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THE WEIGHTED HARDY SPACES ASSOCIATED TO SELF-ADJOINT OPERATORS AND THEIR DUALITY ON PRODUCT SPACES

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Abstract. Let L be a non-negative self-adjoint operator acting on $L^2(\mathbb{R}^n)$ satisfying a pointwise Gaussian estimate for its heat kernel. Let w be an A_r weight on $\mathbb{R}^n \times \mathbb{R}^n$, $1 < r < \infty$. In this article we obtain a weighted atomic decomposition for the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 < p \leqslant 1$ associated to L. Based on the atomic decomposition, we show the dual relationship between $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathrm{BMO}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$.

Keywords: weighted Hardy space; operator; Gaussian estimate; duality; product space

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1. Introduction

The theory of Hardy spaces has been a central part of modern harmonic analysis. The theory of Hardy spaces on product domains was initiated by Gundy and Stein in [23]. The atomic decompositions for Hardy spaces on product domains were obtained by Chang and Fefferman in [11], [13]. Later, Fefferman in [21], Krug in [29], Sato in [36] and others established the weighted theory of the classical Hardy spaces on product domains. See [42], [12], [8] for more results on product domain.

The classical Hardy spaces on $\mathbb{R}^n$ can be characterized by certain estimates via the Laplacian, but there are some important situations in which the theory of classical Hardy spaces is not applicable. Some interesting operators were found to be out of the Calderón-Zygmund class. See [17], [2], [25], [4], [5], [6]. In order to handle such beyond Calderón-Zygmund operators, many researchers study Hardy spaces that

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are adapted to a linear operator L. For some results on the one-parameter Hardy spaces adapted to an operator, we refer the reader to [3], [20], [7], [25], [19], [26], [24], [9], [16], [38], [28] and their references. For the theory of weighted Hardy spaces associated with an operator, we refer the reader to a series of papers [1], [39], [31], [33], [34].

In [32], the authors studied the atomic decomposition of weighted Hardy spaces $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ associated to self-adjoint operators on product spaces. Inspired by the work [32], it is natural to study the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 < p < 1$. Let L be a non-negative self-adjoint operator acting on $L^2(\mathbb{R}^n)$ satisfying a pointwise Gaussian estimate for its heat kernel. Let w be an A_r weight on $\mathbb{R}^n \times \mathbb{R}^n$ (see Section 2 for its definition), where $1 < r < \infty$. The main purpose of this paper is to introduce a weighted Hardy space $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ and prove a weighted atomic decomposition for $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$. See Theorem 4.1. And then, we introduce weighted BMO spaces $\mathrm{BMO}^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ on product spaces. Using the atomic decomposition for $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$, we show that the dual space of the weighted Hardy space $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ is $\mathrm{BMO}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$. See Theorem 5.2. When L is the Laplacian of $\mathbb{R}^n$, the weighted atomic decompositions in product spaces were obtained in [29], [30]. Here, we generalize the results of [29], [30].

Since there is no Whitney decomposition in product domains, the situation becomes considerably complicated. As pointed out by [10], the product atoms should be supported in open sets rather than rectangles, which leads to many difficulties. The important tool is a weighted version of Journé's covering lemma, which is a good substitute in product domains for Whitney decomposition. It will be used to prove (i) of Theorem 4.1.

The layout of the article is as follows. In Section 2, we introduce some basic assumptions and state some preliminary results. In Section 3, we define the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 < p \leqslant 1$ associated to a non-negative self-adjoint operator with Gaussian upper bounds on its heat kernel. We give an atomic characterization of the weighted Hardy spaces on the product domain associated to the operator. In Section 4, we state and prove the relationship between $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathrm{BMO}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$, which is the main result of this paper.

Throughout this article, the letter "C" or "c" will denote (possibly different) constants that are independent of the essential variables.

2.1. Assumption (H). Assume that L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ and that each of the semigroups e^{-tL}, generated by $-L$ on $L^2(\mathbb{R}^n)$, has the kernel $p_t(x,y)$ which satisfies the Gaussian upper bound. That is, there exist constants $C, c > 0$ such that

$$\text{(GE)} \qquad |p_t(x,y)| \leqslant \frac{C}{t^{n/2}} \exp\Big(-\frac{|x-y|^2}{ct}\Big).$$

We note that such estimates are typical for elliptic or sub-elliptic differential operators of second order (see for example, [14] and [18]).

Lemma 2.1. *Let L be an operator satisfying the assumption* (H). *For every $k = 0, 1, \ldots$, there exist two positive constants C_k, c_k such that the kernel $p_{t,k}(x,y)$ of the operator $(t^2L)^k \mathrm{e}^{-t^2L}$ satisfies*

$$(2.1) \qquad |p_{t,k}(x,y)| \leqslant \frac{C_k}{(4\pi t)^n} \exp\Big(-\frac{|x-y|^2}{c_k t^2}\Big)$$

for all $t > 0$ and almost every $x, y \in \mathbb{R}^n$.

