Czechoslovak Mathematical Journal, 68 (143) (2018), 433–444

ON g_c-COLORINGS OF NEARLY BIPARTITE GRAPHS

YUZHUO ZHANG, XIA ZHANG, Jinan

Received September 11, 2016. First published February 8, 2018.

Abstract. Let G be a simple graph, let $d(v)$ denote the degree of a vertex v and let g be a nonnegative integer function on $V(G)$ with $0 \leqslant g(v) \leqslant d(v)$ for each vertex $v \in V(G)$. A g_c-coloring of G is an edge coloring such that for each vertex $v \in V(G)$ and each color c, there are at least $g(v)$ edges colored c incident with v. The g_c-chromatic index of G, denoted by $\chi'_{g_c}(G)$, is the maximum number of colors such that a g_c-coloring of G exists. Any simple graph G has the g_c-chromatic index equal to $\delta_g(G)$ or $\delta_g(G)-1$, where $\delta_g(G) = \min\limits_{v \in V(G)} \lfloor d(v)/g(v) \rfloor$. A graph G is nearly bipartite, if G is not bipartite, but there is a vertex $u \in V(G)$ such that $G-u$ is a bipartite graph. We give some new sufficient conditions for a nearly bipartite graph G to have $\chi'_{g_c}(G) = \delta_g(G)$. Our results generalize some previous results due to Wang et al. in 2006 and Li and Liu in 2011.

Keywords: edge coloring; nearly bipartite graph; edge covering coloring; g_c-coloring; edge cover decomposition

MSC 2010: 05C15

1. INTRODUCTION

Our terminology and notation will be standard, except where indicated. Readers are referred to [1] for undefined terms. Throughout this paper, the word *graph* refers to simple graph. A *multigraph* may have multiple edges but no loops. Let G be a multigraph with a finite nonempty vertex set $V(G)$ and a finite nonempty edge set $E(G)$. Let $N_G(v)$ denote the neighborhood of v and let the degree $d(v)$ be the number of edges incident with v in the graph G. A multigraph G is *nearly bipartite*, if G is not bipartite, but there is a vertex $u \in V(G)$ such that $G-u$ is bipartite

This research is supported by Shandong Provincial Natural Science Foundation, China (Grant No. ZR2014JL001), the Excellent Young Scholars Research Fund of Shandong Normal University of China and Higher Educational Science and Technology Program of Shandong Province (Grant No. J17KA171).

DOI: 10.21136/CMJ.2018.0477-16

with bipartition (X, Y); such a nearly bipartite graph is denoted by $G(X, Y; u)$. An *edge coloring* of a multigraph G is an assignment of some colors to the edges of G. Let $i_\eta(v)$ (or simply $i(v)$) denote the number of edges of G which are incident with the vertex v and receive color i in an edge coloring η. Let g be a nonnegative integer function defined on $V(G)$ such that $0 \leqslant g(v) \leqslant d_G(v)$ for any $v \in V(G)$. A g_c*-coloring* of G is an edge coloring with the colors in a set C satisfying that, for each vertex $v \in V(G)$ and each color $i \in C$, there are at least $g(v)$ incident edges colored with color i. Let $\chi'_{g_c}(G)$ denote the maximum number of colors for which a g_c-coloring of G exists. We call $\chi'_{g_c}(G)$ the g_c*-chromatic index* of G. An edge coloring η is *proper* if $\alpha(v) \leqslant 1$ for each color $\alpha \in C$ and each vertex $v \in V(G)$. For a d-regular graph G with $g \equiv 1$, G has a proper edge coloring with d colors if and only if G has a g_c-coloring with d colors.

Since the proper edge coloring problem is NP-complete even for regular graphs, see [3], the g_c-coloring problem is NP-complete as well. In our daily life many problems on optimization and network design, for example, coding design, the file transfer problem on computer networks, schedule problems and so on, see [5], are related to the g_c-coloring which was for the first time presented by Song and Liu in [6].

Let

$$\delta(G) = \min_{v \in V(G)} \{d(v)\}, \quad \delta_g(G) = \min_{v \in V(G)} \left\lfloor \frac{d(v)}{g(v)} \right\rfloor,$$
$$V_\delta(G) = \{v \in V(G) \colon\ d(v) = \delta(G)\},$$
$$V_{\delta_g}(G) = \{v \in V(G) \colon\ d(v) = g(v)\delta_g(G)\},$$
$$N^*(u) = \{v \in N_G(u) \colon\ d(v) = \delta(G)\},$$
$$N_g^*(u) = \{v \in N_G(u) \colon\ d(v) = \delta_g(G) g(v)\},$$
$$d^*(u) = |N^*(u)|, \quad d_g^*(u) = |N_g^*(u)| \quad \text{and} \quad \operatorname{sur}(v) = d(v) - g(v)\delta_g(G),$$

