorem 9, we have the following proposition.

Proposition 2.1. *Let X be a connected symmetric graph of prime valency p and G an s-arc-transitive subgroup of $\mathrm{Aut}(X)$ for some $s \geqslant 1$. If a normal subgroup N of G has more than two orbits on $V(X)$ then X_N is also a symmetric graph* *of valency p and N is the kernel of the action of G on the set of orbits of N. Moreover, N is semiregular on V(X) and G/N is an s-arc-transitive subgroup of* $\mathrm{Aut}(X_N)$.

By Guo [15], we have the following statement.

Proposition 2.2. *Let X be a connected (G, s)-transitive graph of valency seven for some* $G \leqslant \mathrm{Aut}(X)$ *and* $s \geqslant 1$. *Let* $v \in V(X)$. *Then* $s \leqslant 3$ *and one of the following holds:*

(1) *If* G_v *is soluble, then* $|G_v| \mid 2^2 \cdot 3^2 \cdot 7$. *Further, the triple* $(s, G_v, |G_v|)$ *lies in the following table:*

$s=1$		$s=2$		$s=3$	
G_v	*Order*	G_v	*Order*	G_v	*Order*
$\mathbb{Z}_7$	7	F_{42}	$2\cdot 3\cdot 7$	$F_{42}\times\mathbb{Z}_6$	$2^2\cdot 3^2\cdot 7$
D_{14}	$2\cdot 7$	$F_{42}\times\mathbb{Z}_2$	$2^2\cdot 3\cdot 7$		
F_{21}	$3\cdot 7$	$F_{42}\times\mathbb{Z}_3$	$2\cdot 3^2\cdot 7$		
D_{28}	$2^2\cdot 7$				
$F_{21}\times\mathbb{Z}_3$	$3^2\cdot 7$				

(2) *If* G_v *is insoluble, then* $s \geqslant 2$ *and* $|G_v| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$. *Further, the triple* $(s, G_v, |G_v|)$ *lies in the following table:*

$s=2$		$s=3$	
G_v	*Order*	G_v	*Order*
$\mathrm{PSL}(3,2)$	$2^3\cdot 3\cdot 7$	$\mathrm{PSL}(3,2)\times S_4$	$2^6\cdot 3^2\cdot 7$
A_7	$2^3\cdot 3^2\cdot 5\cdot 7$	$A_7\times A_6$	$2^6\cdot 3^4\cdot 5^2\cdot 7$
S_7	$2^4\cdot 3^2\cdot 5\cdot 7$	$S_7\times S_6$	$2^8\cdot 3^4\cdot 5^2\cdot 7$
$\mathbb{Z}_2^3\times\mathrm{SL}(3,2)$	$2^6\cdot 3\cdot 7$	$(A_7\times A_6)\rtimes\mathbb{Z}_2$	$2^7\cdot 3^4\cdot 5^2\cdot 7$
$\mathbb{Z}_2^4\times\mathrm{SL}(3,2)$	$2^7\cdot 3\cdot 7$	$\mathbb{Z}_2^6\rtimes(\mathrm{SL}(2,2)\times\mathrm{SL}(3,2))$	$2^{10}\cdot 3^2\cdot 7$
		$([2^{20}]\rtimes(\mathrm{SL}(2,2)\times\mathrm{SL}(3,2))$	$2^{24}\cdot 3^2\cdot 7$

From [3], pages 12–14, 3-prime factor simple groups can be found. And by [13], pages 134–136, one can obtain the following proposition by checking the orders of nonabelian simple groups.

Proposition 2.3. *Let p, q be distinct odd primes, and let G be a nonabelian simple group of order* $|G| = 2^i \cdot 3^j \cdot 5^k \cdot 7 \cdot p \cdot q$ *with* $1 \leqslant i \leqslant 26$, $0 \leqslant j \leqslant 4$, $0 \leqslant k \leqslant 2$ *and* $7 \mid |G|$. *Then G has 3-prime factor, 4-prime factor, 5-prime factor or 6-prime factor, and is one of the groups in Table* 1.

G	Order	G	Order
PSL(2,7)	$2^3 \cdot 3 \cdot 7$	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
PSL(2,8)	$2^3 \cdot 3^2 \cdot 7$	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
PSU(3,3)	$2^5 \cdot 3^3 \cdot 7$	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\mathrm{PSL}(2,2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	$\mathrm{PSL}(2,2^9)$	$2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	$\mathrm{PSL}(2,5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$
PSL(2,13)	$2^2 \cdot 3 \cdot 7 \cdot 13$	PSL(4,4)	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$
PSL(2,27)	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	PSL(5,2)	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
PSL(3,4)	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	PSp(4,8)	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$
PSL(3,8)	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$	${}^2D_4(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
PSU(3,5)	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$
PSU(3,8)	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	$G_2(8)$	$2^{18} \cdot 3^5 \cdot 7^2 \cdot 19 \cdot 73$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
$D_4(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	PSL(3,16)	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$
PSp(6,2)	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	$\mathrm{PSL}(2,t)$	$t = \pm 1 \pmod 7$ and $t > 13$
PSp(8,2)	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7$		
PSL(2,49)	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$		

Table 1. Nonabelian simple $\{2,3,5,7,p,q\}$-groups.

P r o o f. Clearly, we have

$$2^{27} \nmid |G|,\ 3^6 \nmid |G|,\ 5^4 \nmid |G|,\ 7^3 \nmid |G|,\ 7 \mid |G|,\ t^2 \nmid |G| \tag{2.1}$$

where $t \in \{q, p\}$ and $t \geqslant 11$.

From [3], pages 12–14, 3-prime factor simple groups can be found. If $7 \mid |G|$, one has $G \cong \mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$ or $\mathrm{PSU}(3,3)$. Specially, if $7^2 \mid |G|$, $3^5 \mid |G|$ or $5^3 \mid |G|$, then $|G|$ has at most five prime divisors. By [31], page 3, each finite nonabelian simple group is isomorphic to A_n with $n \geqslant 5$, one of 26 sporadic simple groups, or a classical group or an exceptional group of Lie type. For the orders of these simple groups, one can see [13], Table 2.4, pages 134–136, and for more details, see [31], Sections 3, 4, 5.

For A_n with $n \geqslant 5$, since $3^6 \nmid |G|$ and $7 \mid |G|$, we have $G \cong A_7, A_8, A_9, A_{10}, A_{11}$ or A_{12}. For the 26 sporadic simple groups, by equation (2.1) we have $G \cong M_{22}$, M_{23}, M_{24}, J_1, J_2, HS.

For the groups of Lie type, since each odd prime divisor of $|G|$ has power at most 5, by [13], Table 2.4, pages 134–136, $G \cong D_4(2), {}^2D_4(2), {}^3D_4(2)$, $\mathrm{PSL}(n,t)$ with $n \geqslant 2$, $\mathrm{PSU}(n,t)$ with $n \geqslant 3$, $\mathrm{PSp}(2n,t)$ with $n \geqslant 2$, or $Sz(2^{2n+1})$ with $n \geqslant 1$, where t is a prime power.

Let $G \cong \mathrm{PSL}(n,t)$. Then $|G| = (n,t-1)^{-1}t^{n(n-1)/2}\prod_{i=2}^{n}(t^i-1)$. First assume $n \geqslant 3$. Then $n(n-1)/2 \geqslant 3$, and by equation (2.1), we have $n = 3$ and $t = 3, 5$ or 2^i with $i < 9$, $n = 4$ and $t = 2^i$ with $i < 5$, or $n = 5$ and $t = 2^i$ with $i < 3$. For each case, by checking orders with equation (2.1) again, we have $G \cong \mathrm{PSL}(3,4)$, $\mathrm{PSL}(3,8)$, $\mathrm{PSL}(3,16)$, $\mathrm{PSL}(4,2)$ ($\cong A_8$), $\mathrm{PSL}(4,4)$ or $\mathrm{PSL}(5,2)$. Now assume $n = 2$. Then $|G| = (2,t-1)^{-1}t(t^2-1)$. If $t = 2^i$ then $i \leqslant 26$ by equation (2.1). Similarly, if $t = 3^i$ then $i \leqslant 5$; if $t = 5^i$ then $i \leqslant 3$; if $t = 7^i$ then $i \leqslant 2$; if $t = s^i$ with $s > 7$ and $s \in \{q,p\}$ then $i = 1$. For each case, checking the orders of $\mathrm{PSL}(2,t)$ again, we have $G \cong \mathrm{PSL}(2,2^6)$, $\mathrm{PSL}(2,2^9)$, $\mathrm{PSL}(2,27)$, $\mathrm{PSL}(2,125)$, $\mathrm{PSL}(2,49)$, or $\mathrm{PSL}(2,t)$ with some prime $t \geqslant 13$ and $t \in \{q,p\}$.