Proof. For the proof, we refer the reader to [14] and [35], Theorem 6.17. □

2.2. Muckenhoupt weights. We review some needed background on Muckenhoupt weights.

A weight w is a non-negative locally integrable function on $\mathbb{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subseteq \mathbb{R}^n$,

$$\Big(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B w^{-1/(p-1)}(x)\,\mathrm{d}x\Big)^{p-1} \leqslant C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subseteq \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w(y)\,\mathrm{d}y \leqslant Cw(x) \quad \text{for a.e. } x \in B.$$

We denote the Hardy-Littlewood maximal function by

$$\mathcal{M}(f)(x) := \sup_{x \in B} |B|^{-1} \int_B |f(y)|\,\mathrm{d}y.$$

We sum up some of the properties of the above classes.

Proposition 2.2. *The following assertions hold:*

(i) $A_1 \subseteq A_p \subseteq A_q$ *for* $1 \leqslant p \leqslant q < \infty$.

(ii) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $1 < q < p$ *such that* $w \in A_q$.

(iii) $A_\infty = \bigcup\limits_{1 \leqslant p < \infty} A_p$.

(iv) *If* $1 < p < \infty$, $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.

(v) *Let* $w \in A_p$, $p \geqslant 1$. *Then for any ball* B *and* $\lambda > 1$, *there exists a constant* C *(independent of* B *and* λ*) such that*

$$w(\lambda B) \leqslant C\lambda^{np} w(B).$$

(vi) *If* $1 < p < \infty$, $w \in A_p$, *then*

$$\int_{\mathbb{R}^n} (\mathcal{M}(f)(x))^p w(x)\,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x.$$

Proof. Properties (i)–(vi) are standard, see for instance [22] and [40]. □

2.3. Muckenhoupt weights on product spaces. In this section, we recall some basic facts on the product $A_p(\mathbb{R}^n \times \mathbb{R}^n)$.

A non-negative locally integrable function $w(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $A_p(\mathbb{R}^n \times \mathbb{R}^n)$, $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|R|}\iint_R w(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)\left(\frac{1}{|R|}\iint_R w(x,y)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}y\right)^{p-1} \leqslant C,$$

where R runs over all rectangles in $\mathbb{R}^n \times \mathbb{R}^n$. When $p = 1$, we say that $w \in A_1(\mathbb{R}^n \times \mathbb{R}^n)$ if there exists a constant C such that

$$\frac{1}{|R|}\iint_R w(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant C \operatorname*{ess\,inf}_{(x,y)\in R} w(x,y) \quad \text{for all } R.$$

We also define $A_\infty(\mathbb{R}^n \times \mathbb{R}^n) := \bigcup\limits_{r \geqslant 1} A_r(\mathbb{R}^n \times \mathbb{R}^n)$. For any $1 \leqslant p < \infty$, the weighted Lebesgue spaces $L^p_w(\mathbb{R}^n \times \mathbb{R}^n)$ can be defined by

$$\left\{f\colon \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x,y)|^p w(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty\right\}$$

with the norm

$$\|f\|_{L^p_w} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x,y)|^p w(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}.$$

We denote the strong maximal function by

$$\mathcal{M}_s(f)(x,y) := \sup_{(x,y)\in R} |R|^{-1} \iint_R |f(u,v)|\,\mathrm{d}u\,\mathrm{d}v,$$

where R is any rectangle in $\mathbb{R}^n \times \mathbb{R}^n$.

The following properties of product A_p weights are well known, see [36], [30].

Proposition 2.3. *The following assertions hold:*

(i) *If $w(x,y) \in A_r(\mathbb{R}^n \times \mathbb{R}^n)$, then $w(x,\cdot)$ satisfies $A_r(\mathbb{R}^n)$ condition uniformly for a.e. x; similarly for $w(\cdot,y)$.*

(ii) *Product weights satisfy $A_1 \subseteq A_r \subseteq A_t \subseteq A_\infty$ for $1 \leqslant r \leqslant t < \infty$.*

(iii) *If $w \in A_r(\mathbb{R}^n \times \mathbb{R}^n)$, $1 < r < \infty$, there exists $1 < s < r$ such that $w \in A_s(\mathbb{R}^n \times \mathbb{R}^n)$.*

(iv) *If $1 < r < \infty$, $w \in A_r(\mathbb{R}^n \times \mathbb{R}^n)$ if and onl if $w^{1-r'} \in A_{r'}(\mathbb{R}^n \times \mathbb{R}^n)$.*

(v) *If $1 < r < \infty$, $w \in A_r(\mathbb{R}^n \times \mathbb{R}^n)$, then*

$$\Big\| \Big(\sum_k \mathcal{M}_s^2(f_k) \Big)^{1/2} \Big\|_{L^r_w} \leqslant C \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_{L^r_w}.$$

3. Weighted product Hardy spaces and atoms associated to operators

3.1. Weighted product Hardy spaces and weighted product atoms. Given a function f on $\mathbb{R}^n \times \mathbb{R}^n$, $K = [n/2] + 1$, let us use Q_{t_1,t_2} to denote the operator $(t_1^2 L)^K \mathrm{e}^{-t_1^2 L} \otimes (t_2^2 L)^K \mathrm{e}^{-t_2^2 L}$. The area function associated with operators L is defined by

$$S_{\alpha_1,\alpha_2} f(x) := \Bigg(\iint_{\substack{|x_1-y_1|<\alpha_1 t_1 \\ |x_2-y_2|<\alpha_2 t_2}} |Q_{t_1,t_2} f(y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t_1^{n+1} t_2^{n+1}} \Bigg)^{1/2}, \tag{3.1}$$

where $\alpha_i > 0$, $x = (x_1,x_2)$, $y = (y_1,y_2)$, $t = (t_1,t_2)$ and $x_i, y_i \in \mathbb{R}^n$, $t_i \in (0,\infty)$ for $i = 1,2$. Here, we replace $(t_i^2 L)\mathrm{e}^{-t_i^2 L}$ by $(t_i^2 L)^K \mathrm{e}^{-t_i^2 L}$ for technical reasons. For simplicity, we write $S(f) := S_{1,1}(f)$. For the theory of unweighted Hardy spaces associated to the operator L on product domains, we refer to [15], [37].