in which $\lfloor d(v)/g(v) \rfloor$ is the largest integer not larger than $d(v)/g(v)$, and $\operatorname{sur}(v)$ is the surplus of $d(v)$. Clearly, $\operatorname{sur}(v) \geqslant 0$ for each $v \in V(G)$. Let $d(v)/g(v) = \infty$ when $g(v) = 0$. So $1 \leqslant \delta_g(G) \leqslant \infty$. When $\delta_g(G) = \infty$, for any given color set C, any edge coloring with colors in C is a g_c-coloring of G. So $\chi'_{g_c}(G) = \infty$. When $\delta_g(G) = 1$, we have $\chi'_{g_c}(G) = 1$. In this paper, we just consider the nontrivial cases, i.e. the graphs with $2 \leqslant \delta_g(G) < \infty$. It is easy to verify that $d(v) \geqslant \delta_g(G) g(v)$ for each $v \in V(G)$ and $\chi'_{g_c}(G) \leqslant \delta_g(G)$. The *multiplicity* $\mu(u, v)$ of a pair of distinct vertices u and v is the number of edges of G joining u and v. Let $\mu(v) = \max\{\mu(v, u) \colon\ u \in V(G)\}$. Song and Liu studied g_c-chromatic indices of multigraphs and obtained the following result in [7].

Theorem 1 ([7]). *Let G be a multigraph associated with a positive integer function $g\colon V(G) \to \mathbb{Z}_{\geqslant 0}$. Then*

$$\min_{v\in V(G)} \left\lfloor \frac{d(v)-\mu(v)}{g(v)} \right\rfloor \leqslant \chi'_{g_c}(G) \leqslant \delta_g(G).$$

When G is a graph, we have $\mu(v) = 1$ for each $v \in V(G)$. Therefore the following corollary holds.

Corollary 2. *Let G be a graph associated with a positive integer function $g\colon V(G) \to \mathbb{Z}_{\geqslant 0}$. Then*

$$\delta_g(G) - 1 \leqslant \chi'_{g_c}(G) \leqslant \delta_g(G).$$

We say that a graph G is of g_c-*class* 1 if $\chi'_{g_c} = \delta_g(G)$, and G is of g_c-*class* 2 otherwise. The problem of deciding whether a graph G is of g_c-class 1 or g_c-class 2 is called the *classification problem* on g_c-colorings.

When $g \equiv 1$, a g_c-coloring of a graph G is exactly an *edge covering coloring* of G, and the g_c-chromatic index of G is denoted by $\chi'_c(G)$ simply. If $g \equiv 1$, Corollary 2 is the famous theorem of Gupta in [2].

Theorem 3 ([2]). *Let G be a graph. Then $\delta(G) - 1 \leqslant \chi'_c(G) \leqslant \delta(G)$.*

We say a graph G is of CI *class* if $\chi'_c(G) = \delta(G)$, otherwise G is of CII *class*.

For the g_c-chromatic index of bipartite graphs, Song and Liu in [7] obtained the following result.

Theorem 4 ([7]). *Let G be a bipartite multigraph associated with a positive integer function $g\colon V(G) \to \mathbb{Z}_{\geqslant 0}$. Then $\chi'_{g_c}(G) = \delta_g(G)$.*

Wang, Zhang and Liu in [8] gave some sufficient conditions for a nearly bipartite graph to be of CI class.

Theorem 5 ([8]). *Let $G(X, Y; u)$ be a nearly bipartite graph with $\delta(G) \geqslant 3$. Then $G(X, Y; u)$ is of* CI *class if one of the following conditions is satisfied:*

(1) $d(u) \geqslant 2\delta(G) - 1$,

(2) $d^*(u) \leqslant 1$,

(3) $N^*(u) \subseteq X$ (or Y) and $d(u) \geqslant \delta(G) + d^*(u) - 1$.

Xu and Jia in [9] gave a sufficient condition for a nearly bipartite graph to be of CI class.

Theorem 6 ([9]). *Let $G(X,Y;u)$ be a nearly bipartite graph with $\delta(G) \geqslant 3$. If there exits a vertex $y \in N_G(u)$, such that $d(u)+d(y) \geqslant 3\delta - 1$, then $G(X,Y;u)$ is of* CI *class.*

Li and Liu in [4] gave some sufficient conditions for a nearly bipartite graph to be of g_c-class 1.

Theorem 7 ([4]). *Let $G(X,Y;u)$ be a nearly bipartite graph associated with a positive integer function $g\colon V(G) \to \mathbb{Z}_{\geqslant 0}$. Then $G(X,Y;u)$ is of g_c-class 1 if one of the following conditions is satisfied:*

(1) $\mathrm{sur}(u) \geqslant \delta_g(G) - 1$,

(2) *there exists a vertex $y \in N_G(u)$ such that*

$$d(u) + d(y) \geqslant \delta_g(G)(g(u)+1) + \delta_g(G)g(y) - 1.$$

In this paper, we obtain the following main result for a nearly bipartite graph to be of g_c-class 1. Our results generalize Theorems 5 and 7 (1).

