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A GENERALIZATION TO THE HARDY-SOBOLEV SPACES $H^{k,p}$ OF AN L^p-L^1 LOGARITHMIC TYPE ESTIMATE

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Abstract. The main purpose of this article is to give a generalization of the logarithmic-type estimate in the Hardy-Sobolev spaces $H^{k,p}(G)$; $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and G is the open unit disk or the annulus of the complex space $\mathbb{C}$.

Keywords: annular domain; Poisson kernel; Hardy-Sobolev space; logarithmic estimate

MSC 2010: 30H10, 30C40, 35R30

1. Introduction

The purpose of this paper is to establish a logarithmic estimate of optimal-type in the Hardy-Sobolev space $H^{k,p}(G)$; $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and G is either the open unit disk $\mathbb{D}$ or the annulus G_s of radii $(s,1)$, $0 < s < 1$ of the complex space $\mathbb{C}$. More precisely, we study the behavior on the boundary of G with respect to the L^p-norm of any function f in the unit ball of the Hardy-Sobolev $H^{k,p}(G)$ starting from its behavior on any open connected subset $I \subset \partial G$ of the boundary of G with respect to the L^1-norm. Our result can be viewed as an extension of those established in [5], [7], [8], [12], [14], [13], [19], [20].

Control problems in Hardy spaces have been motivated by an interpolation scheme for analytic functions in $\mathbb{D}$ from boundary values on the unit circle $\mathbb{T}$. In this context a log-log/log-type inequality with respect to L^2-norm has been proved in the Hardy-Sobolev space $H^{1,2}$ of the unit disk $\mathbb{D}$, see [5]. A similar estimate of $1/\log^{\alpha}$-type, $0 < \alpha < 1$, has been established in more general planar domain, see [1].

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The case of the annular domain $G = G_s$, $0 < s < 1$, was considered in [19]. Based on the Hilbertian properties of the Hardy-Sobolev space $H^{1,2}(G_s)$, the authors established a $1/\log$-type estimate of a function's behavior on the inner boundary $s\mathbb{T}$ from its behavior on the outer boundary $\mathbb{T}$. They also showed that if the L^2-norm of a bounded $H^{1,2}(G_s)$ function is known to be small on a strict subset I of ∂G_s, it remains also small on the whole boundary ∂G_s. In the same context, an explicit logarithmic inequality exhibiting the dependence with respect to the inner radius s was proved in [20]. These estimates are interestingly used to prove logarithmic stability results for an inverse Robin's problem. The uniform case, $p = \infty$, has been considered in [12]. The author proved, by a quite different method, an optimal logarithmic estimate of a function's behavior on the whole boundary ∂G_s from its behavior on any open connected subset $I \subset \partial G_s$. The proof is based on some estimates established on the Poisson kernel of the annulus G_s.

For bounded analytic functions in the open unit disk $\mathbb{D}$, a similar logarithmic estimate has been established in [7]. The authors proved a $1/\log^k$ inequality in the Hardy-Sobolev spaces $H^{k,\infty}(\mathbb{D})$, for any integer k, and have shown that their estimate is of optimal type. Their results are interestingly used to establish logarithmic stability estimates for the Cauchy problem of identifying the Robin coefficient with a Laplace operator, and to prove an error estimate for the inverse problem of the identification of a Robin coefficient. Similar estimates of optimal type for the Hilbertian spaces $H^{k,2}(\mathbb{D})$ have been proved in [14], where the authors derive also logarithmic stability results with respect to the L^2-norm for the same inverse problem of identifying Robin coefficients and for a recovering interpolation scheme in Hardy-Sobolev space $H^{1,2}(\mathbb{D})$ with interpolation points located on the boundary $\mathbb{T}$ of the unit disk.

The outline of this paper is as follows. We begin in Section 2 by establishing the necessary notation and definitions, then we state our main result. Some basic properties are given in Section 3. Section 4 is devoted to the proof of our main result in both cases of the unit disk and of the annulus domain. Finally, some concluding remarks and perspectives for future work are presented in Section 5.

2. Notation, definitions and main result

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ with boundary $\mathbb{T}$ and let G_s denote the annulus of radii $(s, 1)$, $0 < s < 1$,

$$G_s = \{z \in \mathbb{C};\ s < |z| < 1\}.$$

The boundary of the annular domain G_s consists of two pieces $s\mathbb{T}$ and $\mathbb{T}$:

$$\partial G_s = s\mathbb{T} \cup \mathbb{T}.$$

In the sequel, we denote by G the open unit disk $\mathbb{D}$ or the annulus G_s; $0 < s < 1$ and by I an open connected subset of the boundary ∂G. We also equip the boundary ∂G_s with the usual Lebesgue measure μ normalized so that the circles $\mathbb{T}$ and $s\mathbb{T}$, each have unit measure. Furthermore, we denote $\lambda = \mu(I)/2\pi$, we assume that $\lambda \in]0,1[$ and we define by

$$\|f\|_{L^1(I)} = \frac{1}{2\pi\lambda}\int_I |f(r\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta,$$

the L^1-norm of f on I, where $r = s$ if $I \subset s\mathbb{T}$ and $r = 1$ if $I \subset \mathbb{T}$.

For $1 \leqslant p \leqslant \infty$ and for an analytic function f on G, the integral

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta\right)^{1/p}$$

and

$$M_\infty(f,r) = \sup_{0\leqslant\theta\leqslant 2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|$$

provide one measure of growth and lead to a simple definition of Hardy spaces:

The Hardy space $H^p(G)$ is defined as the set of all analytic functions f in G such that $M_p(f,r)$ remains bounded as $r \to 1$ if $G = \mathbb{D}$ and $M_p(f,r)$ remains bounded as $r \to 1$ and $r \to s$ in the annulus case.

The Hardy space $H^p(G_s)$ can be identified with the direct sum:

$$H^p(G_s) = H^p(\mathbb{D}) \oplus H_0^p(\mathbb{C} \setminus s\overline{\mathbb{D}}),$$

where the Hardy space $H_0^p(\mathbb{C}\setminus s\overline{\mathbb{D}})$ is defined as the set of analytic functions in $\mathbb{C}\setminus s\overline{\mathbb{D}}$, with zero limit at infinity.

For $p = \infty$, the Hardy space $H^\infty(G_s)$ is defined as the space of bounded analytic functions on G_s. According to [10], Theorem 7.1, it can be identified with the closed subspace $H^\infty(\partial G_s)$ of $L^\infty(\partial G_s)$.

For $1 < p < \infty$, the space $H^p(\partial G_s)$ is defined to be the closure in the complex Banach space $L^p(\partial G_s)$ of the set $R(\partial G_s)$ consisting of rational functions whose poles lie in the complement of $\overline{G_s}$. We have a natural one-to-one isomorphic correspondence between the spaces $H^p(G_s)$ and $H^p(\partial G_s)$ and this induces a Banach space structure on $H^p(G_s)$. For more details concerning the definitions and properties of Hardy spaces, we refer the reader to [9], [11], [16], [25], [23], [24].

For $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, we designate by $H^{k,p}(G)$ the Hardy-Sobolev space

$$H^{k,p}(G) = \{f \in H^p(G)\colon\ f^{(j)} \in H^p(G); 1 \leqslant j \leqslant k\}.$$

We endow the Banach space $H^{k,p}(G)$ with the usual Sobolev norm

$$\|f\|_{H^{k,p}(G)}^{p} = \sum_{j=0}^{k} \|f^{(j)}\|_{L^p(\partial G)}^{p},$$

where

$$\|f\|_{L^p(\partial G)}^{p} = \|f\|_{L^p(\mathbb{T})}^{p} + \|f\|_{L^p(s\mathbb{T})}^{p} \quad \text{if } G = G_s,$$
$$\|f\|_{L^p(\partial G)} = \|f\|_{L^p(\mathbb{T})} \quad \text{if } G = \mathbb{D}.$$

Let $B_{k,p}(G) = \{f \in H^{k,p}(G);\ \|f\|_{H^{k,p}(G)} \leqslant 1\}$ be the closed unit ball of $H^{k,p}(G)$.

We now state our main result.

Theorem 2.1. *Let $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and let I be a subarc of ∂G of length $2\pi\lambda$; $\lambda \in]0,1[$. There exist two non-negative constants α and Γ, depending only on k, p and s, such that for every $f \in B_{k,p}(G)$ satisfying $\|f\|_{L^1(I)} \leqslant \mathrm{e}^{-\Gamma}$, we have*

$$\|f\|_{L^p(\partial G)} \leqslant \frac{\alpha}{|\lambda \log \|f\|_{L^1(I)}|^k}. \tag{2.1}$$

Note that Theorem 2.1 can be extended to any bounded subset of functions in $H^{k,p}(G)$. Note also that this kind of results generalizes those established in [7], [12], [14], [13], [19], [20] and also improves upon [5], Lemma 4.2, since the upper bound in (2.1) has no log-log term in the numerator.

Actually, Theorem 2.1 is of optimal type as shown by the following proposition.

Proposition 2.2. *Assume $I = \{\mathrm{e}^{\mathrm{i}\theta}, \pi/2 \leqslant \theta \leqslant 3\pi/2\}$ and for $a > 1$, consider the sequence of normalized functions in $B_{k,p}(G)$,*

$$f_n = u_n / \|u_n\|_{H^{k,p}(G)}, \quad u_n(z) = (z-a)^n, \quad n \in \mathbb{N}^*.$$

Then

$$\lim_{n\to\infty} \|f_n\|_{L^p(\partial G)} |\log \|f_n\|_{L^1(I)}|^k \geqslant \Big(\frac{1+a}{2}\Big)^k \log^k\Big(\frac{(1+a)^2}{1+a^2}\Big)(1+o(1)). \tag{2.2}$$

We deduce clearly from Proposition 2.2 that the estimate (2.1) is of optimal type: it is impossible to find a function ε which tends to zero at zero such that for all $f \in B_{k,p}(G)$,

$$\|f\|_{L^p(\partial G)} \leqslant \frac{1}{|\log(\|f\|_{L^1(I)})|^k}\, \varepsilon(\|f\|_{L^1(I)}).$$

Note that the estimate (2.1) of Theorem 2.1 is false in the general setting where $f \in H^p$ only (we can consider the H^p normalized function of u_n).