Definition 3.1. Let L be an operator satisfying the assumption (H) and $0 < p \leqslant 1$. Suppose that $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$. The weighted product Hardy space $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ associated to L is defined as the completion of

$$\{f \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \colon \|S(f)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^n)} < \infty\}$$

with respect to the norm

$$\|f\|_{H^p_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)} = \|S(f)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Remark 3.2. An argument similar to Theorem 1 of Chapter IV in [41] implies the following result. Suppose $w \in A_r(\mathbb{R}^n \times \mathbb{R}^n)$, $1 \leqslant r < \infty$. Then for all $\alpha_1 \geqslant 1$, $\alpha_2 \geqslant 1$, there exist positive constants $c_{\alpha_1,\alpha_2,p}$ and $C_{\alpha_1,\alpha_2,p}$ such that

$$c_{\alpha_1,\alpha_2,p}\|S(f)\|_{L^1_w(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant \|S_{\alpha_1,\alpha_2}(f)\|_{L^1_w(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C_{\alpha_1,\alpha_2,p}\|S(f)\|_{L^1_w(\mathbb{R}^n\times\mathbb{R}^n)}.$$

We now introduce the product (p, r, M, w)-atom associated to operators.

Definition 3.3. Suppose that $M \in \mathbb{N}$, $0 < p \leqslant 1$ and $w \in A_r$, $1 \leqslant r < \infty$. A function $a \in L^2(\mathbb{R}^{2n})$ is called a product (p, r, M, w)-atom associated to L, if it satisfies

(1) $\operatorname{supp} a \subseteq \Omega$, where Ω is an open set with finite measure;

(2) a can be further decomposed into $a = \sum_{R\in m(\Omega)} a_R$, where for each $R \in m(\Omega)$ there exists a function $b_R \in \mathcal{D}(L^M \otimes L^M)$ such that

  (i) $a_R = (L^M \otimes L^M)b_R$;

  (ii) $\operatorname{supp}(L^{k_1} \otimes L^{k_2})b_R \subseteq 10R$, $k_1, k_2 = 0, 1, \dots, M$;

  (iii) $\|a\|_{L^r_w(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant w(\Omega)^{1/r-1/p}$ and for $k_1,\ k_2 = 0, 1, \dots, M$,

$$\sum_{R\in m(\Omega)} \ell(I)^{-2rM}\ell(J)^{-2rM}\|(\ell(I)^2L)^{k_1} \otimes (\ell(J)^2L)^{k_2}b_R\|^r_{L^r_w(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant w(\Omega)^{1-r/p},$$

where $m(\Omega)$ denotes the set of maximal dyadic subrectangles of Ω; $L^0 := \mathbb{I}$ denotes the identity operator on $\mathbb{R}^n$; $R = I \times J$ denotes the dyadic rectangle of $\mathbb{R}^n \times \mathbb{R}^n$, I, J denote the dyadic cube of $\mathbb{R}^n$; $10R$ denotes the 10-fold dilate of R concentric with R.

Now we can define a weighted product Hardy space $H^{p,r,M}_{L,w}$ by atoms.

Definition 3.4. Let M and w be the same as in Definition 3.3. The weighted product Hardy space $H^{p,r,M}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ is defined as follows. We say that $f = \sum_j \lambda_j a_j$ is a product atomic (p, r, M, w)-representation of f if $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, each a_j is a product (p, r, M, w)-atom, and the sum converges in $L^2(\mathbb{R}^{2n})$. Set

$$\mathbb{H}^{p,r,M}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n) = \{f\colon\ f \text{ has a product atomic } (p, r, M, w)\text{-representation}\},$$

with the norm given by

$$\|f\|_{\mathbb{H}^{p,r,M}_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)} = \inf\bigg\{\bigg(\sum_{j=0}^\infty |\lambda_j|^p\bigg)^{1/p}\colon\ f = \sum_j \lambda_j a_j \text{ has a product atomic } (p, r, M, w)\text{-representation}\bigg\}.$$

The space $H^{p,r,M}_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$ is then defined as the completion of $\mathbb{H}^{p,r,M}_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$ with respect to this norm.

3.2. Some auxiliary lemmas. For any $\alpha>0$, let us introduce the product cone $\Gamma(x):=\Gamma(x_1)\times\Gamma(x_2)$, where $\Gamma(x_i):=\{(y_i,t_i)\in\mathbb{R}^{n+1}_+ \colon |x_i-y_i|<t_i\}$, $i=1,2$. Suppose $\Omega\subseteq\mathbb{R}^n\times\mathbb{R}^n$ is an open set of finite measure. Denote by $m(\Omega)$ the maximal dyadic subrectangles of Ω. Let $m_1(\Omega)$ denote those dyadic subrectangles $R\subseteq\Omega$, $R=I\times J$ which are maximal in the x_1 direction. In other words, if $S=I'\times J\supseteq R$ is a dyadic subrectangle of Ω, then $I=I'$. Define $m_2(\Omega)$ similarly.