Theorem 8. *Let $G(X,Y;u)$ be a nearly bipartite graph associated with a non-negative integer function $g\colon V(G) \to \mathbb{N}$, $g(u) \geqslant 1$ and $\delta_g(G) < \infty$. If $d_g^*(u) \leqslant d(u) - \delta_g(G) + 1$, then $G(X,Y;u)$ is of g_c-class 1.*

2. Preliminary results

Theorem 1 and Corollary 2 are based on an integer function $g\colon V(G) \to \mathbb{Z}_{\geqslant 0}$. When $g\colon V(G) \to \mathbb{N}$, $V_0 = \{v \in V(G)\colon g(v) = 0\} \neq \emptyset$ and $1 \leqslant \delta_g(G) < \infty$, Zhang in [10] constructed an auxiliary graph G' from G as follows: for each $v \in V_0$, stick a new complete graph $H_v = K_{2\delta_g(G)+2}$ at v in such a way that v is identified with an arbitrary vertex of H_v. Define a function $h\colon V(G') \to \mathbb{Z}_{\geqslant 0}$ in such a way that

$$\begin{aligned} h(v) &= g(v), && v \in V(G) \setminus V_0;\\ h(v) &= 2, && \text{otherwise.} \end{aligned}$$

Zhang in [10] proved that $\delta_g(G) = \delta_h(G')$, $V_{\delta_g}(G) = V_{\delta_h}(G')$ and $\chi'_{g_c}(G) = \chi'_{h_c}(G')$. So Theorem 1 and Corollary 2 are still true for a function $g\colon V(G) \to \mathbb{N}$.

Theorem 1′. *Let G be a multigraph associated with a nonnegative integer function $g\colon V(G)\to\mathbb{N}$ and let $\delta_g(G)<\infty$. Then*

$$\min_{v\in V(G)}\Big\lfloor\frac{d(v)-\mu(v)}{g(v)}\Big\rfloor\leqslant\chi'_{g_c}(G)\leqslant\delta_g(G).$$

Corollary 2′ ([10])**.** *Let G be a graph associated with a nonnegative integer function $g\colon V(G)\to\mathbb{N}$ and let $\delta_g(G)<\infty$. Then*

$$\delta_g(G)-1\leqslant\min_{v\in V(G)}\Big\lfloor\frac{d(v)-1}{g(v)}\Big\rfloor\leqslant\chi'_{g_c}(G)\leqslant\delta_g(G).$$

Next, we prove that the result in Theorem 4 is still true when g is a nonnegative integer function.

Theorem 4′. *Let G be a bipartite multigraph associated with a nonnegative integer function $g\colon V(G)\to\mathbb{N}$ and let $\delta_g(G)<\infty$. Then $\chi'_{g_c}(G)=\delta_g(G)$.*

Proof. Let $G=G(X,Y)$, $V_0=\{v\in V(G)\colon g(v)=0\}$, $V_1=V_0\cap X$, $V_2=V_0\cap Y$, $T=\{x,y\}$, $T\cap V(G)=\emptyset$. When $V_0=\emptyset$, by Theorem 4, we are done. When $V_0\neq\emptyset$, we can construct an auxiliary graph G' from G as follows: add x,y to $G(X,Y)$, join $\delta_g(G)$ multiedges between vertices x and v for each $v\in V_2$ and join $\delta_g(G)$ multiedges between vertices y and v for each $v\in V_1$. In graph G', define a function $h\colon V(G')\to\mathbb{Z}_{\geqslant 0}$ in such a way that

$$\begin{aligned}&h(v)=g(v), && v\in V(G)\setminus V_0;\\ &h(v)=1, && \text{otherwise}.\end{aligned}$$

It is easy to see that G' is a bipartite multigraph with bipartition (X_1,Y_1), where $X_1=X\cup\{x\}$, $Y_1=Y\cup\{y\}$. In the graph G, $\delta_g(G)=\min\limits_{v\in V(G)\setminus V_0}\lfloor d(v)/g(v)\rfloor$. In the graph G', $d_{G'}(v)=d_G(v)$, $h(v)=g(v)$ for each $v\in V(G)\setminus V_0$, $d_{G'}(v)\geqslant\delta_g(G)$ and $h(v)=1$ for each $v\in V_0\cup T$. Thus $\delta_h(G')=\delta_g(G)$. By Theorem 4, we have $\chi'_{h_c}(G')=\delta_h(G')=\delta_g(G)$. Thus we can find an h_c-coloring η of G' with $\delta_g(G)$ colors in $C=\{1,2,\ldots,\delta_g(G)\}$. Then $\alpha(v)\geqslant g(v)$ for each $\alpha\in C$ and each $v\in V(G')$ in η. Restricting the coloring η of G' to G, we get a g_c-coloring of G with $\delta_g(G)$ colors. Thus, $\chi'_{g_c}(G)=\delta_g(G)$. $\square$