3. Basic properties

In this section, we give some basic properties which will be useful throughout the paper. We start with the following Hardy's convexity theorem (cf. [11], page 9). We can also consult [16], [24] for more details.

Theorem 3.1. *Let f be analytic in $\overline{G}$ and $0 < p \leqslant \infty$. Let $r > 0$ be such that $s \leqslant r \leqslant 1$ if $G = G_s$ and $0 < r \leqslant 1$ if $G = \mathbb{D}$. Then $\log M_p(f, r)$ is a convex function of $\log r$, which means that if*

$$\log r = \alpha \log r_1 + (1 - \alpha) \log r_2 \quad \textit{with } s \leqslant r_1 \leqslant r_2 \leqslant 1,\ 0 \leqslant \alpha \leqslant 1,$$

then

$$M_p(f, r) \leqslant [M_p(f, r_1)]^{\alpha} [M_p(f, r_2)]^{1-\alpha}.$$

For $p \in [1, \infty]$ and $n \in \mathbb{N}$, denote by $L^p_n(\mathbb{T})$ the set of all 2π-periodic functions f such that $f^{(n-1)}$ is locally absolutely continuous and $f^{(n)} \in L^p(\mathbb{T})$. The next lemma, stated in [6], [18] deals with a variant of Kolmogorov-type inequality involving the L^p means on $\mathbb{T}$ of a 2π-periodic function and its derivatives of higher-order. The reader can consult [2], [3], [22] for further details concerning the Kolmogorov-type inequality.

Lemma 3.2. *Let $1 \leqslant k < n$ be two integers. There exists a non-negative constant $C_p(n, k)$ such that for all functions f in the space $L^p_n(\mathbb{T})$, we have*

$$\|f^{(k)}\|_{L^p(\mathbb{T})} \leqslant C_p(n, k) \|f\|^{1-k/n}_{L^p(\mathbb{T})} \|f^{(n)}\|^{k/n}_{L^p(\mathbb{T})}. \tag{3.1}$$

Note that for an analytic function f having an order n zero at the origin, Hardy, Landau and Littlewood proved inequality (3.1) in the case $p = 2$ when the derivatives are taken with respect to the complex variable z; (cf. [17]).

We recall also the following inequality, linked to [11], Theorem 5.6, and [13], Lemma 3.4, which will be useful for the proof of Lemma 4.3 and Lemma 4.8.

Lemma 3.3. *Let $1 \leqslant p < \infty$ and let f be an analytic function in G. Then for $s < r < 1$ if $G = G_s$ and $0 < r < 1$ if $G = \mathbb{D}$, we have*

$$M'_p(f, r) \leqslant M_p(f', r). \tag{3.2}$$

4. Proofs of Theorem 2.1 and Proposition 2.2

This section is devoted to the proof of Theorem 2.1 and Proposition 2.2. To avoid ambiguity, we can divide the proof into two steps:

4.1. Proof of main result in the case of the unit disk.

In this section, we need to recall some preliminary results. We can start by the lemma about the mean growth of the derivative of an analytic function of the unit disk (cf. [11], page 80, or [14], Lemma 2.3).

Lemma 4.1. *Let f be analytic in $\mathbb{D}$ and let $0 < r < \varrho \leqslant 1$. Then*

$$M_p(f', r) \leqslant \frac{M_p(f, \varrho)}{\varrho^2 - r^2}. \tag{4.1}$$

Referring to [5], Lemma 4.1, and [14], Lemma 2.4, we get

Lemma 4.2. *Let I be a subarc of $\mathbb{T}$ of length $2\pi\lambda$, $0 < \lambda < 1$ and let f be a bounded analytic function in $\mathbb{D}$ such that $\|f\|_{L^\infty(\mathbb{D})} \leqslant 1$. Then, for every $z \in \overline{\mathbb{D}}$, we have*

$$|f(z)| \leqslant \|f\|_{L^1(I)}^{\lambda(1-|z|)/2}.$$

We now prove the following lemma which will be the basis for the proof of Theorem 2.1.

Lemma 4.3. *Let k be a positive integer, $1 \leqslant p < \infty$, and let $f \in H^{k,p}(\mathbb{D})$ be such that $M_p(f^{(k)}, 1) \leqslant 1$. Then, for $0 < r < 1$, we have*

$$\|f\|_{L^p(\mathbb{T})} \leqslant \sum_{s=0}^{k-1} \frac{(1-r)^s}{s!} M_p(f^{(s)}, r) + \Big[\frac{\log r}{\log M_p(f^{(k)}, r)}\Big]^k. \tag{4.2}$$

Proof. We first consider a function g in $H^{1,p}$ such that $M_p(g', 1) \leqslant 1$; from (3.2) we get

$$M_p'(g, r) \leqslant M_p(g', r). \tag{4.3}$$

Applying Theorem 3.1 with $r = t$, $r_1 = r$, $r_2 = 1$, $\alpha \log r = \log t$ and the fact that $M_p(g', 1) \leqslant 1$, we obtain

$$M_p(g', t) \leqslant [M_p(g', r)]^{\log t / \log r}, \quad 0 < r < t \leqslant 1. \tag{4.4}$$

Since

$$M_p(g,s) - M_p(g,r) = \int_r^s M_p'(g,t)\,\mathrm{d}t, \tag{4.5}$$

we derive that

$$\begin{aligned} M_p(g,s) - M_p(g,r) &\leqslant \frac{[t^{\log M_p(g',r)/\log r+1}]_r^s}{\log M_p(g',r)/\log r + 1} \\ &\leqslant \frac{\log r}{\log M_p(g',r)}\ s^{\log M_p(g',r)/\log r}. \end{aligned} \tag{4.6}$$

Now, by applying (4.6) to the function $g = f^{(k-1)} \in H^{1,p}(\mathbb{D})$, and assuming $0 < r < t \leqslant 1$, we get

$$M_p(f^{(k-1)},t) \leqslant M_p(f^{(k-1)},r) + \frac{\log r}{\log M_p(f^{(k)},r)}\ t^{\log M_p(f^{(k)},r)/\log r}. \tag{4.7}$$

Writing (4.5) for $f^{(k-2)}$, making use of (4.3), and integrating both sides of the inequalities (4.7) with respect to t, $0 < r \leqslant t \leqslant s \leqslant 1$, we obtain

$$\begin{aligned} M_p(f^{(k-2)},s) - M_p(f^{(k-2)},r) &\leqslant (s-r)M_p(f^{(k-1)},r) \\ &\quad + \Big[\frac{\log r}{\log M_p(f^{(k)},r)}\Big]^2 s^{\log M_p(f^{(k)},r)/\log r+1}. \end{aligned}$$

Hence, after one integration, and for $0 < r < t \leqslant 1$, (4.7) leads to

$$\begin{aligned} M_p(f^{(k-2)},t) &\leqslant M_p(f^{(k-2)},r) + (t-r)M_p(f^{(k-1)},r) \\ &\quad + \Big[\frac{\log r}{\log M_p(f^{(k)},r)}\Big]^2 t^{\log M_p(f^{(k)},r)/\log r}. \end{aligned}$$

Thus, by repeating these integration argument $(k-2)$ times and for $t = 1$ we obtain (4.2), which proves the lemma. □

We need also the following lemma.

Lemma 4.4. *Let $g \in B_{k,p}(\mathbb{D})$ and let us for $r \in\,]0,1[$ define dilated functions h_r by*

$$h_r(\theta) = g(r\mathrm{e}^{\mathrm{i}\theta}), \quad \theta \in \mathbb{R}.$$

Then

$$h_r^{(k)}(\theta) = \mathrm{i}^k \sum_{j=1}^{k} c_{j,k} r^j \mathrm{e}^{\mathrm{i}j\theta} g^{(j)}(r\mathrm{e}^{\mathrm{i}\theta}), \tag{4.8}$$

where $c_{1,k} = c_{k,k} = 1$ and $c_{j,k}$ satisfies the recurrent relation $c_{j,k} = jc_{j,k-1} + c_{j-1,k-1}$.

P r o o f. The proof is obvious for $k=1$. Suppose now that equality (4.8) is true for all integers $s \leqslant k$ and let us derive the function $h_r (k+1)$ times; then we get

$$\begin{aligned} h_r^{(k+1)}(\theta) &= \mathrm{i}^{k+1}\Big(\sum_{j=1}^{k} j c_{j,k} r^j \mathrm{e}^{\mathrm{i}j\theta} g^{(j)}(r\mathrm{e}^{\mathrm{i}\theta}) + \sum_{j=1}^{k} c_{j,k} r^{j+1} \mathrm{e}^{\mathrm{i}(j+1)\theta} g^{(j+1)}(r\mathrm{e}^{\mathrm{i}\theta})\Big) \\ &= \mathrm{i}^{k+1}\sum_{j=1}^{k+1} c_{j,k+1} r^j \mathrm{e}^{\mathrm{i}j\theta} g^{(j)}(r\mathrm{e}^{\mathrm{i}\theta}), \end{aligned}$$

and (4.8) is proved for $s=k+1$. □

Next, we establish the following control lemma.

Lemma 4.5. *Let $k \in \mathbb{N}^*$, $1 \leqslant p < \infty$, $f \in B_{k,p}(\mathbb{D})$ and let $g = f/m$, where m is a non-negative constant chosen such that $g \in B_{k,p}(\mathbb{D})$ and $\|g\|_{L^\infty(\mathbb{T})} \leqslant 1$. Then for every $r \in]0,1[$, we have*

$$M_p(g^{(k)}, r) \leqslant \frac{\beta_k N_I^{(1-r)/(k+1)}}{r^k (1-r)^{k/(k+1)}}, \quad N_I = \|g\|_{L^1(I)}^{\lambda/2} \tag{4.9}$$

where β_k is a non-negative constant depending only on k and p.