In order to prove our main result, we need some auxiliary results.

Lemma 3.5. *Let $w\in A_\infty$ and let Ω be an open set of $\mathbb{R}^n\times\mathbb{R}^n$. For any $R=I\times J\in m_2(\Omega)$, we set*

$$\gamma_1(R):=\sup_{\substack{I\subseteq I'\\ I'\times J\subseteq\Omega^*}}|I'|/|I|,$$

where $\Omega^*=\{x\in\mathbb{R}^n\times\mathbb{R}^n\colon \mathcal{M}_s(\chi_\Omega)(x)>1/2\}$. Then for any $\delta>0$,

$$\sum_{R\in m_2(\Omega)} w(R)\gamma_1^{-\delta}(R)\leqslant c_\delta w(\Omega),$$

where c_δ is a constant depending only on δ, not on Ω.

Similarly, for any $R=I\times J\in m_1(\Omega)$, we set

$$\gamma_2(R):=\sup_{\substack{J\subseteq J'\\ I\times J'\subseteq\Omega^*}}|J'|/|J|.$$

Then for any $\delta>0$,

$$\sum_{R\in m_1(\Omega)} w(R)\gamma_2^{-\delta}(R)\leqslant c_\delta w(\Omega),$$

where c_δ is a constant depending only on δ, not on Ω.

Proof. For the proof, we refer to [21], Lemma 2, and [27]. In fact, this is a weighted version of Journé's lemma. □

We also need the following results.

Lemma 3.6. *Let L be an operator satisfying the assumption* (H). *Then for any $1<r<\infty$ and $w\in A_r(\mathbb{R}^n\times\mathbb{R}^n)$, there exist constants C_1,C_2 such that*

$$C_1\|f\|_{L^r_w(\mathbb{R}^n\times\mathbb{R}^n)}\leqslant\|S(f)\|_{L^r_w(\mathbb{R}^n\times\mathbb{R}^n)}\leqslant C_2\|f\|_{L^r_w(\mathbb{R}^n\times\mathbb{R}^n)}\tag{3.2}$$

holds for all $f\in L^r_w(\mathbb{R}^n\times\mathbb{R}^n)$.

P r o o f. For the details of the proof, we refer the reader to [32]. □

Lemma 3.7. *Fix $M \in \mathbb{N}$ and $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $0 < p \leqslant 1$. Suppose that T is a non-negative sublinear operator satisfying the weak-type $(2,2)$ bound*

$$\omega\{x \in \mathbb{R}^n \times \mathbb{R}^n \colon |Tf(x)| > \eta\} \leqslant C_T \eta^{-2} \|f\|^2_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}, \qquad \eta > 0,$$

and for every product (p,r,M,w)-atom a we have

$$\|Ta\|_{L^p_w(\mathbb{R}^n \times \mathbb{R}^n)} \leqslant C,$$

with constant C independent of a. Then T is bounded from $H^{p,r,M}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ to $L^p_w(\mathbb{R}^{2n})$, and

$$\|Tf\|_{L^p_w(\mathbb{R}^{2n})} \leqslant C\|f\|_{H^{p,r,M}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)}.$$

P r o o f. The proof is similar to that of Lemma 4.3 in [24] and so we skip it here. □

4. The atomic characterizations of weighted product Hardy spaces

In this section we will state and prove the atomic characterizations of weighted product Hardy spaces $H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 < p \leqslant 1$.

Theorem 4.1. *Suppose that $w \in A_s(\mathbb{R}^n \times \mathbb{R}^n)$, $1 < s < \infty$ and $0 < p \leqslant 1$.*

(i) *Suppose that $M \in \mathbb{N}$, $M > (r/p - 1)n/2$ and $r \geqslant s$. Let $f = \sum\limits_{i=0}^{\infty} \lambda_i a_i$, where $\{\lambda_i\}_{i=0}^\infty \in l^p$, the a_i are product (p,r,M,w)-atoms, and the sum converges in $L^2(\mathbb{R}^{2n})$. Then $f \in H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^{2n})$ and*

$$\Big\|\sum_{i=0}^{\infty} \lambda_i a_i\Big\|_{H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)} \leqslant C\Big(\sum_{i=0}^{\infty} |\lambda_i|^p\Big)^{1/p}.$$

(ii) *Let $M \in \mathbb{N}$ and $1 < r < \infty$. If $f \in H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^{2n})$, then there exist a family of product (p,r,M,w)-atoms $\{a_i\}_{i=0}^\infty$ and a sequence of numbers $\{\lambda_i\}_{i=0}^\infty$ such that f can be represented in the form $f = \sum\limits_{i=0}^{\infty} \lambda_i a_i$, and the sum converges in the sense of $L^2(\mathbb{R}^{2n})$-norm. Moreover,*

$$\Big(\sum_{i=0}^{\infty} |\lambda_i|^p\Big)^{1/p} \leqslant C\|f\|_{H^p_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)}.$$

The proof of (ii) in Theorem 4.1 is very similar to the case $p=1$ considered by [32] (in fact, one just needs to set $\lambda_i := c_\Psi 2^i w(\Omega_i)^{1/p}$ to get the result), we do not discuss it here. In this paper, we give the details of the proof of (i).