Corollary 9. *Let G be a bipartite graph associated with a nonnegative integer function $g\colon V(G)\to\mathbb{N}$ and let $\delta_g(G)<\infty$. Then G is of g_c-class 1.*

3. Main results

In the remaining part of this paper, we just concentrate on the graphs G with $2 \leqslant \delta_g(G) < \infty$. In [11], Zhang defined a class of auxiliary graphs, the splitting graphs, for investigating f-colorings of graphs, which are edge-colorings of graphs such that each vertex v has at most $f(v)$ incident edges colored with a same color. Here, we give a similar definition for investigating g_c-colorings of graphs. In a graph G, let $u \in V(G)$, $N_G(u) = \{v_1, v_2, \ldots, v_{d(u)}\}$. Let $T = \{u_1, u_2, \ldots, u_t\}$, $T \cap V(G) = \emptyset$. Let $N_i \subset N_G(u)$, $1 \leqslant i \leqslant t$, $\bigcup\limits_{1 \leqslant i \leqslant t} N_i = N_G(u)$ and $N_i \cap N_j = \emptyset$ for every $i, j \in \{1, 2, \ldots, t\}$, $i \neq j$. Construct an auxiliary graph G' from G as follows:

$$\begin{aligned} &V(G') = V(G) \setminus \{u\} \cup T; \\ &E(G') = E(G) \setminus (\{uv\colon\ v \in N_G(u)\} \cup \{u_i v\colon\ v \in N_i,\ 1 \leqslant i \leqslant t\}). \end{aligned} \tag{3.1}$$

G' is called a *splitting graph* of G.

In the graph G, let $S = \{x_1, x_2, \ldots, x_s\} \subset V(G)$, $Q \subseteq E(G)$ and $Q \neq \emptyset$. We use $G[Q]$ to denote the subgraph of G induced by Q. A *partial edge-coloring* of G is an edge-coloring of a subgraph $G[Q]$ of G. Identifying $x_1, x_2, \ldots, x_s$ with x means removing the vertices in the set S of G, adding a new vertex x to $G - S$ and joining x to each vertex in $N_G(S) \setminus S$ by an edge. The resulting graph is called an *identifying graph* of G.

Theorem 10. *Let $G(X, Y; u)$ be a nearly bipartite graph associated with a non-negative integer function $g\colon V(G) \to \mathbb{N}$ and let $\delta_g(G) < \infty$. If $g(u) = 0$, then $G(X, Y; u)$ is of g_c-class 1.*

P r o o f. Let $N_1 = N_G(u) \cap Y$, $N_2 = N_G(u) \cap X$, $T = \{u_1, u_2\}$ and $T \cap V(G) = \emptyset$. We can construct a splitting graph G' of G so that it satisfies the conditions in (3.1). In the graph G', define a function $h\colon V(G') \to \mathbb{N}$ in such a way that

$$\begin{aligned} h(u_1) = h(u_2) = 0; \\ h(v) = g(v), \quad v \in V(G') \setminus T. \end{aligned}$$

It is easy to see that $\delta_h(G') = \min\limits_{v \in V(G') \setminus T} \lfloor d_{G'}(v)/h(v) \rfloor$ and G' is a bipartite graph with bipartition (X_1, Y_1), where $X_1 = X \cup \{u_1\}$, $Y_1 = Y \cup \{u_2\}$. Since $d_{G'}(v) = d_G(v)$, $h(v) = g(v)$ for each $v \in V(G') \setminus T$, we have $\delta_h(G') = \delta_g(G)$. By Corollary 9, we know that $\chi'_{h_c}(G') = \delta_h(G') = \delta_g(G)$. Thus we can find an h_c-coloring η of G' with $\delta_g(G)$ colors in $C = \{1, 2, \ldots, \delta_g(G)\}$. By identifying u_1, u_2 of G' with u, we get the graph G and an edge-coloring η' of G with $\delta_g(G)$ colors in C. In η', we have $\alpha(x) \geqslant g(x)$ for each $\alpha \in C$ and each vertex $x \in V(G)$. So G is of g_c-class 1. $\square$

Next, we just consider nearly bipartite graphs $G(X,Y;u)$ with $g(u) \geqslant 1$.

An (α,β) exchange chain L of G is a sequence $(v_0,e_1,v_1,e_2,\dots,v_{r-1},e_r,v_r)$ of vertices and edges of G in which

(i) for $1 \leqslant i \leqslant r$, the adjacent vertices v_{i-1} and v_i are distinct and $e_i = v_{i-1}v_i$;

(ii) all the edges are distinct and are colored alternately by α and β;

(iii) e_1 is colored α and $\alpha(v_0) > \beta(v_0)$; $\mu(v_r) > \overline{\mu}(v_r)$, where μ denotes the color of e_r and $\overline{\mu}$ denotes the other color of $\{\alpha,\beta\}$.