Let $G \cong \mathrm{PSU}(n,t)$ with $n \geqslant 3$. Then

$$|G| = (n,t+1)^{-1}t^{n(n-1)/2}\prod_{i=2}^{n}(t^i-(-1)^i).$$

Since $n(n-1)/2 \geqslant 3$, we have $n = 3$ and $t = 3, 5$ or $t = 2^i$ with $i < 9$, $n = 4$ and $t = 2^i$ with $i < 5$, or $n = 5$ and $t = 2^i$ with $i < 3$ by equation (2.1). Hence, $G \cong \mathrm{PSU}(3,8)$, $\mathrm{PSU}(3,5)$. For the other two infinite families $\mathrm{PSp}(2n,t)$ of order $(n,t-1)^{-1}t^{n^2}\prod_{i=2}^{n}(t^{2i}-1)$ with $n \geqslant 2$ and $Sz(2^{2n+1})$ of order $2^{4n+2}(2^{4n+2}+1) \times (2^{2n+1}-1)$ with $n \geqslant 1$, one can similarly obtain that $G \cong \mathrm{PSp}(4,8), Sz(8)$. □

From [30], page 417, we have the following proposition.

Proposition 2.4. *Let p be a prime, and $q = p^n \geqslant 5$. Then a maximal subgroup of $\mathrm{PSL}(2,q)$ is isomorphic to one of the following groups:*

(1) $D_{2(q-1)/d}$, *where $d = (2,q-1)$ and $q \neq 5,7,9,11$;*
(2) $D_{2(q+1)/d}$, *where $d = (2,q-1)$ and $q \neq 7,9$;*
(3) $\mathbb{Z}_q \rtimes \mathbb{Z}_{(q-1)/d}$;
(4) A_4, *when $q = p = 5$, or $q = p \equiv 3, 13, 27, 37 \pmod{40}$;*
(5) S_4, *when $q = p \equiv \pm 1 \pmod 8$;*
(6) A_5, *when $q = p \equiv \pm 1 \pmod 5$, or $q = p^2 \equiv -1 \pmod 5$ with p an odd prime;*
(7) $\mathrm{PSL}(2,r)$, *when $q = r^m$ with m an odd prime;*
(8) $\mathrm{PGL}(2,r)$, *when $q = r^2$.*

To extract a classification of connected valency seven symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [2], we introduce the graphs $G(2p, r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \ldots, p-1\}$ and $V' = \{0', 1', \ldots, (p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of Z_p^* of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \colon x - y \in H(p, r)\}$.

Proposition 2.5. *Let p be a prime, and let X be a connected valency seven symmetric graph of order $2p$. Then one of the following situations occurs:*

(1) $X \cong K_{7,7}$, *the complete bipartite graph of order* 14, *and* $\mathrm{Aut}(K_{7,7}) = (S_7 \times S_7) \rtimes \mathbb{Z}_2$;

(2) $X \cong G(2p, 7)$ *with* $p \equiv 1 \pmod 7$, *and* $\mathrm{Aut}(G(2p, 7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.

Finally, we introduce the so called Cayley graph. For a finite group G and a subset S of G such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \colon g \in G, \ s \in S\}$. Given $g \in G$, right multiplication $x \mapsto xg$ (for $x \in G$) is a permutation $R(g)$ on G, and the homomorphism from G to $\mathrm{Sym}(G)$ taking each g to $R(g)$ is called the *right regular representation* of G. The image $R(G) = \{R(g) \colon g \in G\}$ of G is a regular permutation group on G, and is isomorphic to G, which can therefore be regarded as a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. In particular, the Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive. Moreover, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \colon S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$, indeed of the stabilizer $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$ of the vertex 1. Also, a Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. By [33], Propositions 1.3 and 1.5, a Cayley graph $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$, or equivalently, if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is isomorphic to the semidirect product $R(G) \rtimes \mathrm{Aut}(G, S)$.

Now we introduce an infinite family of one-regular Cayley graphs on the dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Let m and l be integers such that $l^6 + l^5 + l^4 + l^3 + l^2 + l + 1 \equiv 0 \pmod m$. Define

$$\mathcal{CD}^l_{2m} = \mathrm{Cay}(D_{2m}, S), \tag{2.2}$$

where $S = \{b, ab, a^{l+1}b, a^{l^2+l+1}b, a^{l^3+l^2+l+1}b, a^{l^4+l^3+l^2+l+1}b, a^{l^5+l^4+l^3+l^2+l+1}b\}$.

By [10], we have the following propositions.

Proposition 2.6 ([10], Theorem 3.5)**.** *Let n be a square-free integer and X a connected valency seven one-regular graph of order n. Then $n = 2 \cdot 7^t \cdot p_1 p_2 \ldots p_s$, where $t \leqslant 1$, $s \geqslant 1$, and p_i's are distinct primes such that $7 \mid (p_i - 1)$. Furthermore, X is isomorphic to one of $\mathcal{CD}^l_n$ and there are exactly 6^{s-1} nonisomorphic such graphs of order n.*

Now we introduce the so called coset graph (see [22], [28]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. Denote by H_G the largest normal subgroup of G in H. The coset graph $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\}\colon g \in G,\ d \in D\}$. The action of G on $V(\mathrm{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is faithful if and only if $H_G = 1$. Furthermore, $\mathrm{Aut}(G, H, D) = \{\alpha \in \mathrm{Aut}(G)\colon H^\alpha = H, D^\alpha = D\}$ induces a group of automorphisms, which lies in the stabilizer of H in $\mathrm{Aut}(\mathrm{Cos}(G, H, D))$. Clearly, $\mathrm{Cos}(G, H, D) \cong \mathrm{Cos}(G, H^\alpha, D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$. Note that the concept of a coset graph is equivalent to the concept of an orbital graph (see [29]). Conversely, by [28] we have the following statement.

Proposition 2.7. *Let X be a graph and let G be a vertex-transitive subgroup of $\mathrm{Aut}(X)$. Then X is isomorphic to a coset graph $\mathrm{Cos}(G, H, D)$, where $H = G_u$ is the stabilizer of $u \in V(X)$ in G and D consists of all elements of G which map u to one of its neighbors. Further,*

(1) *X is connected if and only if D generates the group G;*
(2) *X is G-arc-transitive if and only if D is a single double coset. In particular, if $g \in G$ interchanges u and one of its neighbors, then $g^2 \in H$ and $D = HgH$;*
(3) *the valency of X is equal to $|D|/|H| = |H : H \cap H^g|$.*

3. Constructions

In this section, we construct valency seven symmetric graphs of order $2pq$, where p and q are distinct primes.

Example 3.1. Let G be a subgroup of S_{14} such that $G \cong \mathrm{PSL}(2, 13)$, and G contains the following elements:

$$\begin{aligned}
a &= (1, 12)(2, 6)(3, 13)(4, 7)(8, 9)(10, 11),\\
b &= (1, 12, 2, 10, 14, 11, 6)(3, 9, 5, 8, 13, 4, 7),\\
g_2 &= (1, 6)(2, 4)(3, 8)(5, 7)(9, 10)(13, 14),\\
g_2 &= (1, 8)(3, 5)(4, 12)(6, 7)(9, 10)(11, 13).
\end{aligned}$$

By Magma [1], $G = \langle a, b, g_i\rangle$ for each $1 \leqslant i \leqslant 2$ and $H = \langle a, b\rangle$. Define the following coset graphs:

$$\mathcal{C}^i_{78} = \mathrm{Cos}(G, H, Hg_iH), \quad 1 \leqslant i \leqslant 2.$$

Again by Magma [1], the two coset graphs $\mathcal{C}_{78}^i$ $(i = 1, 2)$ are pairwise nonisomorphic connected valency seven 1-transitive graphs of order 78 with $\mathrm{Aut}(\mathcal{C}_{78}^1) = \mathrm{PSL}(2, 13)$ and $\mathrm{Aut}(\mathcal{C}_{78}^2) = \mathrm{PGL}(2, 13)$.