P r o o f. The proof is by induction on $k \in \mathbb{N}^*$. For $k=1$ and $0<r<1$, we consider as in the proof of [13], Theorem 2.1, the dilated function h_r,

$$h_r(\theta) = g(r\mathrm{e}^{\mathrm{i}\theta}), \quad \theta \in \mathbb{R}.$$

Then we have

$$h_r'(\theta) = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta} g'(r\mathrm{e}^{\mathrm{i}\theta}) \quad \text{and} \quad h_r''(\theta) = \mathrm{i}^2 (r\mathrm{e}^{\mathrm{i}\theta} g'(r\mathrm{e}^{\mathrm{i}\theta}) + r^2 \mathrm{e}^{2\mathrm{i}\theta} g''(r\mathrm{e}^{\mathrm{i}\theta})), \tag{4.10}$$

thus

$$\|h_r''\|_{L^p(\mathbb{T})} \leqslant 2(M_p(g', r) + M_p(g'', r)). \tag{4.11}$$

Applying Lemma 4.1 to the derivative g' and g'' with $\varrho = 1$, we obtain

$$M_p(g', r) \leqslant \frac{M_p(g,1)}{1-r^2} \quad \text{and} \quad M_p(g'', r) \leqslant \frac{M_p(g',1)}{1-r^2}. \tag{4.12}$$

From (4.11), (4.12) and the fact that $M_p(g,1) + M_p(g',1) \leqslant 1$, we get

$$\|h_r''\|_{L^p(\mathbb{T})} \leqslant 2\Big(\frac{M_p(g,1)}{1-r^2} + \frac{M_p(g',1)}{1-r^2}\Big) \leqslant \frac{2}{1-r}. \tag{4.13}$$

The Kolmogorov inequality (3.1) applied to the function h_r yields

$$\|h_r'\|_{L^p(\mathbb{T})} \leqslant C_p(2,1)\|h_r\|_{L^p(\mathbb{T})}^{1/2}\|h_r''\|_{L^p(\mathbb{T})}^{1/2}. \tag{4.14}$$

Furthermore, since $\|g\|_{L^\infty(\mathbb{T})} \leqslant 1$, we deduce from Lemma 4.2 that

$$\|h_r\|_{L^p(\mathbb{T})} = M_p(g,r) \leqslant N_I^{1-r}, \quad N_I = \|g\|_{L^1(I)}^{\lambda/2} \leqslant 1. \tag{4.15}$$

In the sequel, inequalities (4.13), (4.14), (4.15) and the relation

$$\|h_r'\|_{L^p(\mathbb{T})} = rM_p(g',r) \tag{4.16}$$

give

$$M_p(g',r) \leqslant \frac{\beta_1 N_I^{(1-r)/2}}{r(1-r)^{1/2}}, \quad \text{where } \beta_1 = \sqrt{2}C_p(2,1). \tag{4.17}$$

For $k \geqslant 2$, we suppose that (4.9) is true for all integers s less than $k-1$. Then, from Lemma 4.4 and by using the convex inequality

$$\Big(\sum_{i=1}^{n} a_i\Big)^p \leqslant n^p \sum_{i=1}^{n} a_i^p, \tag{4.18}$$

we get

$$M_p(g^{(s+1)},r) \leqslant (s+1)\bigg(\frac{\|h_r^{(s+1)}\|_{L^p(\mathbb{T})}}{r^{s+1}} + \sum_{j=1}^{s} c_{j,s+1}\frac{M_p(g^{(j)},r)}{r^{s-j+1}}\bigg). \tag{4.19}$$

Furthermore, Lemma 3.2 applied to the function h_r gives

$$\|h_r^{(s+1)}\|_{L^p(\mathbb{T})} \leqslant C_p(s+2,s+1)\|h_r\|_{L^p(\mathbb{T})}^{1/(s+2)}\|h_r^{(s+2)}\|_{L^p(\mathbb{T})}^{(s+1)/(s+2)}, \tag{4.20}$$

while Lemma 4.4, the convex inequality (4.18) and Lemma 4.1 give

$$\|h_r^{(s+2)}\|_{L^p(\mathbb{T})} \leqslant (s+2)\sum_{j=1}^{s+2} c_{j,s+2}M_p(g^{(j)},r) \leqslant \frac{s+2}{1-r^2}\sum_{j=0}^{s+1} c_{j+1,s+2}M_p(g^{(j)},r).$$

Since $\sum\limits_{j=0}^{s+1} M_p(g^{(j)},1) \leqslant 1$, we get

$$\|h_r^{(s+2)}\|_{L^p(\mathbb{T})} \leqslant \frac{\mu_s}{1-r}, \quad \text{where} \quad \mu_s = (s+2)\sum_{j=1}^{s+2} c_{j,s+2}. \tag{4.21}$$

Plugging (4.15) and (4.21) into (4.20), we derive the following control of the first term on the right hand side of (4.19):

$$\|h_r^{(s+1)}\|_{L^p(\mathbb{T})} \leqslant \mu_s^{(s+1)/(s+2)} C_p(s+2,s+1)\frac{N_I^{(1-r)/(s+2)}}{(1-r)^{(s+1)/(s+2)}}. \tag{4.22}$$

For the second term on the right hand side of (4.19) we deduce from the recurrent hypothesis that for $s=1,\ldots,k-1$ and for every $j=1,\ldots,s$, there exists a non negative constant β_j such that

$$M_p(g^{(j)},r) \leqslant \frac{\beta_j}{r^j(1-r)^{j/(j+1)}}\; N_I^{(1-r)/(j+1)}, \quad 0<r<1.$$

Since $j \leqslant k$, we have

$$N_I^{(1-r)/(j+1)} \leqslant N_I^{(1-r)/(k+1)}.$$

In the sequel, by using the monotonicity of the function $t \to r^t(1-r)^{t/(t+1)}$ we get for all $j=1,\ldots,s$ that

$$\frac{M_p(g^{(j)},r)}{r^{s-j+1}} \leqslant \frac{\beta_j}{r^k(1-r)^{k/(k+1)}}\; N_I^{(1-r)/(k+1)}, \quad 0<r<1. \tag{4.23}$$

Plugging (4.22) and (4.18) into (4.20) we conclude the proof of the lemma. □

P r o o f of Theorem 2.1 in the case of the unit disk. The uniform case $p=\infty$ has been proved by Chaabane and Feki in [7].

Let $1 \leqslant p < \infty$, $f \in B_{k,p}(\mathbb{D})$ and let $g=f/m$, where m is a non-negative constant chosen such that

$$\sum_{j=0}^{k} M_p(g^{(j)},1) \leqslant 1 \quad \text{and} \quad \|g\|_{L^\infty(\mathbb{T})} \leqslant 1. \tag{4.24}$$

From Lemma 4.5 we obtain for every $r \in]0,1[$ the inequality

$$M_p(g^{(k)},r) \leqslant \frac{\beta_k N_I^{(1-r)/(k+1)}}{r^k(1-r)^{k/(k+1)}}. \tag{4.25}$$

Let us choose r satisfying

$$N_I^{1-r} = \frac{1}{(\log(1/N_I))^{(k+1)C_k}}, \quad \text{where } C_k = \frac{\beta_k}{(1-2/\mathrm{e})^k}. \tag{4.26}$$

Then r is equal to

$$r = 1-(k+1)C_k\frac{\log\log(1/N_I)}{\log(1/N_I)}.$$

If we suppose that $C_k \geqslant 2$ and if we choose N_I to be small enough in such a way that

$$N_I < \mathrm{e}^{-\Gamma}, \quad \text{where } \Gamma \geqslant \mathrm{e} \text{ and } \frac{\log \Gamma}{\Gamma} = \frac{2}{\mathrm{e}(k+1)C_k}, \tag{4.27}$$

then we obtain

$$1 - \frac{2}{\mathrm{e}} \leqslant r < 1. \tag{4.28}$$

Using the concavity of the function log, we get for $r \in [1 - 2/\mathrm{e}, 1[$ that

$$A(r-1) \leqslant \log r \leqslant r - 1 \quad \text{where } A = -\frac{\log(1 - 2/\mathrm{e})}{2/\mathrm{e}} = 1.808\ldots \tag{4.29}$$

From (4.25) and the fact that $1 - 2/\mathrm{e} \leqslant r$, we obtain

$$M_p(g^{(k)}, r) \leqslant \frac{C_k N_I^{(1-r)/(k+1)}}{(1-r)^{k/(k+1)}}. \tag{4.30}$$

Also we have

$$\frac{\log M_p(g^{(k)}, r)}{\log r} \geqslant \frac{1}{k+1} \frac{\log N_I^{1-r}}{\log r} + \frac{\log C_k}{\log r} - \frac{k}{k+1} \frac{\log(1-r)}{\log r}. \tag{4.31}$$

For the first term on the right hand side of (4.31), we get from the first inequality in (4.29) that

$$\frac{\log N_I^{1-r}}{\log r} \geqslant \frac{(1-r) \log N_I}{A(r-1)} = \frac{\log(1/N_I)}{A}. \tag{4.32}$$

For the last two terms of (4.31), applying the second inequality of (4.29), we have

$$\frac{\log C_k}{\log r} - \frac{k}{k+1} \frac{\log(1-r)}{\log r} \geqslant \frac{\log(1-r)}{1-r} - \frac{\log C_k}{1-r}. \tag{4.33}$$

By substituting the value of r, we get

$$\frac{1}{(k+1)C_k}\Big(-\log(1/N_I) + \frac{\log(1/N_I)(\log(k+1) + \log\log(1/N_I))}{\log\log(1/N_I)}\Big).$$

Since we assume $N_I \leqslant \mathrm{e}^{-\Gamma}$ with $\Gamma \geqslant \mathrm{e}$, the second term in the parentheses is positive and we finally get the inequality

$$\frac{\log C_k}{\log r} - \frac{k}{k+1} \frac{\log(1-r)}{\log r} \geqslant -\frac{\log(1/N_I)}{(k+1)C_k}. \tag{4.34}$$

Plugging (4.32) and (4.34) into (4.31), we obtain

$$\frac{\log M_p(g^{(k)},r)}{\log r} \geqslant \Big(\frac{1}{A}-\frac{1}{C_k}\Big)\frac{\log(1/N_I)}{k+1}, \tag{4.35}$$

where the constant in the parentheses is positive.