Proof of (i) of Theorem 4.1. Let $f(x) = \sum_j \lambda_j a_j(x)$, where the a_j are product (p,r,M,w)-atoms and $\Big(\sum_{i=0}^{\infty} |\lambda_i|^p\Big)^{1/p} < \infty$. By applying Lemma 3.7 and the fact that a (p,s,M,w)-atom is also a (p,r,M,w)-atom for $s \geqslant r > 1$, it shall be enough to show that for every product (p,r,M,w)-atom a there exists a constant C (independent of a) such that

$$\|S(a)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C.$$

By the assumption on M, we can choose N such that $n(r/p-1) < N < 2M$. We assume that $a = \sum_{R\in m(\Omega)} a_R$ is supported in some open set Ω with finite measure. For any $R = I \times J \subseteq \Omega$, let I' be the biggest dyadic cube containing I such that

$$I' \times J \subseteq \widetilde{\Omega} = \{x \in \mathbb{R}^n \times \mathbb{R}^n \colon \mathcal{M}_s(\chi_\Omega)(x) > 1/2\}.$$

Then $I' \times J$ is in $m_1(\widetilde{\Omega})$ and let S be the biggest dyadic cube such that $S \supseteq J$ and $I' \times S \subseteq \widetilde{\widetilde{\Omega}}$, where $\widetilde{\widetilde{\Omega}} = \{x \in \mathbb{R}^n \times \mathbb{R}^n \colon \mathcal{M}_s(\chi_{\widetilde{\Omega}})(x) > 1/2\}$. Let $\breve{R}$ be the 10-fold dilate of $I' \times S$ concentric with $I' \times S$. By (v) of Proposition 2.3, we have $w(\bigcup \breve{R}) \leqslant Cw(\Omega)$.

Making use of Hölder's inequality and the definition of product atoms, we have

$$\begin{aligned}\int_{\bigcup \breve{R}} S(a)^p(x) w(x)\,\mathrm{d}x &\leqslant \int_{\bigcup \breve{R}} S(a)(x) w^{p/r}(x) w^{1-p/r}(x)\,\mathrm{d}x\\ &\leqslant C\Big(\int_{\bigcup \breve{R}} w\,\mathrm{d}x\Big)^{1-p/r} \|S(a)\|^p_{L^r_w}\\ &\leqslant Cw(\Omega)^{1-p/r}\|a\|^p_{L^r_w}\\ &\leqslant Cw(\Omega)^{1-1/p} w(\Omega)^{(1/r-1/p)p}\\ &\leqslant C,\end{aligned}$$

where in the third inequality we have used Lemma 3.6.

It remains to prove

$$\int_{(\bigcup \breve{R})^c} S(a)^p(x) w(x)\,\mathrm{d}x \leqslant C. \tag{4.1}$$

By the definition of product atoms, one can write

$$\begin{aligned}\tag{4.2} \int_{(\bigcup \breve{R})^c} S(a)^p(x)w(x)\,\mathrm{d}x &\leqslant \sum_{R\in m(\Omega)} \int_{\breve{R}^c} S(a_R)^p(x)w(x)\,\mathrm{d}x \\ &\leqslant \sum_{R\in m(\Omega)} \int_{(10I')^c\times\mathbb{R}^n} S(a_R)^p(x)w(x)\,\mathrm{d}x \\ &\quad + \sum_{R\in m(\Omega)} \int_{\mathbb{R}^n\times(10S)^c} S(a_R)^p(x)w(x)\,\mathrm{d}x \\ &=: \mathrm{D}+\mathrm{E}.\end{aligned}$$

We only estimate the term D since the term E can be estimated similarly.

For the term D, we have

$$\begin{aligned}\int_{(10I')^c\times\mathbb{R}^n} S(a_R)^p(x)w(x)\,\mathrm{d}x &= \int_{(10I')^c\times(10J)^c} S(a_R)^p(x_1,x_2)w(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad + \int_{(10I')^c\times 10J} S(a_R)^p(x_1,x_2)w(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &=: \mathrm{D}_1+\mathrm{D}_2.\end{aligned}$$

By the same analysis as that of Theorem 4.1 in [32], we know that

$$\begin{aligned}\tag{4.3} \mathrm{D}_1 &\leqslant C\Big(\frac{|R|}{w(R)^{1/p}}\|c_R\|_{L^p_w}\Big)^p \\ &\quad\times \int\limits_{(10I')^c\times(10J)^c} w(x)\frac{\ell(I)^{pN}}{|x_1-x_I|^{p(n+N)}}\frac{\ell(J)^{pN}}{|x_2-x_J|^{p(n+N)}}\,\mathrm{d}x\end{aligned}$$

where $c_R := |a_R| + \ell(J)^{-2M}|a_{R,1}| + \ell(I)^{-2M}|a_{R,2}| + \ell(I)^{-2M}\ell(J)^{-2M}|b_R|$, and we write $a_{R,1} := (L^M\otimes \mathbb{I})b_R$ and $a_{R,2} := (\mathbb{I}\otimes L^M)b_R$.