Lemma 3.2. *Each connected valency seven symmetric graph X of order 78 admitting $\mathrm{PSL}(2, 13)$ as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{78}^i$ $(i = 1, 2)$. Furthermore, X is 1-transitive and $\mathrm{Aut}(X) \cong \mathrm{PSL}(2, 13)$ or $\mathrm{PGL}(2, 13)$.*

P r o o f. Let $G = \mathrm{PSL}(2, 13)$. As X is a G-arc-transitive graph of order 78, one has $|G_v| = 14$ for any vertex $v \in V(X)$, and by Proposition 2.4, we have $H = G_v \cong D_{14}$. The simplicity of G and the maximality of H imply that $H = N_G(H)$. Take an involution x in H, and set $\langle x\rangle = L$. Since G has one conjugacy class of involutions, by Proposition 2.4, $N_G(L) = D_{12}$. Clearly, $H \cap H^g = L$ and $N_H(L) \cong L$. Thus, there exists an involution g such that $g \in N_G(L)$ and $g \notin N_H(L)$. Furthermore, $g \notin H$, $|HgH|/|H| = 7$ and $\langle H, g\rangle = G$. This implies that $\mathrm{Cos}(G, H, HgH)$ is a connected valency seven symmetric graph of order 78.

Let X be a connected valency seven symmetric graph of order 78 admitting $G = \mathrm{PSL}(2, 13)$ as an arc-transitive automorphism group. Note that $G_v \cong D_{14}$ for any $v \in V(X)$. To complete the proof, it suffices to show that $X \cong \mathrm{Cos}(G, H, HgH)$. By Magma [1], G has one conjugacy class of D_{14} and since G_v has seven subgroups isomorphic to $\mathbb{Z}_2$, each of the subgroups fixes a vertex adjacent to v. By Proposition 2.7, one may assume that $X = \mathrm{Cos}(G, H, HfH)$ such that $H \cap H^f = L$ and $f \in N_G(L)$. By [5], Theorem 2.1, f can be chosen to be a 2-element, and hence f is an involution in $N_G(L) \cong D_{12}$. By the connectivity of X, $f \notin N_H(L) \cong \mathbb{Z}_2$. Thus, f has six choices and by Magma [1], the six coset graphs $\mathrm{Cos}(G, H, HfH)$ corresponding to the six involutions have two nonisomorphism classes. It follows that $X = \mathrm{Cos}(G, H, HfH) \cong \mathcal{C}_{78}^1$ or $\mathcal{C}_{78}^2$, as required. □

Example 3.3. Let $G = S_8$. Then G has a subgroup $H \cong \mathbb{Z}_2^3 \rtimes \mathrm{SL}(3, 2)$ and an involution g such that $|HgH|/|H| = 7$ and $\langle H, g\rangle = G$. The coset graph $\mathrm{Cos}(G, H, HgH)$ is denoted by $\mathcal{C}_{30}$.

Lemma 3.4. *Each connected valency seven symmetric graph X of order 30 admitting S_8 as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{30}$. Furthermore, X is 2-transitive and $\mathrm{Aut}(X) \cong S_8$.*

P r o o f. Let $G = S_8$. Clearly, G has a maximal subgroup $T \cong A_8$ containing a maximal subgroup H such that $H \cong \mathbb{Z}_2^3 \rtimes \mathrm{SL}(3, 2)$. Let $L = \mathbb{Z}_2^3 \rtimes S_4$ be a subgroup of H. By Magma [1], $N_G(L) = L \cdot \mathbb{Z}_2$ and $N_T(L) = L$, and by [19], one has $N_G(L) = S_2 \wr S_4$. Let $g \in N_G(L) \setminus L$ be an involution. Then $N_G(L) = L \cup Lg$, $L = H \cap H^g$, $|HgH|/|H| = 7$ and $\langle H, g\rangle = G$. It follows that the coset graph

$\mathrm{Cos}(G, H, HgH)$ is a connected valency seven symmetric graph of order 30. (Note that H has yet another conjugacy class of order $|\mathbb{Z}_2^3 \rtimes S_4|$, which is not isomorphic to $\mathbb{Z}_2^3 \rtimes S_4$. By Magma [1], $N_G(L) = L$, no graph arises.)

Let X be a connected valency seven symmetric graph of order 30 admitting $G = S_8$ as an arc-transitive automorphism group. Then $G_v \cong \mathbb{Z}_2^3 \rtimes \mathrm{SL}(3, 2)$ for any $v \in V(X)$. To complete the proof, it suffices to show that $X \cong \mathrm{Cos}(G, H, HgH)$. Since G has one conjugacy class of $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3, 2)$ and $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3, 2)$ has seven subgroups isomorphic to $\mathbb{Z}_2^3 \rtimes S_4$, by Proposition 2.7, one may assume that $X = \mathrm{Cos}(G, H, HfH)$ such that $H \cap H^f = L$ and $f \in N_G(L)$. Since $N_G(L) = L \cup Lg$, one has $f = lg$ for some $l \in L$. It follows that $HfH = HgH$, that is, $X = \mathrm{Cos}(G, H, HfH) \cong \mathrm{Cos}(G, H, HgH)$. By Magma [1], $\mathrm{Aut}(X) = G$. □

Example 3.5. Let $G = \mathrm{Aut}(\mathrm{PSL}(5, 2)) = \mathrm{PSL}(5, 2).\mathbb{Z}_2$. Then G has a subgroup $H \cong \mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times \mathrm{SL}(3, 2))$ and an involution g such that $|HgH|/|H| = 7$ and $\langle H, g\rangle = G$. The coset graph $\mathrm{Cos}(G, H, HgH)$ is denoted by $\mathcal{C}_{310}$.

Lemma 3.6. *Each connected valency seven symmetric graph X of order 310 admitting $\mathrm{Aut}(\mathrm{PSL}(5, 2))$ as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{310}$. Furthermore, X is 3-transitive and $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{PSL}(5, 2))$.*

P r o o f. By Atlas [3], $\mathrm{Aut}(\mathrm{PSL}(5, 2)) = \mathrm{PSL}(5, 2).\mathbb{Z}_2$. Let $G = \mathrm{Aut}(\mathrm{PSL}(5, 2))$. Clearly, G has an index two maximal subgroup $T \cong \mathrm{PSL}(5, 2)$ containing a maximal subgroup H such that $H \cong \mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times \mathrm{SL}(3, 2))$. Let $L = \mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times S_4)$ be a subgroup of H. By Magma [1], $N_G(L) = L.\mathbb{Z}_2$ and $N_T(L) = L$. Let $g \in N_G(L) \setminus L$ be an involution. Then $N_G(L) = L \cup Lg$, $L = H \cap H^g$, $|HgH|/|H| = 7$ and $\langle H, g\rangle = G$. It follows that the coset graph $\mathrm{Cos}(G, H, HgH)$ is a connected valency seven symmetric graph of order 310. (Note that $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times \mathrm{SL}(3, 2))$ has yet another conjugacy class of order $|\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times S_4)|$, which is not isomorphic to $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times S_4)$. By Magma [1], $N_G(L) = L$, no graph arises.)

Let X be a connected valency seven symmetric graph of order 310 admitting G as an arc-transitive automorphism group. Then $G_v \cong \mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times \mathrm{SL}(3, 2))$ for any $v \in V(X)$. To complete the proof, it suffices to show that $X \cong \mathrm{Cos}(G, H, HgH)$. Since T has two conjugacy classes of maximal parabolic subgroups $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times \mathrm{SL}(3, 2))$, and has a graph automorphism g, which is of order 2, g fuses the two conjugacy classes of maximal parabolic subgroups. By Magma [1], $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times \mathrm{SL}(3, 2))$ has a conjugacy class of $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2, 2) \times S_4)$. By Proposition 2.7, one may assume that $X = \mathrm{Cos}(G, H, HfH)$ so that $H \cap H^f = L$ and $f \in N_G(L)$. Since $N_G(L) = L \cup Lg$, one has $f = lg$ for some $l \in L$. It follows that $HfH = HgH$, that is, $X = \mathrm{Cos}(G, H, HfH) \cong \mathrm{Cos}(G, H, HgH)$. By Magma [1], $\mathrm{Aut}(X) = G$. □

4. Main results

In this section, we classify valency seven symmetric graphs of order $2pq$ for p and q primes. First, we consider valency seven symmetric graphs of order $4p$, where p is a prime.

Theorem 4.1. *Let p be a prime. Then X is a connected valency seven symmetric graph of order $4p$ if and only if $X \cong K_8$, a complete graph of order 8.*

P r o o f. For $p = 2$, K_8 is a unique symmetric graph of valency seven. For $p = 3$, by [21], there is no symmetric graph of valency seven. Thus, in what follows, we assume that $p \geqslant 5$. Let $A = \mathrm{Aut}(X)$ and $v \in V(X)$. By Guo [15], $|A_v| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$, and hence $|A| \mid 2^s \cdot 3^t \cdot 5^r \cdot 7 \cdot p$ with $2 \leqslant s \leqslant 26$, $0 \leqslant t \leqslant 4$ and $0 \leqslant r \leqslant 2$. We divide our discussion into the following two cases. Let N be a minimal normal subgroup of A.