Furthermore, we know from Lemma 4.3 that

$$\|g\|_{L^p(\mathbb{T})} \leqslant \sum_{j=0}^{k-1}\frac{(1-r)^j}{j!}M_p(g^{(j)},r)+\Big[\frac{\log r}{\log M_p(g^{(k)},r)}\Big]^k. \tag{4.36}$$

Inequality (4.35) gives an upper bound for the above bracketed term. It remains to control the means $M_p(g^{(j)},r)$ of the derivatives of orders $j=0,\ldots,k-1$, $0<r<1$.

The case $k=1$ is reduced to the single term $M_p(g,r)$ for which inequalities (4.15), (4.26) together with the condition $N_I<1/\mathrm{e}$ give

$$M_p(g,r)\leqslant\frac{1}{(\log(1/N_I))^k}.$$

We assume now that $k\geqslant 2$. Then from Lemma 4.4 we get

$$M_p(g^{(j)},r)\leqslant\frac{j}{r^j}\Big(\|h_r^{(j)}\|_{L^p(\mathbb{T})}+\sum_{l=1}^{j-1}c_{l,j}M_p(g^{(l)},r)\Big),\quad 0\leqslant j\leqslant k-1. \tag{4.37}$$

For the first term on the right hand side of (4.37) we obtain from the Kolmogorov inequality, the fact that $M_p(g,r)=\|h_r\|_{L^p(\mathbb{T})}$ and Lemma 4.4 that

$$\begin{aligned}\|h_r^{(j)}\|_{L^p(\mathbb{T})} &\leqslant C_p(j+1,j)\|h_r\|_{L^p(\mathbb{T})}^{1/(j+1)}\|h_r^{(j+1)}\|_{L^p(\mathbb{T})}^{j/(j+1)}\\ &\leqslant C_p(j+1,j)(M_p(g,r))^{1/(j+1)}\Big((j+1)\sum_{l=1}^{j+1}c_{l,j+1}M_p(g^{(l)},r)\Big)^{j/(j+1)}.\end{aligned} \tag{4.38}$$

Now, applying Theorem 3.1 to the function $g^{(l)}$, with $r_1=r^{1/\alpha}$, $r_2=1$, $0<\alpha<1$ and since $M_p(g^{(l)},1)\leqslant 1$, we get

$$M_p(g^{(l)},r)\leqslant(M_p(g^{(l)},r_1))^\alpha. \tag{4.39}$$

Furthermore, Lemma 4.1 applied to the derivatives $g^{(l)}$ with $r=r_1$ and $\varrho=r$ gives

$$(M_p(g^{(l)},r_1))^\alpha\leqslant\frac{M_p^\alpha(g^{(l-1)},r)}{(r^2-r_1^2)^\alpha}. \tag{4.40}$$

By repeating arguments (4.39) and (4.40) successively to the derivatives $g^{(j)}$ for $j = (l-1), \dots, 1$, we get

$$M_p(g^{(l)}, r) \leqslant \frac{M_p^{\alpha^l}(g, r)}{(r^2 - r_1^2)^{\sigma_l}}, \quad \text{where } \sigma_l = \sum_{s=1}^{l} \alpha^s. \tag{4.41}$$

Thus, we have

$$\sum_{l=1}^{j+1} c_{l,j+1} M_p(g^{(l)}, r) \leqslant C_j \frac{M_p^{\alpha^{j+1}}(g, r)}{(r^2 - r_1^2)^{\sigma_{j+1}}}, \quad \text{where } C_j = \sum_{l=1}^{j+1} c_{l,j+1}. \tag{4.42}$$

Plugging inequality (4.42) into (4.38), we deduce that there exists a non-negative constant γ_j such that

$$\|h_r^{(j)}\|_{L^p(\mathbb{T})} \leqslant \gamma_j \frac{(M_p(g, r))^{(1+j\alpha^{j+1})/(j+1)}}{(r^2 - r_1^2)^{j\sigma_{j+1}/(j+1)}}, \tag{4.43}$$

and choosing $\alpha = (1 - 1/k)^{1/(j+1)}$, we get

$$\|h_r^{(j)}\|_{L^p(\mathbb{T})} \leqslant \gamma_j \frac{(M_p(g, r))^{1-1/k}}{(r^2 - r_1^2)^{j\alpha}}, \tag{4.44}$$

where we have used the inequalities

$$\sigma_{j+1} \leqslant (j+1)\alpha, \quad 1 + j\Big(1 - \frac{1}{k}\Big) > (j+1)\Big(1 - \frac{1}{k}\Big), \qquad j = 0, \dots, k-1.$$

For the last term of (4.37) , we obtain from Theorem 3.1 applied with $0 < r_1 < r < 1$, $0 < \alpha < 1$ and Lemma 4.1 the inequalities

$$\begin{aligned} \sum_{l=1}^{j-1} c_{l,j} M_p(g^{(l)}, r) &\leqslant \sum_{l=1}^{j-1} c_{l,j} (M_p(g^{(l)}, r_1))^{\alpha} \\ &\leqslant \frac{(M_p(g, r))^{\alpha^{j-1}}}{(r^2 - r_1^2)^{(j-1)\alpha}} \sum_{l=1}^{j-1} c_{l,j}. \end{aligned} \tag{4.45}$$

Plugging (4.44) and (4.45) into (4.37), we obtain

$$M_p(g^{(j)}, r) \leqslant A_j \frac{(M_p(g, r))^{1-1/k}}{r^j (r^2 - r_1^2)^j}, \quad \text{where } A_j = j\Big(\gamma_j + \sum_{l=1}^{j-1} c_{l,j}\Big). \tag{4.46}$$

Next, from (4.26) and the inequalities

$$M_p(g,r) \leqslant N_I^{1-r}, \quad N_I < \frac{1}{\mathrm{e}}, \quad (k+1)\Big(1-\frac{1}{k}\Big)C_k \geqslant k, \quad k \geqslant 2,$$

we deduce that

$$(M_p(g,r))^{1-1/k} \leqslant \frac{1}{(\log(1/N_I))^k}.$$

Then

$$\sum_{j=0}^{k-1} \frac{(1-r)^j}{j!} M_p(g^{(j)},r) \leqslant \sum_{j=0}^{k-1} \frac{A_j}{j!} \frac{(1-r)^j}{(r(r^2-r_1^2))^j} \frac{1}{(\log(1/N_I))^k}.$$

For the second fraction in the sum, we have

$$\frac{1-r}{r(r^2-r_1^2)} = \frac{1}{r^3} \frac{1-r}{1-r^{2(1/\alpha\ -1)}}$$

which is upper bounded by some constant B depending only on k and j since r satisfies the inequalities in (4.28). Consequently,

$$\sum_{j=0}^{k-1} \frac{(1-r)^j}{j!}\ M_p(g^{(j)},r) \leqslant \frac{A\mathrm{e}^{\widetilde{C}}}{(\log(1/N_I))^k}, \quad \text{where } \mathrm{A} = \max_{0\leqslant \mathrm{j} \leqslant \mathrm{k}-1} (\mathrm{A_j}). \tag{4.47}$$

Making use of (4.35) and (4.47) in (4.36), we get that there exists a non-negative constant β_k depending only on k such that

$$\|g\|_{L^p(\mathbb{T})} \leqslant \frac{\beta_k}{(\log(1/N_I))^k}.$$

From the relation $g = f/m$ and the definition of N_I in (4.15), we derive that there exists a non-negative constant α_k depending only on k such that

$$\|f\|_{L^p(\mathbb{T})} \leqslant \frac{\alpha_k}{(\lambda \log(m/\|f\|_{L^1(I)}))^k},$$

with $\alpha_k = 2^k m\beta_k$. This concludes the proof of (2.1). □

P r o o f of Proposition 2.2 in the case of the unit disk. To prove (2.2), we consider the sequence of functions

$$u_n(z) = (z-a)^n, \quad a > 1 \text{ and } n \in \mathbb{N}^*.$$

Let $I_n := \|u_n\|^p_{L^p(\mathbb{T})}$, then by making use of the Laplace method [21], Chapter 3, we derive the following asymptotic estimate:

$$I_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+a^2-2a\cos\theta)^{np/2}\,\mathrm{d}\theta = (2\pi anp)^{-1/2}(1+a)^{np+1}(1+o(1)),$$

Also we derive the estimate of the Sobolev norm:

$$\begin{aligned}\|u_n\|^p_{H^{k,p}(\mathbb{D})} &= I_n + n^p I_{n-1} + \ldots + n^p(n-1)^p \ldots (n-k+1)^p I_{n-k}\\ &= (2\pi anp)^{-1/2} n^{kp}(1+a)^{pn-kp+1}(1+o(1)).\end{aligned}$$

In the sequel, let $f_n = u_n/\|u_n\|_{H^{k,p}(\mathbb{D})}$ be the $H^{k,p}(\mathbb{D})$ normalized function of u_n. Then

$$\|f_n\|^p_{L^p(\mathbb{T})} = n^{-kp}(1+a)^{kp}(1+o(1)), \quad \text{as } n \to \infty. \tag{4.48}$$

Moreover,

$$\|u_n\|^p_{L^\infty(I)} = (1+a^2)^{np/2}.$$

This implies that

$$\|f_n\|^p_{L^\infty(I)} = (2\pi anp)^{1/2} n^{-kp}(1+a)^{-np+kp-1}(1+a^2)^{np/2}(1+o(1)), \tag{4.49}$$

as n tends to infinity.