By (iii), L must be a closed chain with odd edges when $v_0 = v_r$. So any even closed chain would not be an exchange chain. If $v_0 \neq v_r$, then exchanging the colors on L makes $\alpha(v_0)$ decrease by one, $\beta(v_0)$ increase by one while $\alpha(v)$ and $\beta(v)$ remain unchanged for each $v \in \{v_1,v_2,\dots,v_{r-1}\}$; if $v_0 = v_r$, then exchanging the colors on L makes $\alpha(v_0)$ decrease by two, $\beta(v_0)$ increase by two while $\alpha(v)$ and $\beta(v)$ remain unchanged for each $v \in \{v_1,v_2,\dots,v_{r-1}\}$.

In an edge coloring of a graph G, an edge colored by i is called an *i-edge*. We now prove the main result of this paper.

Proof of Theorem 8. If $\delta_g(G) = 1$, then $G(X,Y;u)$ is of g_c-class 1. Next, consider the cases with $\delta_g(G) \geqslant 2$.

If $d_g^*(u) = 0$, let $G' = G \setminus u$. In the graph G', define a function $h\colon V(G') \to \mathbb{N}$ in such a way that $h(v) = g(v)$ for each $v \in V(G\setminus u)$. Note that $N_G(u)\setminus V_{\delta_g}(G) = N_G(u)$ and, for each $v \in N_G(u)$, $d_G(v) \geqslant \delta_g(G)g(v) + 1$. So $d_{G'}(v) \geqslant \delta_g(G)g(v)$ for each $v \in N_G(u)$. Let $V_0 = \{v \in V(G)\colon \lfloor d_G(v)/g(v)\rfloor = \delta_g(G)\}$, it is easy to see that $V_0 \neq \emptyset$. If $V_0 = \{u\}$, it may be the case that $\delta_h(G') > \delta_g(G)$; otherwise, $\delta_h(G') = \delta_g(G)$. So $\delta_h(G') \geqslant \delta_g(G)$. By Corollary 9, we have $\chi'_{h_c}(G') = \delta_h(G')$. Thus we can find an h_c-coloring η of G' with $\delta_h(G')$ colors in $C' = \{1,2,3,\dots,\delta_h(G')\}$. If $\delta_h(G') = \delta_g(G)$, let $\eta' = \eta$. If $\delta_h(G') > \delta_g(G)$, we can find an h_c-coloring η' of G' with $\delta_g(G)$ colors in $C = \{1,2,3,\dots,\delta_g(G)\}$ from the coloring η by recoloring the i-edges, $\delta_g(G)+1 \leqslant i \leqslant \delta_h(G')$, by the color $\delta_g(G)$. In η', we have $\alpha(v) \geqslant g(v)$, for each $\alpha \in C$ and $v \in V(G')$. Based on the coloring η' of G' and $d_G(u) \geqslant g(u)\delta_g(G)$, we can color all the uncolored edges incident by u by the colors in C such that every color at a vertex u appears at least $g(u)$ times. Then we get a g_c-coloring of G by $\delta_g(G)$ colors. Thus G is of g_c-class 1.

If $1 \leqslant d_g^*(u) \leqslant d(u) - \delta_g(G) + 1$, then equivalently $1 \leqslant d_g^*(u) \leqslant (g(u)-1)\delta_g(G) + \mathrm{sur}(u) + 1$ since $d(u) = g(u)\delta_g(G) + \mathrm{sur}(u)$. Let $N_g^*(u) = \{v_1,v_2,\dots,v_{d_g^*(u)}\}$, $N_3 = \{v_1,v_2,\dots,v_t\}$, where $t = \min\{\mathrm{sur}(u), d_g^*(u)\}$, $N_1 = (N_g^*(u) \setminus N_3) \cap Y$, $N_2 = (N_g^*(u) \setminus N_3) \cap X$, $T = \{u_1,u_2,u_3\}$ and $T \cap V(G) = \emptyset$. Construct a new graph G_1 (see Figure 1) from G as follows:

$$V(G_1) = V(G) \setminus \{u\} \cup T;$$
$$E(G_1) = E(G) \setminus (\{uv\colon\ v \in N_G(u)\} \cup \{u_iv\colon\ v \in N_i,\ 1 \leqslant i \leqslant 3\}).$$

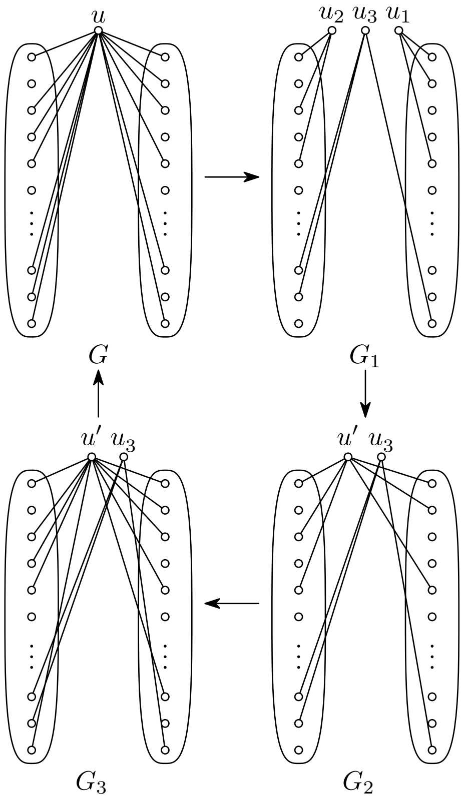

Figure 1.