Assume that N is solvable. Then N is elementary abelian. By Proposition 2.1, N is semiregular on $V(X)$, and the quotient graph X_N of X relative to the orbits of N has valency seven. Since $|V(X)| = 4p$, A has no normal subgroup of order 4 or p.

It follows that $N \cong \mathbb{Z}_2$, forcing that $N \leqslant Z(A)$, the center of A. By Proposition 2.1, X_N is a connected valency seven symmetric graph of order $2p$ with A/N as an arc-transitive subgroup of $\mathrm{Aut}(X_N)$. By Proposition 2.5, either $X_N \cong K_{7,7}$ or $X_N \cong G(2p, 7)$ with $7 \mid p-1$. Take a minimal normal subgroup of A/N, say M/N. Let $X_N \cong K_{7,7}$. Clearly, $p = 7$. Suppose that M/N is solvable. Then $M/N \cong \mathbb{Z}_2, \mathbb{Z}_7$ or $\mathbb{Z}_7^2$. If $M/N \cong \mathbb{Z}_2$ or $\mathbb{Z}_7$, then A has a normal subgroup of order 4 or 7 because $N \cong \mathbb{Z}_2$, a contradiction. If $M/N \cong \mathbb{Z}_7^2$ then $M \cong \mathbb{Z}_2 \times \mathbb{Z}_7^2$. It is easy to see that M has two orbits on $V(X)$, and since M is abelian and $M_v \cong \mathbb{Z}_7$, one has $X \cong 2K_{7,7}$, a union of two copies of $K_{7,7}$, which contradicts the connectivity of X. Suppose that M/N is nonsolvable. Then $M/N \cong A_7$ or $A_7 \times A_7$. Obviously, M/N has two orbits on $V(X_N)$. Since $(M/N)_u \trianglelefteq (A/N)_u$ for any $u \in X_N$, by the primitivity of $(A/N)_u$ on the neighborhood of u one has $7 \mid |(M/N)_u|$, implying that $49 \mid |M/N|$. Thus, $M/N \cong A_7 \times A_7$. Let $B/N \cong A_7$ and $B/N \trianglelefteq M/N$. Similarly, B/N has two orbits on $V(X_N)$ and $7 \mid |(B/N)_w|$. Thus, $49 \mid |B/N|$, a contradiction. Let $X_N \cong G(2p, 7)$ with $7 \mid p-1$. Then a normal Sylow p-subgroup of $\mathrm{Aut}(X_N)$ must be PN/N because each Sylow p-subgroup of A/N is a Sylow p-subgroup of $\mathrm{Aut}(X_N)$. It follows that $P \trianglelefteq A$ because P is characteristic in PN, which is impossible because A has no normal subgroup of order p.

If A has a solvable nontrivial normal subgroup, then A has a solvable minimal normal subgroup isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_p$, which is impossible by the above argument. Thus, in what follows we assume that A has no solvable nontrivial normal subgroups.

Now assume that N is nonsolvable. Then $N \cong T^m$, where T is a nonabelian simple group. By Proposition 2.1, N has at most two orbits on $V(X)$. Then $|N|$ is divisible by $2p \cdot 7$, and since $|N| \mid 2^s \cdot 3^t \cdot 5^r \cdot 7 \cdot p$ with $1 \leqslant s \leqslant 26$, $0 \leqslant t \leqslant 4$ and $0 \leqslant r \leqslant 2$. One has $N = T$ except $p = 7$.

If $p = 5$, then $|N|$ is a factor of $2^{26} \cdot 3^4 \cdot 5^3 \cdot 7$ and $|N|$ is divisible by $2 \cdot 5 \cdot 7$. By Table 1,

$$N \cong A_7,\ A_8,\ A_9,\ A_{10},\ \mathrm{PSL}(3,4),\ \mathrm{PSU}(3,5),\ J_2,\ \mathrm{PSp}(6,2). \tag{4.1}$$

For $p = 7$, one has $7^2 \mid |N|$ and $|N| \mid 2^{26} \cdot 3^4 \cdot 5^2 \cdot 7^2$. If $N \cong T^2$, then $T \cong \mathrm{PSL}(3,4), \mathrm{PSL}(2,7), A_7, A_8, \mathrm{PSL}(2,8)$ by Table 1. Clearly, N has a normal subgroup isomorphic to T, say S. Since $S \trianglelefteq N$, one has $7 \mid |S_v|$ and S has an orbit of length $7, 14$ or 28, implying that $49 \mid |S|$, a contradiction. Thus, $N = T$. In this case, $|N|$ has at most four primes, and $7^2 \mid |N|$. Again by Table 1, one has

$$N \cong \mathrm{PSL}(2,49). \tag{4.2}$$

Let $p > 7$. We first consider $N \cong \mathrm{PSL}(2,p)$, the infinite family listed in Table 1. By the subgroup structure of $\mathrm{PSL}(2,p)$, one has N_v is solvable and $|N_v| \mid 2^2 \cdot 3^2 \cdot 7$, and $5 \nmid |N_v|$. Then $|N|$ is a factor of $2^4 \cdot 3^2 \cdot 7 \cdot p$ and $|N|$ is divisible by $2 \cdot p \cdot 7$. Hence $|N| = |\mathrm{PSL}(2,p)| = \frac{1}{2}p(p-1)(p+1)$ and $(\frac{1}{2}p(p+1), \frac{1}{2}(p-1)) = 1$. If $7 \mid p-1$, then $p+1 = 2^i \cdot 3^j$, where $1 \leqslant i \leqslant 4$, $0 \leqslant j \leqslant 2$. It follows that $p = 71$. Similarly, if $7 \mid p+1$, then $p = 13$. Combining with Table 1, N is one of the following:

$$\mathrm{PSL}(2,13),\ \mathrm{PSL}(2,71),\ \mathrm{PSL}(2,27),\ \mathrm{PSU}(3,8),\ Sz(8),\ A_{11},\ \mathrm{M}_{22},\ \mathrm{PSL}(2,2^6), \tag{4.3}$$

$$\mathrm{PSL}(4,4),\ \mathrm{PSL}(5,2),\ {}^2D_4(2),\ G_2(4). \tag{4.4}$$

Since N is nonsolvable, N has at most two orbits. We may assume that N is a group listed in (4.1)–(4.4). Let N be transitive on $V(X)$. By Proposition 2.7, $X \cong \mathrm{Cos}(N, H, HaH)$, where $H = N_v$, $a \in N \setminus H$ and $a^2 \in H$. By the Atlas [3], $N = A_7$ ($p = 5$) has no subgroup of order $|H| = |N|/|V(X)|$. Thus, $N \neq A_7$. Similarly, $N \neq \mathrm{PSL}(2,2^6)$ ($p = 13$). For $N = A_8$ ($p = 5$), $|N|/|V(X)|$ is not the order of the vertex stabilizer by Proposition 2.2, a contradiction. It follows that $N \neq A_8$. Similarly, $N \neq A_9$ ($p = 5$), A_{10} ($p = 5$), $\mathrm{PSL}(3,4)$ ($p = 5$), $\mathrm{PSU}(3,5)$ ($p = 5$), J_2 ($p = 5$), $\mathrm{PSp}(6,2)$ ($p = 5$), $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(2,13)$ ($p = 13$), $\mathrm{PSL}(2,71)$ ($p = 71$), $\mathrm{PSL}(2,27)$ ($p = 13$), $\mathrm{PSU}(3,8)$ ($p = 19$), $Sz(8)$ ($p = 13$), A_{11} ($p = 11$), M_{22} ($p = 11$), $\mathrm{PSL}(4,4)$ ($p = 17$), $\mathrm{PSL}(5,2)$ ($p = 31$), ${}^2D_4(2)$ ($p = 17$), $G_2(4)$ ($p = 13$).