Furthermore, we deduce from (4.37) and (4.38) that

$$\lim_{n\to\infty} \|f_n\|_{L^p(\mathbb{T})} \log^k\left(\frac{1}{\|f_n\|_{L^\infty(I)}}\right) = \left(\frac{1+a}{2}\right)^k \log^k\left(\frac{(1+a)^2}{1+a^2}\right)(1+o(1)),$$

from which the assertion (2.2) follows. □

4.2. Proof of main result in the case of the annulus.

As in the proof of the unit disk, we need to start with a preliminary lemma. First of all, we recall a point-wise estimate based on a lower bound for the Poisson kernel of the annulus G_s (cf. [12], Lemma 3.3).

Lemma 4.6. *Let I be a subarc of ∂G_s of length $2\pi\lambda$ and let f be a bounded analytic function in G_s such that $m \geqslant \|f\|_{L^\infty(\partial G_s)}$. Then, for every $z \in \overline{G_s}$, we have*

$$\begin{aligned} |f(z)| &\leqslant m \left\|\frac{f}{m}\right\|_{L^1(I)}^{(2\lambda C_s/\log s)(\log s - \log|z|)} && \text{if } s < |z| \leqslant \sqrt{s},\\ |f(z)| &\leqslant m \left\|\frac{f}{m}\right\|_{L^1(I)}^{(2\lambda C_s/\log s)\log|z|} && \text{if } \sqrt{s} \leqslant |z| < 1.\end{aligned}$$

Referring to [11], page 80, and [14], Lemma 2.3, we have the following lemma.

Lemma 4.7. *Let f be analytic in G_s and let $0 < s \leqslant \varrho < r < \delta \leqslant 1$. Then*

$$M_p(f', r) \leqslant \frac{M_p(f,\delta)}{\delta^2 - r^2} + \frac{M_p(f,\varrho)}{r^2 - \varrho^2}.$$

We now prove the following lemma which will be the basis for the proof of our main result.

Lemma 4.8. *Let $k \in \mathbb{N}^*$, $1 \leqslant p < \infty$, and let $f \in H^{k,p}(G_s)$ be such that $\|f^{(k)}\|_{L^p(\partial G_s)} \leqslant 1$. Then, for $0 < s < r < 1$, we have*

$$\|f\|_{L^p(\partial G_s)} \leqslant \sum_{j=0}^{k-1} \frac{(1-r)^j + (r-s)^j}{j!}\ M_p(f^{(j)}, r) + \Big[\frac{\log r + 2\log(s/r)}{\log M_p(f^{(k)}, r)}\Big]^k. \tag{4.50}$$

P r o o f. First of all, we observe that

$$\|f\|_{L^p(\partial G_s)} \leqslant M_p(f,1) + M_p(f,s). \tag{4.51}$$

So the proof can be divided into two steps:

For the first term on the left hand side of (4.51), let us consider a function g in $H^{1,p}$ such that $M_p(g',1) \leqslant 1$; then, as in the proof of Lemma 4.3 we obtain for $0 < s < r < t \leqslant 1$ the inequality

$$M_p(g', t) \leqslant (M_p(g', r))^{\log t/\log r}. \tag{4.52}$$

Since for $0 < s < r < u \leqslant 1$

$$M_p(g,u) - M_p(g,r) = \int_r^u M_p'(g,t)\,\mathrm{d}t, \tag{4.53}$$

we derive from (4.52) and Lemma 3.3 that

$$M_p(g,u) - M_p(g,r) \leqslant \frac{\log r}{\log M_p(g',r)}\ u^{\log M_p(g',r)/\log r}. \tag{4.54}$$

Now by applying (4.54) to the function $g = f^{(k-1)} \in H^{1,p}(G_s)$, and for $0 < s < r < t \leqslant 1$, we get

$$M_p(f^{(k-1)}, t) \leqslant M_p(f^{(k-1)}, r) + \frac{\log r}{\log M_p(f^{(k)}, r)},\quad t^{\log M_p(f^{(k)},r)/\log r}. \tag{4.55}$$

Writing (4.53) for $f^{(k-2)}$, making use of Lemma 3.3 and integrating both sides of the inequalities (4.55) with respect to t, $0 < s < r < t \leqslant u \leqslant 1$, we obtain

$$\begin{aligned}(4.56)\qquad M_p(f^{(k-2)}, u) \leqslant M_p(f^{(k-2)}, r) + (u-r)M_p(f^{(k-1)}, r)\\ + \Big[\frac{\log r}{\log M_p(f^{(k)}, r)}\Big]^2 u^{\log M_p(f^{(k)},r)/\log r}.\end{aligned}$$

Thus by repeating the integration argument $(k-2)$ times and for $t = 1$, we conclude that

$$(4.57)\qquad M_p(f, 1) \leqslant \sum_{j=0}^{k-1} \frac{(1-r)^j}{j!} M_p(f^{(j)}, r) + \Big[\frac{\log r}{\log M_p(f^{(k)}, r)}\Big]^k.$$

For the second term onn the left hand side of (4.51), we obtain by using Theorem 3.1 and the fact that $M_p(g', s) \leqslant 1$, the inequality

$$(4.58)\qquad M_p(g', t) \leqslant M_p(g', r)^{(\log s - \log t)/(\log s - \log r)}, \quad 0 < s < t < r \leqslant 1.$$

From (4.3) and the triangle inequality we get

$$(4.59)\qquad M_p(g, u) \leqslant M_p(g, r) + \int_u^r M_p(g', t)\,\mathrm{d}t, \quad 0 < s < u < t < r \leqslant 1,$$

then, if we suppose that $M_p(g', r) \leqslant (s/r)^2$, we deduce from (4.58) and (4.59) that

$$\begin{aligned}(4.60)\quad M_p(g, u) \leqslant M_p(g, r) + (M_p(g', r))^{\log s/\log(s/r)}\big[2\log(s/r)/\log M_p(g', r)\big]\\ \times u^{\log M_p(g',r)/(\log(r/s)+1)}.\end{aligned}$$

Now, by applying (4.60) to the function $g = f^{(k-1)} \in H^{1,p}(G_s)$ and for $0 < s < t < r \leqslant 1$, we get

$$\begin{aligned}(4.61)\qquad M_p(f^{(k-1)}, t) \leqslant M_p(f^{(k-1)}, r) + (M_p(f^{(k)}, r))^{\log s/\log(s/r)}\\ \times \frac{2\log(s/r)}{\log M_p(f^{(k)}, r)} t^{\log M_p(f^{(k)},r)/(\log r - \log s)}.\end{aligned}$$

Writing (4.59) for $f^{(k-2)}$ and integrating both sides of the previous inequality with respect to t, $0 < s \leqslant u < t < r \leqslant 1$, we obtain

$$\begin{aligned}M_p(f^{(k-2)}, u) \leqslant M_p(f^{(k-2)}, r) + (r-u)M_p(f^{(k-1)}, r) + (M_p(f^{(k)}, r))^{\log s/(\log(s/r))}\\ \times \Big[\frac{2\log(s/r)}{\log M_p(f^{(k)}, r)}\Big]^2 u^{\log M_p(f^{(k)},r)/\log r}.\end{aligned}$$

Thus, by repeating this integration argument $(k-2)$ times and for $u=s$, we obtain

$$M_p(f,s) \leqslant \sum_{j=0}^{k-1} \frac{(r-s)^j}{j!} M_p(f^{(j)},r) + \Big[\frac{2\log(s/r)}{\log M_p(f^{(k)},r)}\Big]^k. \tag{4.62}$$

Combining inequalities (4.62) and (4.57), we obtain (4.50), which proves the lemma. □

Next, we prove the following control lemma.

Lemma 4.9. *Let $k \in \mathbb{N}^*$, $1 \leqslant p < \infty$ and $f \in B_{k,p}(G_s)$ and let $g = f/m$, where m is a non-negative constant chosen such that $g \in B_{k,p}(G_s)$ and $\|g\|_{L^\infty(\partial G_s)} \leqslant 1$. Then for every $r \in]s,1[$ we have*

$$M_p(g^{(k)},r) \leqslant \frac{\beta_k(1-s)^{1/(k+1)}}{s^k(r-s)^{k/(k+1)}} \frac{N_I^{\log r/((k+1)\log s)}}{(1-r)^{k/(k+1)}}; \quad N_I = \|g\|_{L^1(I)}^{2\lambda C_s} \tag{4.63}$$

where β_k is a non-negative constant depending only on k and p.