It is easy to see that $d_{G_1}(u_3) \leqslant \mathrm{sur}(u)$. Let $R = N_G(u) \setminus V_{\delta_g}(G)$. Note that, for each $v \in R$, $d_G(v) \geqslant \delta_g(G)g(v) + 1$. So $d_{G_1}(v) \geqslant \delta_g(G)g(v)$ for each $v \in R$. In the graph G_1, define a function $h\colon V(G_1) \to \mathbb{N}$ in such a way that

$$h(u_1) = \Big\lfloor \frac{d_{G_1}(u_1)}{\delta_g(G)} \Big\rfloor; \quad h(u_2) = \Big\lfloor \frac{d_{G_1}(u_2)}{\delta_g(G)} \Big\rfloor;$$
$$h(u_3) = 0;$$
$$h(v) = g(v), \quad v \in V(G_1) \setminus T.$$

It is easy to see that $\min\{h(u_1), h(u_2)\} \geqslant 0$, $\delta_h(G_1) = \delta_g(G)$. When $N_{G_1}(u_3) \subseteq X$ or $N_{G_1}(u_3) \subseteq Y$, G_1 is a bipartite graph; when $N_{G_1}(u_3) \cap X \neq \emptyset$ and $N_{G_1}(u_3) \cap Y \neq \emptyset$, G_1 is a nearly bipartite graph with $h(u_3) = 0$. By Corollary 9 or Theorem 10, we know that $\chi'_{h_c}(G_1) = \delta_h(G_1) = \delta_g(G)$. Thus we can find an h_c-coloring ξ of G_1 with $\delta_g(G)$ colors in $C = \{1, 2, \ldots, \delta_g(G)\}$. Identify u_1 and u_2 of G_1 with u' in the coloring ξ. Then we get a graph G_2 (see Figure 1) and an edge coloring ξ' of G_2 with $\delta_g(G)$ colors in C. In ξ', $c(v) \geqslant h(v) = g(v)$ for each $v \in V(G_1) \setminus \{u', u_3\}$ and each

$c \in C$. We claim that there exists an edge-coloring of G_2 satisfying the condition above and $c(u') \leqslant g(u)$ for each $c \in C$. We know that $0 \leqslant d_{G_2}(u') = d_g^*(u) - t \leqslant (g(u)-1)\delta_g(G)+1$. So we have

$$0 \leqslant \left\lfloor \frac{d_{G_2}(u')}{\delta_g(G)} \right\rfloor \leqslant g(u) - 1 \quad \text{and} \quad \frac{d_{G_2}(u')-1}{\delta_g(G)} \leqslant g(u) - 1.$$

If $g(u) = 1$, then $d_{G_2}(u') \leqslant 1$. So ξ' is a required edge-coloring of G_2. Next, consider the cases with $g(u) \geqslant 2$. In ξ', if there exists a color $\alpha \in C$ with $\alpha(u') \geqslant g(u)+1$, there must be some color $\beta \in C$ with $\beta(u') \leqslant g(u) - 2$. We can construct an (α, β) exchange chain L starting at u'. By (iii) of the definition of the exchange chain, L satisfies one of the following conditions:

(1) L ends at u' with an edge colored with α;

(2) L doesn't end at u'.

(According to the definition of an exchange chain and $\alpha(u') \geqslant g(u) + 1 \geqslant 3$, we can always construct a maximal (α, β) exchange chain $L = (u', e_1, v_1, e_2, \ldots, e_r, v_r)$.) Let μ denote the color of e_r and $\overline{\mu}$ the other color of $\{\alpha, \beta\}$. Since L is maximal, there is $\mu(v_r) > \overline{\mu}(v_r)$. (In fact, we do not need to find a maximal exchange chain. When constructing an (α, β) exchange chain starting at u', if entering a vertex w with an α-edge and $\alpha(w) > \beta(w)$ or a β-edge and $\beta(w) > \alpha(w)$, then L ends at w.) We exchange the two colors on L. If case (1) occurs, then exchanging the colors on L makes $\alpha(u')$ decrease by two and $\beta(u')$ increase by two; if case (2) occurs, then exchanging the colors on L makes $\alpha(u')$ decrease by one and $\beta(u')$ increase by one. In either case, we still have $\beta(u') \leqslant g(u)$ after exchanging the colors on L. Use the method iteratively until $c(u') \leqslant g(u)$ for each $c \in C$. Then we obtain a required edge-coloring ξ''. Construct a new graph G_3 from G_2 (see Figure 1) as follows:

$$\begin{aligned} &V(G_3) = V(G_2); \\ &E(G_3) = E(G_2) \cup \{u'v\colon\ v \in R\}. \end{aligned}$$