Let N have two orbits on $V(X)$. Then $|H| = |N|/\frac{1}{2}|V(X)|$. For $N = A_7$ ($p = 5$), by Proposition 2.2, $|N|/\frac{1}{2}|V(X)|$ is not the order of the vertex stabilizer, a contradiction. It follows that $N \neq \mathrm{A}_7$. Similarly, $N \neq A_9$ ($p = 5$), A_{10} ($p = 5$), $\mathrm{PSL}(3,4)$ ($p = 5$), $\mathrm{PSU}(3,5)$ ($p = 5$), J_2 ($p = 5$), $\mathrm{PSp}(6,2)$ ($p = 5$), $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(2,13)$ ($p = 13$), $\mathrm{PSL}(2,71)$ ($p = 71$), $\mathrm{PSL}(2,27)$ ($p = 13$), $\mathrm{PSU}(3,8)$ ($p = 19$), $Sz(8)$ ($p = 13$), M_{22} ($p = 11$), $\mathrm{PSL}(4,4)$ ($p = 17$), $\mathrm{PSL}(5,2)$ ($p = 31$), ${}^2D_4(2)$ ($p = 17$), $G_2(4)$ ($p = 13$). By the Atlas [3], $N = A_8$ has no subgroup of order $|H| = |N|/\frac{1}{2}|V(X)|$. Thus, $N \neq \mathrm{A}_8$ ($p = 5$). Similarly, $N \neq \mathrm{PSL}(2,2^6)$ ($p = 13$). For $N = A_{11}$, one has $|H| = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7$. By Proposition 2.2, $H \cong A_7 \times A_6$, and by [19], A_{11} has no subgroup which is isomorphic to $\mathrm{A}_7 \times \mathrm{A}_6$, a contradiction. This completes the proof. □

Theorem 4.2. *Let X be a connected valency seven symmetric graph of order $2pq$, where $p > q$ are odd primes. Then X is* 1-, 2- *or* 3-*transitive. Furthermore, one of the following situations occurs:*

(1) *X is* 1-*transitive, and $X \cong \mathcal{C}^i_{78}$ ($i = 1,2$) with $\mathrm{Aut}(\mathcal{C}^1_{78}) \cong \mathrm{PSL}(2,13)$ and $\mathrm{Aut}(\mathcal{C}^2_{78}) \cong \mathrm{PGL}(2,13)$, or $X \cong \mathcal{CD}^l_{2pq}$ (defined in equation (2.2)) with $\mathrm{Aut}(X) \cong D_{2pq} \rtimes \mathbb{Z}_7$ for some l satisfying $l^6 + l^5 + l^4 + l^3 + l^2 + l \equiv 0 \pmod{pq}$—the number of pairwise nonisomorphic such graphs of order $2pq$ is*

$$f(p,q) = \begin{cases} 1, & q = 7 \text{ and } 7 \mid p-1; \\ 6, & 7 \mid q-1 \text{ and } 7 \mid p-1; \\ 0, & \textit{otherwise.} \end{cases}$$

(2) *X is* 2-*transitive, and $X \cong \mathcal{C}_{30}$ is a vertex bi-primitive graph with $\mathrm{Aut}(X) \cong S_8$.*

(3) *X is* 3-*transitive, and $X \cong \mathcal{C}_{310}$ is a vertex bi-primitive graph with $\mathrm{Aut}(X) \cong \mathrm{PSL}(5,2) \cdot \mathbb{Z}_2$.*

P r o o f. Let $A = \mathrm{Aut}(X)$ and $v \in V(X)$. By Guo [15], $|A_v| \mid 2^{24} \cdot 3^2 \cdot 5^2 \cdot 7$, and hence $|A| = 2^s \cdot 3^t \cdot 5^r \cdot 7 \cdot q \cdot p$ with $1 \leqslant s \leqslant 25, 0 \leqslant s \leqslant 4$ and $0 \leqslant r \leqslant 2$. We first prove a claim.

Claim: If A has a normal subgroup of order q then $X \cong \mathcal{CD}^l_{2pq}$.

Let Q be a normal subgroup of A of order q. By Proposition 2.1, Q is semiregular on $V(X)$ and the quotient graph X_Q of X relative to Q is a symmetric graph of order $2p$ and valency seven with A/Q as an arc-transitive subgroup of $\mathrm{Aut}(X_Q)$. By Proposition 2.5, one has $X_Q \cong K_{7,7}$ or $X_Q \cong G(2p,7)$ with $7 \mid p-1$.

Suppose that $X_Q \cong K_{7,7}$. Then $p = 7$ and $q = 3$ or 5. Take a minimal normal subgroup of A/Q, say M/Q. Assume that M/Q is nonsolvable. Then $M/Q \cong A_7$ or $A_7 \times A_7$ because $A/Q \leqslant \mathrm{Aut}(K_{7,7}) \cong (S_7 \times S_7) \rtimes \mathbb{Z}_2$. Obviously, M/Q has two orbits on $V(X_Q)$ and $7 \mid |(M/Q)_w|$ for any $w \in V(X_Q)$, implying that $49 \mid |M/Q|$. Thus, $M/Q \cong \mathrm{A}_7 \times \mathrm{A}_7$. Let $B/Q \cong \mathrm{A}_7$ and $B/Q \trianglelefteq M/Q$. Similarly, B/Q has two orbits on $V(X_Q)$ and $7 \mid |(B/Q)_w|$. Thus, $49 \mid |B/Q|$, a contradiction. Now assume that M/Q is solvable. Then $M/Q \cong \mathbb{Z}_2, \mathbb{Z}_7$ or $\mathbb{Z}_7^2$. If $M/Q \cong \mathbb{Z}_2$ then X_M is a symmetric graph of order p and valency seven, a contradiction. If $M/Q \cong \mathbb{Z}_7$ then $M \cong \mathbb{Z}_{21}$ or $\mathbb{Z}_{35}$ and M has two orbits on $V(X)$, implying that X is a bipartite graph. Let $R \leqslant M$ and $R \cong \mathbb{Z}_7$. Then $R \lhd A$, and since $R \leqslant M$, the quotient graph X_R is bipartite and of valency seven. However, $|X_R| = 6$ or 10, a contradiction. If $M/Q \cong \mathbb{Z}_7^2$ then $M \cong Q \times \mathbb{Z}_7^2$ because $Q \cong \mathbb{Z}_3$ or $\mathbb{Z}_5$. Since M is abelian and $M_v \cong \mathbb{Z}_7$, one has $X \cong 3K_{7,7}$ or $5K_{7,7}$, which contradicts the connectivity of X.

Thus, $X_Q \cong G(2p, 7)$ with $7 \mid p-1$. By Proposition 2.5, X_Q is valency seven and 1-regular graph of order $2p$. Since A/Q is arc-transitive on X_Q, one has $A/Q = \mathrm{Aut}(X_Q)$ and X is a valency seven and 1-regular graph of order $2pq$. By Proposition 2.6, $X \cong \mathcal{CD}_{2pq}^l$. This completes the proof of Claim.

If A has a normal subgroup of order 2, then the quotient graph has valency seven and odd order pq, a contradiction.

Let A have a normal subgroup P of order p. By Proposition 2.5, $X_P \cong K_{7,7}$ or $G(2q, 7)$ with $7 \mid q-1$. Let $C := C_A(P)$. Clearly, $P \leqslant C$. If $P = C$ then $A/P \leqslant \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$, implying that A is abelian. It follows that A is regular on $V(X)$, which contradicts the fact that X is symmetric. Hence, $P < C$. Take a minimal normal subgroup of A/P, say M/P, in C/P. Suppose that M/P is solvable. By Proposition 2.1, M/P is semiregular on $V(X_P)$. Then $M/P \cong \mathbb{Z}_2$ or $\mathbb{Z}_q$, which implies that A has a normal subgroup of order 2 or q respectively; we have done two cases. Thus, M/P is nonsolvable, and hence $X_P \cong K_{7,7}$. Then $M/P \cong A_7$ or $A_7 \times A_7$. Obviously, M/P has two orbits on $V(X_P)$, and $7 \mid |(M/P)_u|$ for any $u \in V(X_P)$, implying that $49 \mid |M/P|$. Thus, $M/P \cong A_7 \times A_7$. Let $B/P \cong A_7$ and $B/P \trianglelefteq M/P$. Similarly, B/P has two orbits on $V(X_P)$ and $7 \mid |(B/P)_u|$. Thus, $49 \mid |B/P|$, a contradiction.

If A has a solvable nontrivial normal subgroup, then A has a solvable minimal normal subgroup isomorphic to $\mathbb{Z}_2, \mathbb{Z}_p$ or $\mathbb{Z}_q$, which was done by the above argument. Thus, in what follows we assume that A has no solvable nontrivial normal subgroups.