P r o o f. Let $f \in B_{k,p}(G_s)$ and let $g = f/m$, where m is a non-negative constant chosen such that $g \in B_{k,p}(G_s)$ and $\|g\|_{L^\infty(\partial G_s)} \leqslant 1$. For $k=1$, let us for $r \in]s,1[$ set $h_r(\theta) = g(re^{i\theta})$. Then, as was proved in (4.11), we get

$$\|h_r''\|_{L^p(\mathbb{T})} \leqslant 2(M_p(g',r) + M_p(g'',r)). \tag{4.64}$$

Applying Lemma 4.7 to the derivative g' and g'' with $\delta = 1$ an $\varrho = s$, we obtain

$$M_p(g',r) \leqslant \frac{M_p(g,1)}{1-r^2} + \frac{M_p(g,s)}{r^2-s^2} \tag{4.65}$$

and

$$M_p(g'',r) \leqslant \frac{M_p(g',1)}{1-r^2} + \frac{M_p(g',s)}{r^2-s^2}. \tag{4.66}$$

Hence, from (4.64), (4.65), (4.66), the facts that $M_p(g,1) + M_p(g',1) \leqslant 1$ and $M_p(g,s) + M_p(g',s) \leqslant 1$ we get

$$\|h_r''\|_{L^p(\mathbb{T})} \leqslant 2\Big(\frac{1}{1-r^2} + \frac{1}{r^2-s^2}\Big) \leqslant \frac{1-s}{s(1-r)(r-s)}. \tag{4.67}$$

Since $\|h_r'\|_{L^p(\mathbb{T})} = rM_p(g',r)$, from (4.67) and the Kolmogorov-type inequality (3.1) we obtain

$$rM_p(g',r) \leqslant C_p(2,1)\|h_r\|_{L^p(\mathbb{T})}^{1/2} \frac{(1-s)^{1/2}}{s^{1/2}(1-r)^{1/2}(r-s)^{1/2}}. \tag{4.68}$$

On the other hand, using the second inequality of Lemma 4.6 and the fact that $\|g\|_{L^\infty(\partial G_s)} \leqslant 1$, we obtain for every $r \in]\sqrt{s}, 1[$ the inequality

$$\|h_r\|_{L^p(\mathbb{T})} = M_p(g,r) \leqslant N_I^{\log r/\log s}, \quad N_I = \|g\|_{L^1(I)}^{2\lambda C_s}. \tag{4.69}$$

Hence, plugging (4.69) into (4.68) we derive

$$M_p(g',r) \leqslant \frac{\beta_1(1-s)^{1/2}}{s(r-s)^{1/2}} \frac{N_I^{\log r/2\log s}}{(1-r)^{1/2}}, \quad \text{where } \beta_1 = C_p(2,1). \tag{4.70}$$

For $k \geqslant 2$, we suppose that inequality (4.63) is true for all $j = 1, \dots, k-1$. Then, as was proved in (4.19), we get

$$M_p(g^{(j+1)}, r) \leqslant (j+1)\left(\frac{\|h_r^{(j+1)}\|_{L^p(\mathbb{T})}}{r^{j+1}} + \sum_{l=1}^{j} c_{l,j+1} \frac{M_p(g^{(l)}, r)}{r^{j-l+1}}\right). \tag{4.71}$$

For the first term on the right hand side of (4.71), by applying the Kolmogorov-type inequality (3.1) to the function h_r, we obtain

$$\|h_r^{(j+1)}\|_{L^p(\mathbb{T})} \leqslant C_p(j+2, j+1)\|h_r\|_{L^p(\mathbb{T})}^{1/(j+1)} \|h_r^{(j+2)}\|_{L^p(\mathbb{T})}^{(j+1)/(j+2)}. \tag{4.72}$$

Since

$$\|h_r^{(j+2)}\|_{L^p(\mathbb{T})} \leqslant (j+2)\sum_{l=1}^{j+2} c_{l,j+2} M_p(g^{(l)}, r),$$

by applying Lemma 4.7 to the derivative $g^{(l)}$ with $\delta = 1$, $\varrho = s$ and the fact that

$$\sum_{l=0}^{j+1} M_p(g^{(l)}, 1) + M_p(g^{(l)}, s) \leqslant 1,$$

we obtain

$$\|h_r^{(j+2)}\|_{L^p(\mathbb{T})} \leqslant \frac{j+2}{2} \sum_{l=1}^{j+2} c_{l,j+2} \frac{(1-s)}{s(r-s)(1-r)}. \tag{4.73}$$

Plugging (4.69) and (4.73) into (4.72), we deduce

$$\|h_r^{(j+1)}\|_{L^p(\mathbb{T})} \leqslant \frac{\gamma_j \ (1-s)^{1/(j+2)}}{s^{(j+1)/(j+2)} \ (r-s)^{(j+1)/(j+2)}} \frac{N_I^{\log r/((j+2)\log s)}}{(1-r)^{(j+1)/(j+2)}}, \tag{4.74}$$

where

$$\gamma_j = C_p(j+2, j+1)\left(\frac{j+2}{2}\sum_{l=1}^{j+2} c_{l,j+2}\right)^{(j+1)/(j+2)}.$$

Since we have supposed by induction that (4.63) is true for all $j = 1, \ldots, k-1$, then we get from (4.74) and (4.71) the desired inequality

$$M_p(g^{(k)}, r) \leqslant \frac{\beta_k (1-s)^{1/(k+1)}}{s^k (r-s)^{k/(k+1)}} \frac{N_I^{\log r/((k+1)\log s)}}{(1-r)^{k/(k+1)}}, \quad \text{where } \beta_k > 0.$$

Thus (4.63) is true for $j = k$ and this concludes the proof of the lemma. $\square$

We are now in a position to establish the main control theorem in the Hardy-Sobolev spaces $H^{k,p}(G_s)$ for every integer k and $1 \leqslant p \leqslant \infty$.

P r o o f of Theorem 2.1 in the case of the annulus. The uniform case $p = \infty$ has been proved by [12]. Let $f \in B_{k,p}(G_s)$ and let $g = f/m$, where m is a non-negative constant chosen such that

$$\sum_{j=0}^{k} M_p(g^{(j)}, 1) + M_p(g^{(j)}, s) \leqslant 1 \qquad \text{and} \qquad |g\|_{L^\infty(\partial G_s)} \leqslant 1. \tag{4.75}$$

From Lemma 4.9, we obtain for every $r \in]\sqrt{s}, 1[$ the following inequality

$$M_p(g^{(k)}, r) \leqslant \frac{\beta_k (1-s)^{1/(k+1)}}{s^k (r-s)^{k/(k+1)}} \frac{N_I^{\log r/((k+1)\log s)}}{(1-r)^{k/(k+1)}}. \tag{4.76}$$

Let us choose r satisfying

$$N_I^{\log r/\log s} = \frac{1}{(\log(1/N_I))^{(k+1)\beta_k}}, \tag{4.77}$$

and consequently,

$$\log r = (k+1)\beta_k \log s \frac{\log\log(1/N_I)}{\log(1/N_I)}. \tag{4.78}$$

If we suppose that $\beta_k \geqslant 2$ and if we choose N_I to be small enough in such a way that

$$N_I < \mathrm{e}^{-\Gamma}, \quad \text{where } \Gamma \geqslant \mathrm{e} \text{ and } \frac{\log\Gamma}{\Gamma} = \frac{1}{2(k+1)\beta_k},$$

then we verify that

$$\sqrt{s} \leqslant r < 1. \tag{4.79}$$

Also, we have from the concavity of the function log that

$$\frac{\log s}{2(\sqrt{s}-1)}(r-1) \leqslant \log r \leqslant r - 1 \quad \forall\, r \in [\sqrt{s}, 1[. \tag{4.80}$$

Furthermore, from (4.76), we deduce that the quantity $\log M_p(g^{(k)}, r)/\log r$ is greater than

$$\frac{\log(1/N_I)}{-(k+1)\log s} + \frac{1}{k+1}\Bigg[\frac{\log \frac{\beta_k^{k+1}(1-s)}{s^{k(k+1)}(r-s)^k}}{\log r} - \frac{\log(1-r)^k}{\log r}\Bigg]. \tag{4.81}$$

By using the first inequality of (4.80) and the fact that $r \geqslant \sqrt{s}$, we prove that the bracket term of (4.81) is greater than

$$\bigg(\log \frac{2^k(1-\sqrt{s})^k \log^k r}{\log^k s} - \log \frac{\beta_k^{k+1}(1-s)}{s^{k(k+1)}(\sqrt{s}-s)^k}\bigg)\Big/ -\log r,$$

which is equal to

$$\frac{1}{-\log r}\,\log A\frac{(2(k+1))^k\log^k r}{((k+1)\beta_k)^k\log^k s} \quad \text{where } A = \frac{(s^{k+1}\sqrt{s})^k(1-\sqrt{s})^{2k-1}}{\beta_k(1+\sqrt{s})} < 1. \tag{4.82}$$

By substituting the value of $\log r$, (4.82) becomes

$$\frac{-\log(1/N_I)}{(k+1)\beta_k \log s}\Big(-k + \frac{\log A}{\log\log(1/N_I)} + \frac{k\log 2(k+1) + k\log\log\log(1/N_I)}{\log\log(1/N_I)}\Big).$$

Since, $N_I \leqslant \mathrm{e}^{-\Gamma}$; $\Gamma \geqslant \mathrm{e}$, we observe that the last term in the parentheses is positive, and then we deduce that

$$\frac{\log \frac{\beta^{k+1}(1-s)}{s^{k(k+1)}(r-s)^k}}{\log r} - \frac{\log(1-r)^k}{\log r} \geqslant \frac{-\log(1/N_I)}{(k+1)\beta_k\log s}\Big(-k + \frac{\log A}{\log\Gamma}\Big). \tag{4.83}$$

Plugging (4.83) into (4.81), we get

$$\frac{\log M_p(g^{(k)}, r)}{\log r} \geqslant \frac{-1}{(k+1)\log s}\Big[1 + \frac{1}{\beta_k}\Big(-k + \frac{\log A}{\log\Gamma}\Big)\Big]\log(1/N_I), \tag{4.84}$$

where the constant in the bracket term is positive when we choose $\beta_k \geqslant k - \log A/\log\Gamma$.

We recall that from Lemma 4.8 we have

$$\|g\|_{L^p(\partial G_s)} \leqslant \sum_{j=0}^{k-1}\frac{(1-r)^j + (r-s)^j}{j!}M_p(g^{(j)}, r) + \Big[\frac{\log r + 2\log(s/r)}{\log M_p(g^{(k)}, r)}\Big]^k.$$

Inequality (4.84) gives an upper bound for the above bracketed term. It remains to control the means $M_p(g^{(j)}, r)$ of the derivatives of orders $j = 0, \ldots, k-1$.