So ξ'' is a partial edge coloring of G_3. Clearly, $d_{G_3}(u') = d_G(u) - d_{G_1}(u_3) \geqslant \delta_g(G)g(u)$ and so $d_{G_3}(u')/\delta_g(G) \geqslant g(u)$. Since $\alpha(u') \leqslant g(u)$ for each color $\alpha \in C$ in ξ'', we can color all the uncolored edges incident with the vertex u' of graph G_3 with the colors in C such that every color at the vertex u' appears at least $g(u)$ times. By identifying u', u_3 of G_3 with u, we get the graph G (see Figure 1) and an edge-coloring η of G with $\delta_g(G)$ colors in C. In η, we have $\alpha(x) \geqslant g(x)$, for each $\alpha \in C$ and each vertex $x \in V(G)$. So G is of g_c-class 1. $\square$

Let $G(X, Y; u)$ be a nearly bipartite graph. We define $d_X(u) = |X \cap N_G(u)|$, $d_Y(u) = |Y \cap N_G(u)|$. Using parameters $d_X(u)$ and $d_Y(u)$, we obtain the following result.

Theorem 11. *Let $G(X, Y; u)$ be a nearly bipartite graph associated with a non-negative integer function $g\colon V(G) \to \mathbb{N}$ and let $\delta_g(G) < \infty$. If $\lfloor d_X(u)/\delta_g(G)\rfloor + \lfloor d_Y(u)/\delta_g(G)\rfloor \geqslant g(u)$, then $G(X, Y; u)$ is of g_c-class 1.*

Proof. Let $N_1 = N_G(u) \cap Y$, $N_2 = N_G(u) \cap X$, $T = \{u_1, u_2\}$ and $T \cap V(G) = \emptyset$. We can construct a splitting graph G' of G so that it satisfies the conditions in (3.1). In graph G', define a function $h\colon V(G') \to \mathbb{N}$ in such a way that

$$h(u_1) = \Big\lfloor \frac{d_Y(u)}{\delta_g(G)} \Big\rfloor; \quad h(u_2) = \Big\lfloor \frac{d_X(u)}{\delta_g(G)} \Big\rfloor;$$
$$h(v) = g(v), \quad v \in V(G') \setminus T.$$

It is easy to see that G' is a bipartite graph with $\delta_h(G') = \delta_g(G)$. By Corollary 9, we have $\chi'_{h_c}(G') = \delta_h(G') = \delta_g(G)$. So we can find an h_c-coloring η of G' with $\delta_g(G)$ colors in $C = \{1, 2, 3, \ldots, \delta_g(G)\}$. In η, we have $\alpha(u_1) + \alpha(u_2) \geqslant h(u_1) + h(u_2) \geqslant g(u)$ for each $\alpha \in C$, $\alpha(v) \geqslant g(v)$ for each $v \in V(G') \setminus \{u_1, u_2\}$ and each $\alpha \in C$. By identifying u_1, u_2 of G' with u, we get the graph G and an edge-coloring η' of G with $\delta_g(G)$ colors in C. In η', we have $\alpha(x) \geqslant g(x)$ for each $\alpha \in C$ and each vertex $x \in V(G)$. So G is of g_c-class 1. $\square$

Similarly, we can get the following result.

Corollary 12. *Let $G(X, Y; u)$ be a nearly bipartite graph associated with a non-negative integer function $g\colon V(G) \to \mathbb{N}$ and let $\delta_g(G) < \infty$. Let $i, j \in \mathbb{Z}_{\geqslant 0}$. If $d_X(u) = i\delta_g(G)$ or $d_Y(u) = j\delta_g(G)$, then $G(X, Y; u)$ is of g_c-class 1.*

Proof. If $g(u) = 0$, by Theorem 10 we know that G is of g_c-class 1. So let us consider the case $g(u) \geqslant 1$. Without loss of generality, we can suppose $d_X(u) = i\delta_g(G)$. Clearly $d(u) \geqslant \delta_g(G)g(u)$ by the definition of $\delta_g(G)$. It is easy to see that $d_Y(u) = d(u) - d_X(u) \geqslant (g(u) - i)\delta_g(G)$. Thus, $\lfloor d_X(u)/\delta_g(G)\rfloor + \lfloor d_Y(u)/\delta_g(G)\rfloor \geqslant g(u)$. By Theorem 11, G is of g_c-class 1. $\square$

4. Concluding remarks

In Theorem 8, the condition $d^*_g(u) \leqslant d(u) - \delta_g(G) + 1$ is sharp. Consider the nearly bipartite graph $G(X, Y; u)$ in Figure 2. It is easy to see that $\delta_g(G) = 2$, $V_{\delta_g}(G) = V(G)$ and $d^*_g(u) = d(u) - \delta_g(G) + 2$. Suppose that G is of g_c-class 1, then G has a g_c-coloring η with two colors in $C = \{\alpha, \beta\}$. In η, the number of edges colored with α is $(1 \times 6 + 3)/2 = 9/2$, which is not an integer. This contradicts our assumption. So G is of g_c-class 2. Thus, the condition $d^*_g(u) \leqslant d(u) - \delta_g(G) + 1$ in Theorem 8 is sharp.