Let N be a minimal normal subgroup of A. Then $N \cong T^m$, where T is a nonabelian simple group. By Proposition 2.1, N has at most two orbits on $V(X)$. Since $pq{\cdot}7 \mid |N|$ and $|N| \mid |A| = 2^s \cdot 3^t \cdot 5^r \cdot 7 \cdot q \cdot p$ with $1 \leqslant s \leqslant 25$, $0 \leqslant t \leqslant 4$ and $0 \leqslant r \leqslant 2$, one has $N = T$ except for $p = 7$. Assuming that $p = 7$, one has $q = 3$ or 5. Hence $7^2 \mid |N|$ and $|N| \mid 2^{25} \cdot 3^5 \cdot 5^3 \cdot 7^2$. If $N \cong T^2$, then $T \cong \mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(3,4)$, A_7, A_8 by Table 1. Clearly, N has a normal subgroup isomorphic to T, say S. Since $S \trianglelefteq N$, one has $7 \mid |S_v|$ and S has an orbit of length 7, $7q$ or $14q$, implying that $49 \mid |S|$, a contradiction. Thus, $N = T$. In this case, $|N|$ has at most four primes

$\{2,3,5,7\}$, and $7^2 \mid |N|$. Again by Table 1, one has

$$N \cong \mathrm{PSL}(2,49). \tag{4.5}$$

Next, we assume that $p \neq 7$ and $N = T$. We first consider $N \cong \mathrm{PSL}(2,p)$ $(p > 7)$, the infinite family listed in Table 1. By the subgroup structure of $\mathrm{PSL}(2,p)$, one has N_v is solvable, and by Proposition 2.2, $|N_v| \mid 2^2 \cdot 3^2 \cdot 7$ and $5 \nmid |N_v|$. Thus $|N| \mid 2^3 \cdot 3^2 \cdot 7 \cdot q \cdot p$, implying that N is at most five-prime factor simple group. Hence $|N| = |\mathrm{PSL}(2,p)| = \frac{1}{2}p(p-1)(p+1)$ and $(\frac{1}{2}(p+1), \frac{1}{2}(p-1)) = 1$. For $q \leqslant 7$, if $7 \mid \frac{1}{2}(p+1)$, then $p - 1 = 2^i \cdot 3^j \cdot q$, where $1 \leqslant i \leqslant 3, 0 \leqslant j \leqslant 2$. It follows that $p = 13, 41, 181$. Since $\mathrm{PSL}(2,181)$ is a six-prime factor simple group, one has $p = 13, 41$. Similarly, if $7 \mid \frac{1}{2}(p-1)$, then $p = 29, 71$. For $p > q > 7$, if $q \mid \frac{1}{2}(p+1)$, then $p - 1 = 2^i \cdot 3^j \cdot 7$, where $1 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant 2$. It follows that $p = 43, 127$. Since $2^7 \mid |\mathrm{PSL}(2,127)|$, a contradiction. Thus $p = 43$. Similarly, if $q \mid \frac{1}{2}(p-1)$, then $p = 83, 97, 251, 503$. Hence $2^5 \mid |\mathrm{PSL}(2,97)|$ and $5^3 \mid |\mathrm{PSL}(2,251)|$, a contradiction. Thus $p = 83, 503$.

For $q = 3$, one has $3 \cdot 7 \cdot p \mid |N|$ and $|N| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$. By Table 1, N is one of the following groups:

$$A_7,\ A_8,\ A_9,\ A_{10},\ \mathrm{PSL}(2,27),\ \mathrm{PSL}(3,4),\ \mathrm{PSU}(3,5),\ \mathrm{PSU}(3,8),\ J_2, \tag{4.6}$$
$$D_4(2),\ \mathrm{PSp}(6,2),\ \mathrm{PSp}(8,2),\ A_{11},\ A_{12},\ M_{22}, \mathrm{PSL}(2,2^6),\ \mathrm{PSL}(4,4), \tag{4.7}$$
$$\mathrm{PSL}(4,4),\ \mathrm{PSL}(5,2),\ {}^2D_4(2),\ G_2(4),\ \mathrm{PSL}(2,p)\ (p = 13, 127). \tag{4.8}$$

For $q = 5$, one has $5 \cdot 7 \cdot p \mid |N|$ and $|N| \mid 2^{25} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot p$. By Table 1, N is one of the following groups:

$$Sz(8), A_{11},\ M_{22},\ HS,\ \mathrm{PSL}(2,2^6),\ \mathrm{PSL}(2,5^3),\ \mathrm{PSL}(5,2),\ \mathrm{PSL}(4,4), \tag{4.9}$$
$${}^2D_4(2),\ G_2(4),\ \mathrm{PSL}(2,p)\ (p = 29, 41, 71). \tag{4.10}$$

For $q \geqslant 7$, one has $7 \cdot q \cdot p \mid |N|$ and $|N| \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot q \cdot p$. By Table 1, N is one of the following groups:

$$\mathrm{PSL}(3,8),\ {}^3D_4(2),\ \mathrm{PSL}(2,2^9),\ \mathrm{PSp}(4,8),\ M_{23},\ M_{24},\ J_1,\ \mathrm{PSL}(3,16), \tag{4.11}$$
$$\mathrm{PSL}(2,p)\ (p = 43, 83, 503). \tag{4.12}$$

We may assume that N is a group listed in (4.5)–(4.12). Let $G \leqslant A$ be a transitive subgroup of X. By Proposition 2.7, $X \cong \mathrm{Cos}(G, H, HgH)$, where $H = G_v$, $g \in G \setminus H$, and $g^2 \in H$, implying that a normalizes $R = H \cap H^g$, that is, $g \in N_G(R) \setminus H$. Recall that N has at most two orbits on $V(X)$. First let N be transitive on $V(X)$. Take $G = N$.

If $N = A_7$ ($q = 3, p = 5$), then N_v has order $|N|/|V(X)| = 2^2{\cdot}3{\cdot}7$. However, A_7 has no subgroups of order $2^2{\cdot}3{\cdot}7$ by Atlas [3]. Thus, $N \neq A_7$. Similarly, $N \neq \mathrm{PSL}(2,127)$ ($q = 3$, $p = 127$), $\mathrm{PSL}(2,29)$ ($q = 5$, $p = 29$), $\mathrm{PSL}(2,41)$ ($q = 5$, $p = 41$), $\mathrm{PSL}(2,71)$ ($q = 5$, $p = 71$) and $\mathrm{PSL}(2,503)$ ($q = 251$, $p = 503$) by Proposition 2.4. If $N = A_9$ ($q = 3$, $p = 5$), A_{10} ($q = 3$, $p = 5$), $\mathrm{PSU}(3,8)$ ($q = 3$, $p = 19$), A_{11} ($q = 3$, $p = 11$) or ${}^2D_4(2)$ ($q = 3$, $p = 17$), then $3^3 \parallel |N_v|$ and $3^4 \nmid |N_v|$. By Proposition 2.2, it is not possible. If $N = A_8$ ($q = 3$, $p = 5$), then $|N_v| = 2^5 \cdot 3 \cdot 7$. By Proposition 2.2, there exists a vertex stabilizer whose order is $2^5 \cdot 3 \cdot 7$, a contradiction. Similarly, $N \neq \mathrm{PSL}(2,49)$ ($q = 3, p = 7$), $\mathrm{PSL}(2,49)$ ($q = 5, p = 7$), $\mathrm{PSL}(2,27)$ ($q = 3, p = 13$), $\mathrm{PSL}(3,4)$ ($q = 3$, $p = 5$), $\mathrm{PSU}(3,5)$ ($q = 3$, $p = 5$), J_2 ($q = 3$, $p = 5$), $\mathrm{PSp}(6,2)$ ($q = 3$, $p = 5$), $\mathrm{PSp}(8,2)$ ($q = 3$, $p = 5$), A_{11} ($p = 5, q = 11$), M_{22} ($q = 3$, $p = 11$), $\mathrm{PSL}(2,2^6)$ ($q = 3$, $p = 13$), $\mathrm{PSL}(2,2^6)$ ($q = 5$, $p = 13$), $\mathrm{PSL}(4,4)$ ($q = 3$, $p = 17$), $\mathrm{PSL}(4,4)$ ($q = 5$, $p = 17$), $\mathrm{PSL}(5,2)$ ($q = 3$, $p = 31$), $\mathrm{PSL}(5,2)$ ($q = 5$, $p = 31$), $D_4(2)$ ($q = 3, p = 5$), HS ($q = 5, p = 11$), $\mathrm{PSL}(2,5^3)$ ($q = 5, p = 31$), ${}^2D_4(2)$ ($q = 5$, $p = 17$), $G_2(4)$ ($q = 3$, $p = 13$), $G_2(4)$ ($q = 5$, $p = 13$), $Sz(8)$ ($q = 5$, $p = 13$), $\mathrm{PSL}(2,71)$ ($q = 3$, $p = 71$), $\mathrm{PSL}(3,8)$ ($q = 7$, $p = 73$), ${}^3D_4(2)$ ($q = 7$, $p = 17$), $\mathrm{PSL}(2,2^9)$ ($q = 19$, $p = 73$), $\mathrm{PSp}(4,8)$ ($q = 7$, $p = 13$), M_{23} ($q = 11$, $p = 23$), M_{24} ($q = 11, p = 23$), J_1 ($q = 11$, $p = 19$) and $N \neq \mathrm{PSL}(3,16)$ ($q = 13$, $p = 17$).