Noting that $N > n(r/p-1)$, one can compute

$$\begin{aligned}\tag{4.4} &\int_{(10I')^c\times(10J)^c} w(x)\frac{\ell(I)^{pN}}{|x_1-x_I|^{p(n+N)}}\frac{\ell(J)^{pN}}{|x_2-x_J|^{p(n+N)}}\,\mathrm{d}x \\ &\leqslant C\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\int_{2^{l_1}I'\setminus 2^{l_1-1}I'}\int_{2^{l_2}J\setminus 2^{l_2-1}J} w(x_1,x_2)\frac{\ell(I)^{pN}}{|x_1-x_I|^{p(n+N)}} \\ &\qquad\qquad\times\frac{\ell(J)^{pN}}{|x_2-x_J|^{p(n+N)}}\,\mathrm{d}x_2\,\mathrm{d}x_1\end{aligned}$$

$$\leqslant C\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\frac{\ell(I)^{pN}}{(2^{l_1}\ell(I'))^{p(n+N)}}\frac{\ell(J)^{pN}}{(2^{l_2}\ell(J))^{p(n+N)}}w(2^{l_1}I'\times 2^{l_2}J)$$
$$\leqslant C\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\frac{1}{2^{l_1(p(n+N)-rn)}}\frac{1}{2^{l_2(p(n+N)-rn)}}\frac{\ell(I)^{N}}{\ell(I')^{n+N}}w(R)|J|^{-p}$$
$$\leqslant C\Big(\frac{\ell(I)}{\ell(I')}\Big)^{pN}w(R)|R|^{-p},$$

where in the third inequality above we have used (v) of Proposition 2.2.

Combining (4.3) and (4.4), we write

$$\mathrm{D}_1\leqslant C\gamma_1(R)^{-pN}w(R)^{1-p/r}\|c_R\|^p_{L^r_w}, \tag{4.5}$$

where $\gamma_1(R)=\ell(I')/\ell(I)$.

We turn to estimating the term D_2. By Hölder's inequality, we have

$$\mathrm{D}_2\leqslant\sum_{k=3}^{\infty}\int_{2^{k+1}I'\backslash 2^kI'}\int_{10J}|S(a_R)(x_1,x_2)|^p w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$$
$$\leqslant\sum_{k=3}^{\infty}\Big(\int_{2^{k+1}I'\backslash 2^kI'}\int_{10J}|S(a_R)(x_1,x_2)|^r w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\Big)^{p/r}$$
$$\times\Big(\int_{2^{k+1}I'}\int_{10J}w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\Big)^{1-p/r}.$$

We have

$$\int_{2^{k+1}I'\backslash 2^kI'}\int_{10J}|S(a_R)(x_1,x_2)|^r w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \tag{4.6}$$
$$\leqslant C\int_{2^{k+1}I'\backslash 2^kI'}\int\Big(\iint_{\Gamma(x_1)}|(t_1^2L)^K\mathrm{e}^{-t_1^2L}a_R(y_1,x_2)|^2\frac{\mathrm{d}y_1\,\mathrm{d}t_1}{t_1^{n+1}}\Big)^{r/2}$$
$$\times w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1.$$

By an argument similar to that of $\mathrm{D}_{1,i}$, $i=1,\dots,4$ we obtain

$$\iint_{\Gamma(x_1)}|(t_1^2L)^K\mathrm{e}^{-t_1^2L}a_R(y_1,x_2)|^2\frac{\mathrm{d}y_1\,\mathrm{d}t_1}{t_1^{n+1}}$$
$$\leqslant C\frac{\ell(I)^{2N}}{|x_1-x_I|^{2(n+N)}}\Big[\Big(\int|a_R(z_1,x_2)|\,\mathrm{d}z_1\Big)^2+\Big(\ell(I)^{-2M}\int|a_{R,2}(z_1,x_2)|\,\mathrm{d}z_1\Big)^2\Big],$$

since $2M>N$.

Let $\mathcal{M}^{(1)}$ denote the Hardy–Littlewood maximal operator in the first variable. Then we have

$$\begin{aligned}\int |c_R(z_1,x_2)|\,\mathrm{d}z_1 &= \frac{|2^{k+1}I'|}{|2^{k+1}I'|}\int_{2^{k+1}I'} |c_R(z_1,x_2)|\,\mathrm{d}z_1\\ &\leqslant |2^{k+1}I'| \inf_{z_1\in 2^{k+1}I'} \mathcal{M}^{(1)}c_R(z_1,x_2).\end{aligned}$$

Thus, one can write

$$\begin{aligned}(4.7)\quad &\Big(\int_{2^{k+1}I'\setminus 2^kI'}\int_{10J} |S(a_R)(x)|^r w(x)\,\mathrm{d}x\Big)^{1/r}\\ &\leqslant C\frac{\ell(I)^N 2^{kn}|I'|}{(2^k\ell(I'))^{n+N}}\Big(\int_{2^{k+1}I'}\int\Big[\inf_{z_1\in 2^{k+1}I'}\mathcal{M}^{(1)}c_R(z_1,x_2)\Big]^r\\ &\qquad\qquad \times w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\Big)^{1/r}\\ &\leqslant C\gamma_1(R)^{-N}2^{-kN}\Big(\iint [\mathcal{M}^{(1)}c_R(x_1,x_2)]^r w(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\Big)^{1/r}\\ &\leqslant C\gamma_1(R)^{-N}2^{-kN}\|c_R\|_{L^r_w},\end{aligned}$$

where in the third inequality above we have used (vi) of Proposition 2.2 and (i) of Proposition 2.3.