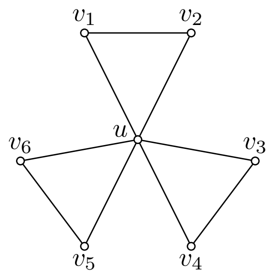

Figure 2. A nearly bipartite graph $G(X,Y;u)$, where $X = \{v_1, v_3, v_5\}$, $Y = \{v_2, v_4, v_6\}$, $g(u) = 3$ and $g(v_i) = 1$, $1 \leqslant i \leqslant 6$.

Let $G(X,Y;u)$ be a nearly bipartite graph. Theorem 7 (1) says that, when $\delta_g(G) \leqslant \operatorname{sur}(u)+1$, G is of g_c-class 1. Theorem 8 implies that, when $\delta_g(G) \leqslant \operatorname{sur}(u) + 1 + g(u)\delta_g(G) - d_g^*(u)$, G is of g_c-class 1. So when $g(u)\delta_g(G) > d_g^*(u)$, Theorem 8 is stronger than Theorem 7 (1).

When $g(v) = 1$ for all $v \in V(G)$, we can get the following result by Theorem 8.

Corollary 13. *Let $G(X,Y;u)$ be a nearly bipartite graph. If $d^*(u) \leqslant d(u) - \delta(G) + 1$, then $G(X,Y;u)$ is of* CI *class.*

It is easy to see that Corollary 13 strictly generalizes Theorem 5 (2) when $u \notin V_\delta(G)$ and it generalizes Theorem 5 (3). (Corollary 13 removed the restrictions that $N^*(u) \subseteq X$ or $N^*(u) \subseteq Y$).

When $g(v) = 1$ for all $v \in V(G)$, we can get the following result by Theorem 11.

Corollary 14. *Let $G(X,Y;u)$ be a nearly bipartite graph. If $d_X(u) \geqslant \delta(G)$ or $d_Y(u) \geqslant \delta(G)$, then $G(X,Y;u)$ is of* CI *class.*

If $d(u) \geqslant 2\delta(G) - 1$, we must have $d_X(u) \geqslant \delta(G)$ or $d_Y(u) \geqslant \delta(G)$, but not vice versa. So, Corollary 14 generalizes Theorem 5 (1).

Acknowledgment. The authors would like to thank the referees for their helpful suggestions and carefully reading the manuscript.

References

[1] *J. A. Bondy, U. S. R. Murty*: Graph Theory with Applications. Macmillan Press, London, 1976. zbl MR
[2] *R. P. Gupta*: On decompositions of multi-graph into spanning subgraphs. Bull. Am. Math. Soc. *80* (1974), 500–502. zbl MR doi
[3] *I. Holyer*: The NP-completeness of edge coloring. SIAM J. Comput. *10* (1981), 718–720. zbl MR doi
[4] *J. Li, G. Liu*: On f-edge cover coloring of nearly bipartite graphs. Bull. Malays. Math. Sci. Soc. (2) *34* (2011), 247–253. zbl MR

[5] *S. Nakano, T. Nishizeki*: Scheduling file transfers under port and channel constrains. Int. J. Found. Comput. Sci. *4* (1993), 101–115. zbl MR doi
[6] *H. Song, G. Liu*: On f-edge cover-coloring of simple graphs. Acta Math. Sci., Ser. B, Engl. Ed. *25* (2005), 145–151. zbl MR
[7] *H. Song, G. Liu*: f-edge cover-coloring of graphs. Acta Math. Sin. *48* (2005), 919–928. (In Chinese.) zbl MR
[8] *J. Wang, X. Zhang, G. Liu*: Edge covering coloring of nearly bipartite graphs. J. Appl. Math. Comput. *22* (2006), 435–440. zbl MR doi
[9] *C. Xu, Y. Jia*: A note on edge-cover coloring of nearly bipartite graphs. Ars Comb. *91* (2009), 423–427. zbl MR
[10] *X. Zhang*: The correlation between the f-chromatic class and the g_c-chromatic class of a simple graph. Ars Comb. *135* (2017), 17–28. MR
[11] *X. Zhang*: Vertex splitting for determining the f-chromatic class of simple graphs. Submitted.

Authors' address: Yuzhuo Zhang, Xia Zhang (corresponding author), School of Mathematics and Statistics, Shandong Normal University, 88 Wenhua E Rd, Lixia, Jinan 250358, P. R. China, e-mail: `pandarhz@sina.com`.