Suppose that $N = A_{12}$ ($q = 3$, $p = 11$). Then $|N_v| = |N|/|V(X)| = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. By Proposition 2.2, $N_v \cong S_7 \times S_6$. By [19], one concludes that N has no subgroup which is isomorphic to $S_7 \times S_6$, a contradiction.

Suppose that $N = \mathrm{PSL}(2,13)$ ($q = 3$, $p = 13$). Then $|N_v| = 2 \cdot 7$. By Proposition 2.2, $N_v \cong D_{14}$, and by Proposition 2.4, N_v is a maximal subgroup. By Example 3.1, $X \cong \mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$.

Suppose that $N = \mathrm{PSL}(2,43)$ ($q = 11$, $p = 43$). Then $|N_v| = 2 \cdot 3 \cdot 7$. By Proposition 2.2, $N_v \cong F_{42}$, and by Proposition 2.4, one concludes that N has a unique conjugacy class D_{42} which has order 42. Clearly, it is isomorphic to F_{42}, a contradiction.

Suppose that $N = \mathrm{PSL}(2,83)$ ($q = 41$, $p = 83$). Then $|N_v| = 2 \cdot 3 \cdot 7$. By Proposition 2.2, $N_v \cong F_{42}$. By Proposition 2.4, one concludes that N has a unique maximal subgroup conjugacy class D_{84} which contains subgroups of order 42. Clearly, the subgroups of order 42 of D_{84} are isomorphic to D_{42} or $\mathbb{Z}_{42}$. They are not isomorphic to F_{42}, a contradiction.

Suppose that $N = M_{22}$ ($q = 5$, $p = 11$). Then $|N_v| = 2^6 \cdot 3^2 \cdot 7$. By Atlas [3], the unique maximal subgroup class of M_{22} which has order divided by $2^6 \cdot 3^2 \cdot 7$ is $L_3(4)$; again by Atlas [3], $L_3(4)$ has no subgroup of order $2^6 \cdot 3^2 \cdot 7$, a contradiction.

Now let N have two orbits on $V(X)$. If $N \neq A_7$, then N_v has order $|N|/\frac{1}{2}|V(X)| = 2^3 \cdot 3 \cdot 7$. However, A_7 has no subgroups of order $2^3 \cdot 3 \cdot 7$ by Atlas [3]. Thus, $N \neq A_7$. Similarly, $N \neq \mathrm{PSL}(2,13)$ ($q = 3$, $p = 13$), $\mathrm{PSL}(2,27)$ ($q = 3$, $p = 13$), $\mathrm{PSL}(2,127)$

($q=3$, $p=127$), $\mathrm{PSL}(2,29)$ ($q=5$, $p=29$), $\mathrm{PSL}(2,41)$ ($q=5$, $p=41$), $\mathrm{PSL}(2,71)$ ($q=5$, $p=71$) and $\mathrm{PSL}(2,43)$ ($q=11$, $p=43$) by Proposition 2.4. If $N=A_9$ ($q=3$, $p=5$), A_{10} ($q=3$, $p=5$), A_{11} ($q=3$, $p=11$), $\mathrm{PSU}(3,8)$ ($q=3$, $p=19$) or ${}^2D_4(2)$ ($q=3$, $p=17$), then $3^3 \parallel |N_v|$. By Proposition 2.2, this is not possible. If $N=A_{12}$ ($q=3$, $p=11$), then $|N_v|=2^9\cdot 3^4\cdot 5^2\cdot 7$. By Proposition 2.2, there exists no vertex stabilizer whose order is $2^9\cdot 3^4\cdot 5^2\cdot 7$, a contradiction. Similarly, $N\neq \mathrm{PSL}(2,49)$ ($q=3$, $p=7$), $\mathrm{PSL}(2,49)$ ($q=5$, $p=7$), $\mathrm{PSL}(2,503)$ ($q=251$, $p=503$), $\mathrm{PSL}(3,4)$ ($q=3$, $p=5$), $\mathrm{PSU}(3,5)$ ($q=3$, $p=5$), J_2 ($q=3$, $p=5$), $\mathrm{PSp}(6,2)$($q=3$, $p=5$), $\mathrm{PSp}(8,2)$ ($q=3$, $p=5$), $Sz(8)$ ($q=5$, $p=13$), A_{11} ($p=5, q=11$), M_{22} ($q=3$, $p=11$), M_{22} ($q=5$, $p=11$), $\mathrm{PSL}(2,2^6)$ ($q=3$, $p=13$), $\mathrm{PSL}(2,2^6)$ ($q=5$, $p=13$), $\mathrm{PSL}(4,4)$ ($q=3$, $p=17$), $\mathrm{PSL}(4,4)$ ($q=5$, $p=17$), $\mathrm{PSL}(5,2)$ ($q=3$, $p=31$), HS ($q=5$, $p=11$), ${}^2D_4(2)$ ($q=5$, $p=17$), $D_4(2)$ ($q=3$, $p=5$), $G_2(4)$ ($q=3$, $p=13$), $G_2(4)$ ($q=5$, $p=13$), $\mathrm{PSL}(2,5^3)$ ($q=5$, $p=31$), $\mathrm{PSL}(2,71)$ ($q=3$, $p=71$), $\mathrm{PSL}(2,71)$ ($q=5$, $p=71$), $\mathrm{PSL}(3,8)$ ($q=7$, $p=73$), ${}^3D_4(2)$ ($q=7$, $p=17$), $\mathrm{PSp}(4,8)$ ($q=7$, $p=13$), $\mathrm{PSL}(2,2^9)$ ($q=19$, $p=73$), M_{23} ($q=11$, $p=23$), M_{24} ($q=11$, $p=23$), J_1($q=11$, $p=19$) and $N\neq \mathrm{PSL}(3,16)$ ($q=13$, $p=17$).

Suppose that $N=\mathrm{PSL}(2,83)$($q=41$, $p=83$). Then $|N_v|=2^2\cdot 3\cdot 7$. By Proposition 2.2, $N_v\cong F_{42}\times \mathbb{Z}_2$. By Proposition 2.4, one concludes that N has a unique conjugacy class D_{84} which has order 84. Clearly, it is isomorphic to $F_{42}\times\mathbb{Z}_2$, a contradiction.

Suppose that $N=A_8$($q=3$, $p=5$). Then $|N_v|=|N|/\frac{1}{2}|V(X)|=2^6\cdot 3\cdot 7$. By Proposition 2.2, $N_v\cong \mathbb{Z}_2^3\times \mathrm{SL}(3,2)$. In this case, N has two orbits on $V(X)$, and $N_v\cap N_v^g\cong \mathbb{Z}_2^3\rtimes S_4$. Let $C=C_A(N)$. Since N is simple, $C\cap N=1$ and $CN=C\times N\trianglelefteq A$. Since $A/CN\lesssim \mathrm{Out}(N)$, we have $A=(C\times N)\cdot O$ with $O\lesssim \mathrm{Out}(N)$, where $\mathrm{Out}(N)$ is the outer automorphism group of N. Hence $|A|\mid 2^{25}\cdot 3^5\cdot 5^3\cdot 7$ and $|N|=2^6\cdot 3^2\cdot 5\cdot 7$. Then $|C|\mid 2^{19}\cdot 3^3\cdot 5^2$. If C is insolvable, by [3], pages 12–14, then C has a minimal normal insolvable subgroup $M\cong A_5, A_6$ or A_5^2. Then $NM\trianglelefteq CN$ has at most two orbits on $V(X)$. Then $|(MN)_v|=|MN|/|V(X)|=2^7\cdot 3^2\cdot 5\cdot 7$ or $|MN|/\frac{1}{2}|V(X)|=2^8\cdot 3^2\cdot 5\cdot 7$ for $M\cong A_5$. By Proposition 2.2, there exists no vertex stabilizer whose order is $|(MN)_v|$, a contradiction. For $M\cong A_6$ or A_5^2, one has $3^3\mid |(MN)_v|$. By Proposition 2.2, this is a contradiction. Thus, C is solvable. Clearly, C is not semiregular on $V(X)$. If it were, X_C would be a connected valency seven graph of order $2pq/|C|$, yielding that $2\nmid |C|$. Furthermost, $C\ncong \mathbb{Z}_3$ or $\mathbb{Z}_5$, because there is no connected valency seven symmetric graph of order 6 or 10. If C has at most two orbits on $V(X)$, then $|C|=15$ or 30. Let $R\cong\mathbb{Z}_5\leqslant C$. Then $R\lhd A$, and then X_C is a connected valency seven graph of order 6, a contradiction. Thus, $C=1$ and $A\leqslant \mathrm{Aut}(A_8)$. Further, $A=S_8$. By Example 3.3, $X\cong\mathcal{C}_{30}$. Hence

N_v is a maximal subgroup of N, and N has two orbits on $V(X)$. Then X is a vertex bi-primitive 3-arc transitive graph.