By virtue of (v) of Proposition 2.2 we have

$$(4.8)\qquad \Big(\int_{2^{k+1}I'}\int_{10J} w(x)\,\mathrm{d}x\Big)^{1-p/r} \leqslant C2^{(k+1)n(r-p)}w(R)^{1-p/r}.$$

Combining (4.7) and (4.8), we have

$$\begin{aligned}\mathrm{D}_2 &\leqslant C\sum_{k=3}^{\infty} 2^{-k(pN-n(r-p))}\|c_R\|^p_{L^r_w}\gamma_1(R)^{-pN}w(R)^{1-p/r}\\ &\leqslant C\|c_R\|^p_{L^r_w}\gamma_1(R)^{-pN}w(R)^{1-p/r},\end{aligned}$$

since $N > n(r/p-1)$.

Estimates of D_1 and D_2, together with Hölder's inequality and weighted Journé's covering lemma (Lemma 3.5), imply that

$$\begin{aligned}\mathrm{D} &\leqslant C\sum_{R\in m(\Omega)} \|c_R\|^p_{L^r_w}\gamma_1(R)^{-pN}w(R)^{1-p/r}\\ &\leqslant C\Big(\sum_{R\in m(\Omega)}\|c_R\|^r_{L^r_w}\Big)^{p/r}\Big(\sum_{R\in m(\Omega)} w(R)\gamma_1(R)^{-Np(r-p)/r}\Big)^{1-p/r}\\ &\leqslant Cw(\Omega)^{p/r-1}w(\Omega)^{1-p/r}\leqslant C.\end{aligned}$$

The desired estimate of (4.1) follows readily. This completes the proof of (i) of Theorem 4.1. □

5. $\mathrm{BMO}_{L,w}$: DUALITY WITH $H^1_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$ SPACES

In this section, we introduce and study the duality of the weighted Hardy spaces $H^1_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$. Given a real-valued function f on $\mathbb{R}^n\times\mathbb{R}^n$ and a weight $w\in A_s(\mathbb{R}^n\times\mathbb{R}^n)$, $1\leqslant s<\infty$, we consider the following situation: if Ω is a bounded open set in $\mathbb{R}^n\times\mathbb{R}^n$, and $R=I\times J$ runs over the collection of the maximal dyadic rectangles contained in Ω, then we introduce the following definition.

Definition 5.1. Let L be a non-negative self-adjoint operator such that the corresponding heat kernel satisfies conditions (GE). For $w\in A_s$, $1\leqslant s<\infty$ and $1\leqslant p<\infty$, an element $f\in L^2$ is said to belong to $\mathrm{BMO}^p_{L,w}$ if

$$\|f\|_{\mathrm{BMO}^p_{L,w}} =: \sup_{\Omega}\biggl(\frac{1}{w(\Omega)}\sum_{R\subset\Omega}\iint_{I\times J}|(\mathbb{I}-(1+\ell(I)^2L\otimes\ell(J)^2L)^{-1})^M f|^p w^{1-p}\,\mathrm{d}x\biggr)^{1/p}<\infty,$$

where the supremum ranges over the whole open set Ω of $\mathbb{R}^n\times\mathbb{R}^n$ with finite measure, and $\mathbb{I}$ denotes the identity operator on $\mathbb{R}^n\times\mathbb{R}^n$. In particular, for $p=1$ we denote $\mathrm{BMO}^1_{L,\omega}=:\mathrm{BMO}_{L,w}$.

We have the following theorem. In fact, it expresses the dual relationship of $\mathrm{BMO}_{L,w}$ and $H^1_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$.

Theorem 5.2. $H^{1,r,M}_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)^*=\mathrm{BMO}^{r'}_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$, $r\geqslant 1$.

Proof. First, we show that each $f\in\mathrm{BMO}^{r'}_{L,w}$ induces a bounded linear functional on $H^{1,r,M}_{L,w}(\mathbb{R}^n\times\mathbb{R}^n)$. Suppose that a is a $(1,r,M,w)$-atom in $H^{1,r,M}_{L,w}(\mathbb{R}^n)$, and let $f\in\mathrm{BMO}^{r'}_{L,w}(\mathbb{R}^n)$. For simplicity, denote $\mathbb{I}-(1+\ell(I)^2L\otimes\ell(J)^2L)^{-1})^M=:P_M$, then by the atomic definition we have

$$\begin{aligned}\biggl|\iint_{\mathbb{R}^n\times\mathbb{R}^n}a(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y\biggr| &\leqslant\sum_R\biggl|\iint_{10R}a_R(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y\biggr|\\ &\leqslant\sum_R\biggl|\iint_{10R}P_Ma_R(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y\biggr|\\ &\quad+\sum_R\biggl|\iint_{10R}(\mathbb{I}-P_M)a_R(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y\biggr|=:\mathrm{J}_1+\mathrm{J}_2.\end{aligned}$$