The case $k = 1$ is reduced to the single term $M_p(g, r)$ for which inequalities (4.69), (4.77), together with the condition $N_I < 1/\mathrm{e}$, give

$$M_p(g, r) \leqslant \frac{1}{(\log(1/N_I))^k}.$$

We assume now that $k \geqslant 2$. Then from Lemma 4.4 we get

$$M_p(g^{(j)}, r) \leqslant \frac{j}{r^j}\bigg(\|h_r^{(j)}\|_{L^p(\mathbb{T})} + \sum_{l=1}^{j-1} c_{l,j} M_p(g^{(l)}, r)\bigg). \tag{4.85}$$

For the first term in the parentheses, we obtain from the Kolmogorov-type inequality (3.1), the fact that $M_p(g, r) = \|h_r\|_{L^p(\mathbb{T})}$, Lemma 4.4, Theorem 3.1 for $r_1 = r^{1/\alpha}$, $r_2 = 1$, $0 < \alpha < 1$ and Lemma 4.7 that

$$\begin{aligned}
\|h_r^{(j)}\|_{L^p(\mathbb{T})} &\leqslant C_p(j+1, j)\|h_r\|_{L^p(\mathbb{T})}^{1/(j+1)}\|h_r^{(j+1)}\|_{L^p(\mathbb{T})}^{j/(j+1)} \\
&\leqslant C_p(j+1, j)(M_p(g, r))^{1/(j+1)}\bigg((j+1)\sum_{l=1}^{j+1} C_{l,j+1} M_p(g^{(l)}, r)\bigg)^{j/(j+1)} \\
&\leqslant C_p(j+1, j)(M_p(g, r))^{1/(j+1)}\bigg((j+1)\sum_{l=1}^{j+1} C_{l,j+1} (M_p(g^{(l)}, r_1))^{\alpha}\bigg)^{j/(j+1)} \\
&\leqslant \frac{\gamma_j (r-s)^{j\alpha/(j+1)}(M_p(g, r))^{(1+j\alpha^{j+1})/(j+1)}}{2^{j\alpha/(j+1)}(s(r-r_1)(r_1-s))^{j\sigma_j/(j+1)}},
\end{aligned}$$

where

$$\sigma_j = \sum_{l=1}^{j+1} \alpha^l \quad \text{and} \quad \gamma_j = C_p(j+1, j)\bigg((j+1)\sum_{l=1}^{j+1} c_{l,j+1}\bigg)^{j/(j+1)}.$$

Choosing $\alpha = (1 - 1/k)^{1/(j+1)}$, we get

$$\|h_r^{(j)}\|_{L^p(\mathbb{T})} \leqslant \frac{\gamma_j (r-s)^{j/(2(j+1))}(M_p(g, r))^{(1-1/k)}}{2^{j/(2(j+1))}(s(r-r_1)(r_1-s))^{j(1-1/k)^{1/(j+1)}}}. \tag{4.86}$$

For the last term of (4.85), we obtain from Theorem 3.1 with $0 < s < r_1 = r^{1/\alpha} < r < 1$, $0 < \alpha < 1$ and Lemma 4.7 that

$$\begin{aligned}
\sum_{l=1}^{j-1} c_{l,j} M_p(g^{(j)}, r) &\leqslant \sum_{l=1}^{j-1} c_{l,j} (M_p(g^{(l)}, r_1))^{\alpha} \\
&\leqslant \frac{(r-s)^{1/2}(M_p(g, r))^{(1-1/k)^{(j-1)/(j+1)}}}{2^{1/2}(s(r-r_1)(r_1-s))^{(j-1)(1-1/k)^{1/(j+1)}}}\sum_{l=1}^{j-1} c_{l,j}.
\end{aligned} \tag{4.87}$$

Plugging (4.86) and (4.87) into (4.85), we get

$$M_p(g^{(j)},r) \leqslant \frac{C_j(r-s)^{j/(2(j+1))}(M_p(g,r))^{(1-1/k)}}{(rs(r-r_1)(r_1-s))^j}, \tag{4.88}$$

where

$$C_j = \frac{j}{2^{j/(2(j+1))}}\Big(\gamma_j + \sum_{l=1}^{j-1} c_{l,j}\Big).$$

Hence, from (4.77), the fact that $M_p(g,r) \leqslant N_I^{\log r/\log s}$ and the inequalities

$$N_I < 1/\mathrm{e}, \qquad (k+1)\Big(1-\frac{1}{k}\Big)\beta_k \geqslant k \quad \text{for } k \geqslant 2,$$

we deduce that

$$(M_p(g,r))^{(1-1/k)} \leqslant \frac{1}{(\log(1/N_I))^k}.$$

Hence,

$$\sum_{j=0}^{k-1} \frac{(1-r)^j+(r-s)^j}{j!} M_p(g^{(j)},r) \\ \leqslant \sum_{j=0}^{k-1} \frac{C_j}{j!} \frac{((1-r)^j+(r-s)^j)(r-s)^{j/(2(j+1))}}{(rs(r-r_1)(r_1-s))^j} \frac{1}{(\log(1/N_I))^k}.$$

Furthermore, we observe that

$$\frac{((1-r)^j+(r-s)^j)(r-s)^{j/(2(j+1))}}{(rs(r-r_1)(r_1-s))^j} \\ = \frac{[(1-r)^j+(r-s)^j](r-s)^{j/(2(j+1))}}{(r^2s)^j(r^{(k/(k-1))^{1/(j+1)}}-s)^j} \frac{1}{(1-r^{((k/(k-1))^{1/(j+1-1)})}},$$

which is upper bounded by a constant $\widetilde{C}$ depending only on k, j and s, since r satisfies the inequalities in (4.79).

Consequently

$$\sum_{j=0}^{k-1} \frac{(1-r)^j+(r-s)^j}{j!}\ M_p(g^{(j)},r) \leqslant \frac{Ce^{\widetilde{C}}}{(\log(1/N_I))^k}, \tag{4.89}$$

where $C = \max(C_j)$ for $j = 0, \ldots, k-1$.

Plugging (4.84) and (4.89) into (4.50) we get that there exists an explicit constant δ depending only on k, s and p such that

$$\|g\|_{L^p(\partial G_s)} \leqslant \frac{\delta}{(\log(1/N_I))^k}.$$

Making use of the relation $g = f/m$ and the definition of N_I in (4.69), we derive that

$$\|f\|_{L^p(\partial G_s)} \leqslant \frac{m\delta/(2C_s)^k}{(\lambda \log(1/\|f\|_{L^1(I)}))^k} \leqslant \frac{\alpha}{(\lambda \log(1/\|f\|_{L^1(I)}))^k},$$

where $\alpha = m\delta/(2C_s)^k$. We conclude the proof of Theorem 2.1 in the annular case.

$\square$

P r o o f of Proposition 2.2 in the case of the annulus. For $a > 1$, we consider the sequence of polynomials

$$u_n(z) = (z - a)^n, \quad n \geqslant 1.$$

As have as in the proof of the open unit disk case we have that

$$\begin{aligned} I_n := \|u_n\|^p_{L^p(\mathbb{T})} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + a^2 - 2a\cos\theta)^{np/2} \, \mathrm{d}\theta \\ &= (2\pi anp)^{-1/2} \, (1 + a)^{np+1} \, (1 + o(1)), \end{aligned}$$

and

$$\begin{aligned} J_n := \|u_n\|^p_{L^p(s\mathbb{T})} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (a^2 + s^2 - 2as\cos\theta)^{np/2} \, \mathrm{d}\theta \\ &= (2\pi anps)^{-1/2} (a + s)^{np+1} (1 + o(1)). \end{aligned}$$

Then we deduce that

$$\begin{aligned} \|u_n\|^p_{H^{k,p}(G_s)} &= I_n + J_n + n^p(I_{n-1} + J_{n-1}) + n^p(n-1)^p \ldots (n-k+1)^p(I_{n-k} + J_{n-k}) \\ &= (2\pi anp)^{-1/2} n^{kp} (a+1)^{1+np-kp} (1 + o(1)), \end{aligned}$$

and also

$$\|f_n\|^p_{L^\infty(I)} = (2\pi anp)^{1/2} n^{-kp} (1+a)^{-np+kp-1} (1 + a^2)^{np/2} (1 + o(1)).$$

Hence,

$$\lim_{n\to\infty} \|f_n\|_{L^p(\mathbb{T})} \log^k(1/\|f_n\|_{L^\infty(I)}) = \Big(\frac{1+a}{2}\Big)^k \log^k\Big(\frac{(1+a)^2}{1+a^2}\Big)(1 + o(1)),$$

which concludes the proof of (2.2). $\square$

Note that Theorem 2.1 still holds when the subset I is the union of a finite number of connected subsets. We get there the following corollary.

Corollary 4.10. *Let $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and let I be the union of a finite number of connected subsets of ∂G of length $2\pi\lambda$; $\lambda \in]0, 1[$. There exists two non-negative* *constants α and Γ, depending only on $k5$, p and s, such that for every $f \in B_{k,p}(G)$ satisfying $\|f\|_{L^1(I)} \leqslant \mathrm{e}^{-\Gamma}$, we have*

$$\|f\|_{L^p(\partial G)} \leqslant \frac{\alpha}{|\lambda \log \|f\|_{L^1(I)}|^k}.$$

Proof. Let $n \in \mathbb{N}^*$ and let for $j = 1, \ldots, n$; I_j be an open connected subset of the boundary of G of length $2\pi\lambda_j$; $0 < \lambda_j < 1$ such that $I = I_1 \cup \ldots \cup I_n$.