Suppose that $N = \mathrm{PSL}(5,2)$ ($q = 5$, $p = 31$). Then $|N_v| = 2^{10} \cdot 3^2 \cdot 7$. By Proposition 2.2, $N_v \cong \mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$. Let $C = C_A(N)$. Similarly to the above proof, one has $A = (C \times N).O$ with $O \lesssim \mathrm{Out}(N)$, where $\mathrm{Out}(N)$ is the outer automorphism group of N. Hence $|A| \mid 2^{25} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 31$ and $|N| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Then $|C| \mid 2^{15} \cdot 3^2 \cdot 5^2$. If C is insolvable, by [3], pages 12–14, then C has a minimal normal insolvable subgroup $M \cong A_5, A_6$ or A_5^2. Then $NM \trianglelefteq CN$ has at most two orbits on $V(X)$. For $M \cong A_5$, one has $3^3 \mid |(MN)_v|$. By Proposition 2.2, this is a contradiction. Thus, $M \ncong A_5$. If $M \cong A_6$, then $|(MN)_v| = |MN|/|V(X)| = 2^{13} \cdot 3^4 \cdot 5 \cdot 7$ or $|MN|/\frac{1}{2}|V(X)| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7$. By Proposition 2.2, there exists no vertex stabilizer whose order is $|(MN)_v|$, a contradiction. Similarly, $M \ncong A_5^2$. Thus, C is solvable. Clearly, C is not semiregular on $V(X)$. If it were, X_C would be a connected valency seven graph of order $2pq/|C|$, yielding that $2 \nmid |C|$. Furthermost, $C \ncong \mathbb{Z}_{31}$ because there is no connected valency seven symmetric graph of order 10. If $C \cong \mathbb{Z}_5$, by Proposition 2.5, there is no connected valency seven symmetric graph of order 62 because $7 \nmid p - 1$ with $p = 31$. Thus C has at most two orbits on $V(X)$, then $|C| = 5p$ or $10p$. Let $R \cong \mathbb{Z}_p < C$. Then $R \lhd A$, and then X_R is a connected valency seven graph of order 10, a contradiction. Thus, $C = 1$ and $A \leqslant \mathrm{Aut}(N)$. Further, $A \cong \mathrm{Aut}(\mathrm{PSL}(5,2) \cdot \mathbb{Z}_2$ because $\mathrm{Out}(N) = \mathbb{Z}_2$. By Example 3.5, $X \cong \mathcal{C}_{310}$. Hence N_v is a maximal subgroup of N, and N has two orbits on $V(X)$. Then X is a vertex bi-primitive 3-arc transitive graph. This completes the proof. □

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References

[1] *W. Bosma, J. Cannon, C. Playoust*: The Magma algebra system. I. The user language. J. Symb. Comput. *24* (1997), 235–265. zbl MR doi

[2] *Y. Cheng, J. Oxley*: On weakly symmetric graphs of order twice a prime. J. Comb. Theory, Ser. B *42* (1987), 196–211. zbl MR doi

[3] *J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson*: Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups. Oxford University Press, Eynsham, 1985. zbl MR

[4] *D. Ž. Djoković, G. L. Miller*: Regular groups of automorphisms of cubic graphs. J. Comb. Theory, Ser. B *29* (1980), 195–230. zbl MR doi

[5] *X. G. Fang, C. E. Praeger*: Finite two-arc transitive graphs admitting a Suzuki simple group. Commun. Algebra *27* (1999), 3727–3754. zbl MR doi

[6] *X. Fang, J. Wang, M. Y. Xu*: On 1-arc-regular graphs. Eur. J. Comb. *23* (2002), 785–791. zbl MR doi

[7] *Y.-Q. Feng, M. Ghasemi, D.-W. Yang*: Cubic symmetric graphs of order $8p^3$. Discrete Math. *318* (2014), 62–70. zbl MR doi

[8] *Y.-Q. Feng, J. H. Kwak*: Cubic symmetric graphs of order a small number times a prime or a prime square. J. Comb. Theory, Ser. B *97* (2007), 627–646. zbl MR doi
[9] *Y.-Q. Feng, J. H. Kwak, M.-Y. Xu*: Cubic s-regular graphs of order $2p^3$. J. Graph Theory *52* (2006), 341–352. zbl MR doi
[10] *Y.-Q. Feng, Y.-T. Li*: One-regular graphs of square-free order of prime valency. Eur. J. Comb. *32* (2011), 265–275. zbl MR doi
[11] *A. Gardiner, C. E. Praeger*: On 4-valent symmetric graphs. Eur. J. Comb. *15* (1994), 375–381. zbl MR doi
[12] *A. Gardiner, C. E. Praeger*: A characterization of certain families of 4-valent symmetric graphs. Eur. J. Comb. *15* (1994), 383–397. zbl MR doi
[13] *D. Gorenstein*: Finite Simple Groups. An Introduction to Their Classification. The University Series in Mathematics, Plenum Press, New York, 1982. zbl MR doi
[14] *S.-T. Guo, Y.-Q. Feng*: A note on pentavalent s-transitive graphs. Discrete Math. *312* (2012), 2214–2216. zbl MR doi
[15] *S. Guo, Y. Li, X. Hua*: (G, s)-transitive graphs of valency 7. Algebra Colloq. *23* (2016), 493–500. zbl MR doi
[16] *S.-T. Guo, J. Shi, Z.-J. Zhang*: Heptavalent symmetric graphs of order $4p$. South Asian J. Math. *1* (2011), 131–136. zbl
[17] *X.-H. Hua, Y.-Q. Feng, J. Lee*: Pentavalent symmetric graphs of order $2pq$. Discrete Math. *311* (2011), 2259–2267. zbl MR doi
[18] *Y. Li, Y.-Q. Feng*: Pentavalent one-regular graphs of square-free order. Algebra Colloq. *17* (2010), 515–524. zbl MR doi
[19] *M. W. Liebeck, C. E. Praeger, J. Saxl*: A classification of the maximal subgroups of the finite alternating and symmetric groups. J. Algebra *111* (1987), 365–383. zbl MR doi
[20] *P. Lorimer*: Vertex-transitive graphs: symmetric graphs of prime valency. J. Graph Theory *8* (1984), 55–68. zbl MR doi
[21] *B. D. McKay*: Transitive graphs with fewer than twenty vertices. Math. Comput. *33* (1979), 1101–1121. zbl MR doi
[22] *R. C. Miller*: The trivalent symmetric graphs of girth at most six. J. Comb. Theory, Ser. B *10* (1971), 163–182. zbl MR doi
[23] *J.-M. Oh*: A classification of cubic s-regular graphs of order $14p$. Discrete Math. *309* (2009), 2721–2726. zbl MR doi
[24] *J.-M. Oh*: A classification of cubic s-regular graphs of order $16p$. Discrete Math. *309* (2009), 3150–3155. zbl MR doi
[25] *J. Pan, B. Ling, S. Ding*: One prime-valent symmetric graphs of square-free order. Ars Math. Contemp. *15* (2018), 53–65.
[26] *J. Pan, B. Lou, C. Liu*: Arc-transitive pentavalent graphs of order $4pq$. Electron. J. Comb. *20* (2013), Researh Paper P36, 9 pages. zbl MR
[27] *P. Potočnik*: A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index $(4, 2)$. Eur. J. Comb. *30* (2009), 1323–1336. zbl MR doi
[28] *G. Sabidussi*: Vertex-transitive graphs. Monatsh. Math. *68* (1964), 426–438. zbl MR doi
[29] *C. C. Sims*: Graphs and finite permutation groups. Math. Z. *95* (1967), 76–86. zbl MR doi
[30] *M. Suzuki*: Group Theory I. Grundlehren der Mathematischen Wissenschaften 247, Springer, Berlin, 1982. zbl MR
[31] *R. A. Wilson*: The Finite Simple Groups. Graduate Texts in Mathematics 251, Springer, London, 2009. zbl MR doi
[32] *J. Xu, M. Xu*: Arc-transitive Cayley graphs of valency at most four on abelian groups. Southeast Asian Bull. Math. *25* (2001), 355–363. zbl MR doi
[33] *M.-Y. Xu*: Automorphism groups and isomorphisms of Cayley digraphs. Discrete Math. *182* (1998), 309–319. zbl MR doi

[34] *J.-X. Zhou, Y.-Q. Feng*: Tetravalent s-transitive graphs of order twice a prime power. J. Aust. Math. Soc. *88* (2010), 277–288. zbl MR doi

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