For the term J_1, by Hölder's inequality and the definitions of atom and $f \in \mathrm{BMO}^{r'}_{L,w}$ we obtain

$$\begin{aligned}\mathrm{J}_1 &\leqslant \Big(\sum_R \|a_R\|^r_{L^r_w}\Big)^{1/r} \sum_R \Big(\iint_{10R} |P_M f(x,y)|^{r'} w^{1-r'}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/r'}\\ &\leqslant C\|f\|_{\mathrm{BMO}^{r'}_{L,1/w}} w(B)^{1/r-1} w(B)^{1/r'}\\ &\leqslant C\|f\|_{\mathrm{BMO}^{r'}_{L,1/w}}.\end{aligned}$$

To analyse J_2, by the condition $a_R = (L^M \otimes L^M)b_R$ and the fact that L is self-adjoint, we write

$$\begin{aligned}&(\mathbb{I} - P_M)a_R = (\mathbb{I} - P_M)(L^M \otimes L^M)b_R = (L^M \otimes L^M)(\mathbb{I} - P_M)b_R\\ &= \sum_{k_1=1}^{M}\sum_{k_2=1}^{M} \frac{M!}{(M-k_1)k_1!}\frac{M!}{(M-k_2)k_2!}(\ell(I)^{-2k_1}L^{M-k_1} \otimes \ell(J)^{-2k_2}L^{M-k_2})P_M b_R\\ &=: \sum_{k_1=1}^{M}\sum_{k_2=1}^{M} C_{k_1}C_{k_2}(\ell(I)^{-2k_1}L^{M-k_1} \otimes \ell(J)^{-2k_2}L^{M-k_2})P_M b_R.\end{aligned}$$

Thus, by the definitions of an atom and $f \in \mathrm{BMO}^{q'}_{L,w}$ and Hölder's inequality, we have

$$\begin{aligned}\mathrm{J}_2 \leqslant& \sum_{k_1=1}^{M}\sum_{k_2=1}^{M} C_{k_1}C_{k_2}\sum_R \bigg|\ell(I)^{-2M}\ell(J)^{-2M}\\ &\times \iint_{I\times J} (\ell(I)^2L)^{M-k_1} \otimes (\ell(J)^2L)^{M-k_2} b_R P_M f(x,y)\,\mathrm{d}x\,\mathrm{d}y\bigg|\\ \leqslant& \sum_{k_1=1}^{M}\sum_{k_2=1}^{M} C_{k_1}C_{k_2}\\ &\times \Big(\sum_R |\ell(I)^{-2rM}\ell(J)^{-2rM}|(\ell(I)^2L)^{M-k_1} \otimes (\ell(J)^2L)^{M-k_2} b_R\|_{L^r_w}\Big)^{1/r}\\ &\times \sum_R \iint_{I\times J} |P_M f(x,y)|^{r'} w^{1-r'}\,\mathrm{d}x\,\mathrm{d}y \leqslant Cw(\Omega)^{1/r-1}w(\Omega)^{1/r'} \leqslant C.\end{aligned}$$

Therefore, for every $h = \sum_j \lambda_j a_j \in H^{1,q,M}_{L,\omega}(\mathbb{R}^n \times \mathbb{R}^n)$, where a_j are weighted atoms, we have

$$\begin{aligned}\Big|\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x,y)h(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| &\leqslant \sum_j |\lambda_j|\Big|\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x,y)a_j(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big|\\ &\leqslant C\sum_j |\lambda_j|\|f\|_{\mathrm{BMO}^{q'}_{L,1/w}} \leqslant C\|h\|_{H^1_{L,w}}\|f\|_{\mathrm{BMO}^{q'}_{L,w}}.\end{aligned}$$

Conversely, suppose that $l \in H^{1,r,M}_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)^*$ can be represented by $f(x,y)$ in the form

$$l(g) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x,y)g(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Let Ω be a bounded open set in $\mathbb{R}^n \times \mathbb{R}^n$, and suppose that $\Omega = \bigcup_R R$, where the R's are the maximal dyadic rectangles contained in Ω, and that

$$\Big(\sum_R \|\varphi\|^r_{L^r_w(R)}\Big)^{1/r} = 1.$$

Set

$$a(x,y) = \frac{1}{w(\Omega)^{1-1/r}} \sum_R (\mathbb{I} - (1 + \ell(I)^2 L \otimes \ell(J)^2 L)^{-1})^M)\varphi.$$

Then it is not difficult to check that a is a $(1,r,M,w)$-atom (see Theorem 6.4 in [24]).

Consequently,

$$\begin{aligned}
\|l\| &\geqslant \|l(a)\| \\
&= \frac{1}{w(\Omega)^{1-1/r}} \sum_R \iint_R (\mathbb{I} - (1 + \ell(I)^2 L \otimes \ell(J)^2 L)^{-1})^M)\varphi(x,y) f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{w(\Omega)^{1-1/r}} \sum_R \iint_R \varphi(x,y)(\mathbb{I} - (1 + \ell(I)^2 L \otimes \ell(J)^2 L)^{-1})^M) f(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Then by duality it readily follows that

$$\Big(\frac{1}{\omega(\Omega)} \sum_R \iint_{I \times J} |(\mathbb{I} - (1 + \ell(I)^2 L \otimes \ell(J)^2 L)^{-1})^M f|^{r'} w^{1-r'}\,\mathrm{d}x\Big)^{1/r'} \leqslant \|l\|,$$

which is what we want to show. □

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