By applying Theorem 2.1 to each I_j, we prove that there exist two non-negative constants α_j and Γ_j, depending only on k, p and s, such that for every $f \in B_{k,p}(G)$ satisfying $\|f\|_{L^1(I)} \leqslant \mathrm{e}^{-\Gamma_j}$, we have

$$\|f\|_{L^p(\partial G)} \leqslant \frac{\alpha_j}{|\lambda_j \log \|f\|_{L^1(I_j)}|^k}. \tag{4.90}$$

Since

$$\|f\|_{L^1(I_j)} \leqslant \frac{\lambda}{\lambda_j} \|f\|_{L^1(I)},$$

hence, if we choose $\Gamma = \sup\limits_{1 \leqslant j \leqslant n} \Gamma_j$, we derive from (4.90) and the monotonicity of the mapping $x \longmapsto 1/\log(1/x)$ that

$$\|f\|_{L^p(\partial G)} \leqslant \frac{\alpha_j}{|\lambda_i \log \|f\|_{L^1(I_j)}|^k} \leqslant \frac{2^k \alpha_j / \lambda_j^k}{|\log \|f\|_{L^1(I)}|^k},$$

where in the latter inequality we have assumed that $\|f\|_{L^1(I)} \leqslant (\lambda_j/\lambda)^2$, which is satisfied if we suppose further that $\mathrm{e}^{-\Gamma} \leqslant (\lambda_j/\lambda)^2$; $j = 1, \ldots, n$. We conclude the proof of the corollary by setting $\alpha = \sup\limits_{1 \leqslant j \leqslant n} (2^k \alpha_j / \lambda_j^k)$. $\square$

5. Concluding remarks

We have established in the present paper an optimal estimate of $1/\log$-type in the Hardy-Sobolev spaces $H^{k,p}(G)$, $1 \leqslant p \leqslant \infty$, and k is a positive integer (G is the unit disk $\mathbb{D}$ or the annulus G_s). More precisely, we have studied the behavior with respect to the L^p-norm of functions, elements of the unit ball of $H^{k,p}(G)$, on the whole boundary of G with respect to the L^1-norm on an open connected subset I of the boundary ∂G.

(1) We have observed that our result still holds in more general situations of a smooth connected domain $\mathfrak{g}$ in $\mathbb{R}^2$:
   (a) For simply-connected domain: Theorem 2.1 remains valid in a simply-connected bounded Jordan domain $\mathfrak{g}$ in $\mathbb{R}^2$ with $C^{1,\beta}$ boundary, $\beta \in]0, 1[$.

It is a well known result (cf. [15], Theorems 3.5 and 3.6) that there exists a conformal mapping ψ from $\mathbb{D}$ onto $\mathfrak{g}$ having a C^1 extension to $\overline{\mathbb{D}}$. Moreover, the derivative ψ' does not vanish on the unit circle $\mathbb{T}$.

(b) For doubly-connected domain: Theorem 2.1 remains also valid in a doubly-connected domain $\mathfrak{g}$ in $\mathbb{R}^2$ with $C^{1,\beta}$ boundary, $0 < \beta < 1$, made of two non-intersecting closed $C^{1,\beta}$ Jordan curves: By applying the extensions of [15], Theorems 3.5 and 3.6, given in [4], Proposition 4.2, and also in [20], we can deduce that there exists a conformal mapping ψ from G_s onto $\mathfrak{g}$ having a C^1 extension to $\overline{G_s}$.

(2) We have also observed in Corollary 4.10 that Theorem 2.1 still holds when the subset I is supposed to be a finite union of connected subsets.

(3) Questions concerning the behaviour of the constants α and Γ mentioned in the main result are of interest and will be undertaken in a subsequent work. Note that in the particular case $G = \mathbb{D}$, $k = 1$ and for the uniform norm where $p = \infty$, an upper bound for the constant α has been established in [7]. Let us mention also that the question under investigation is to give the optimal constant α in inequality (2.1):

$$\alpha = \max_{f \in B_{k,2}(G_s)} \|f\|_{L^2(\partial G_s)} |\log \|f\|_{L^1(I)}|^k.$$

(4) We finally mention [7], [14], where stability of Cauchy's problem for the Laplace operator in the bi-dimensional case and for the inverse problem of identifying Robin's coefficients by boundary measurements have been studied when $G = \mathbb{D}$ and $p = 2, \infty$. In this context, we consider the Cauchy problem

$$(\mathrm{CP}) \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}; \\ \partial_n u = \Phi & \text{on } I; \\ u = f & \text{on } I, \end{cases}$$

where $\partial_n u$ stands for the partial derivative of u with respect to the outer normal, Φ denotes the imposed current flux and f the potential measurement. Then, we establish the following logarithmic stability result:

Let $\Phi \in \mathcal{C}^0(\overline{I})$, $K > 0$ and $1 \leqslant p \leqslant \infty$. We denote by $\mathcal{W}_K$ the set

$$\mathcal{W}_K = \{u \in W^{1,p}(\partial G), \|u\|_{W^{1,p}(\partial G)} \leqslant K\}.$$

Let $u_i \in \mathcal{W}_K$ be the solution of (CP) when $f = f_i$; $i = 1, 2$. If $\|f_1 - f_2\|_{L^1(I)} < \mathrm{e}^{-\Gamma}$, then

$$\|u_1 - u_2\|_{L^p(\mathbb{T})} \leqslant \frac{\beta}{|\log(\|f_1 - f_2\|_{L^1(I)})|},$$

where $\Gamma, \beta > 0$ are constants depending only on Φ, I and K.

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References

[1] *G. Alessandrini, L. Del Piero, L. Rondi*: Stable determination of corrosion by a single electrostatic boundary measurement. Inverse Probl. *19* (2003), 973–984. zbl MR doi
[2] *A. A. Balinskii, S. P. Novikov*: Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. Sov. Math., Dokl. *32* (1985), 228–231. (In English. Russian original.); translation from Dokl. Akad. Nauk SSSR *283* (1985), 1036–1039. zbl MR
[3] *V. F. Babenko, V. A. Kofanov, S. A. Pichugov*: Inequalities of Kolmogrov type and some their applications in approximation theory. Proceedings of the 3rd International Conference on Functional Analysis and Approximation Theory, Palermo: Circolo Matemàtico di Palermo, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 52. 1998, pp. 223–237. zbl MR
[4] *L. Baratchart, F. Mandréa, E. B. Saff, F. Wielonsky*: 2D inverse problems for the Laplacian: a meromorphic approximation approach. J. Math. Pures Appl. (9) *86* (2006), 1–41. zbl MR doi
[5] *L. Baratchart, M. Zerner*: On the recovery of functions from pointwise boundary values in a Hardy-Sobolev class of the disk. J. Comput. Appl. Math. *46* (1993), 255–269. zbl MR doi
[6] *H. Brézis*: Analyse fonctionnelle. Théorie et applications. Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983. (In French.) zbl MR
[7] *S. Chaabane, I. Feki*: Optimal logarithmic estimates in Hardy-Sobolev spaces $H^{k,\infty}$. C. R., Math., Acad. Sci. Paris *347* (2009), 1001–1006. zbl MR doi
[8] *S. Chaabane, I. Fellah, M. Jaoua, J. Leblond*: Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems. Inverse Probl. *20* (2004), 47–59. zbl MR doi
[9] *I. Chalendar, J. R. Partington*: Approximation problems and representations of Hardy spaces in circular domains. Stud. Math. *136* (1999), 255–269. zbl MR doi
[10] *B. Chevreau, C. M. Pearcy, A. L. Shields*: Finitely connected domains G, representations of $H^\infty(G)$, and invariant subspaces. J. Oper. Theory *6* (1981), 375–405. zbl MR
[11] *P. L. Duren*: Theory of H^p Spaces. Pure and Applied Mathematics 38, Academic Press, New York, 1970. zbl MR doi
[12] *I. Feki*: Estimates in the Hardy-Sobolev space of the annulus and stability result. Czech. Math. J. *63* (2013), 481–495. zbl MR doi
[13] *I. Feki, H. Nfata*: On L^p-L^1 estimates of logarithmic-type in Hardy-Sobolev spaces of the disk and the annulus. J. Math. Anal. Appl. *419* (2014), 1248–1260. zbl MR doi
[14] *I. Feki, H. Nfata, F. Wielonsky*: Optimal logarithmic estimates in the Hardy-Sobolev space of the disk and stability results. J. Math. Anal. Appl. *395* (2012), 366–375. zbl MR doi
[15] *D. Gaier, C. Pommerenke*: On the boundary behavior of conformal maps. Mich. Math. J. *14* (1967), 79–82. zbl MR doi
[16] *G. H. Hardy*: The mean value of the modulus of an analytic function. Lond. M. S. Proc. (2) *14* (1915), 269–277. zbl doi
[17] *G. H. Hardy, E. Landau, J. E. Littlewood*: Some inequalities satisfied by the integrals or derivatives of real or analytic functions. Math. Z. *39* (1935), 677–695. zbl MR doi
[18] *B. E. Klots*: Approximation of differentiable functions by functions of large smoothness. Math. Notes *21* (1977), 12–19. (In English. Russian original.); translation from Mat. Zametki *21* (1977), 21–32. zbl doi
[19] *J. Leblond, M. Mahjoub, J. R. Partington*: Analytic extensions and Cauchy-type inverse problems on annular domains: stability results. J. Inverse Ill-Posed Probl. *14* (2006), 189–204. zbl MR doi

[20] *H. Meftahi, F. Wielonsky*: Growth estimates in the Hardy–Sobolev space of an annular domain with applications. J. Math. Anal. Appl. *358* (2009), 98–109. zbl MR doi
[21] *P. D. Miller*: Applied Aymptotic Analysis. Graduate Studies in Mathematics 75, American Mathematical Society, Providence, 2006. zbl MR doi
[22] *L. Nirenberg*: An extended interpolation inequality. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. *20* (1966), 733–737. zbl MR
[23] *W. Rudin*: Analytic functions of class H_p. Trans. Am. Math. Soc. *78* (1955), 46–66. zbl MR doi
[24] *D. Sarason*: The H^p spaces of an annulus. Mem. Amer. Math. Soc. *56* (1965), 78 pages. zbl MR doi
[25] *H.-C. Wang*: Real Hardy spaces of an annulus. Bull. Aust. Math. Soc. *27* (1983), 91–105. zbl MR doi

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