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ORDER OF THE SMALLEST COUNTEREXAMPLE TO GALLAI'S CONJECTURE

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Abstract. In 1966, Gallai conjectured that all the longest paths of a connected graph have a common vertex. Zamfirescu conjectured that the smallest counterexample to Gallai's conjecture is a graph on 12 vertices. We prove that Gallai's conjecture is true for every connected graph G with $\alpha'(G) \leqslant 5$, which implies that Zamfirescu's conjecture is true.

Keywords: longest path; matching number

MSC 2010: 05C38, 05C70, 05C75

1. INTRODUCTION

Graphs in this paper are simple (without loops or parallel edges), finite and undirected. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. Let v be a vertex of $V(G)$. The *neighborhood* of v in G, denoted by $N_G(v)$, is the set of vertices in $V(G)$ which are adjacent to v. The *degree* of v in G, denoted by $d_G(v)$, equals to $|N_G(v)|$. A *matching* in a graph is a set of pairwise nonadjacent edges. A *maximum matching* is a matching with the largest number of edges. The *matching number* of G, denoted by $\alpha'(G)$, is the number of edges in the maximum matching of G.

The research on the intersection of longest paths in a graph has a long history. In particular, Gallai in [6] proposed the following conjecture in 1966.

Conjecture 1.1 (Gallai [6]). If G is a connected graph, then all the longest paths of G have a common vertex.

Three years later, Walther in [9] disproved Gallai's conjecture by exhibiting a counterexample on 25 vertices. Up to now, the smallest counterexample to Gallai's

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conjecture is a graph on 12 vertices (see Figure 1), which was found by Walther in [10] and Zamfirescu in [12], independently. One may find that this graph is somewhat interesting: for each vertex v of it there is a longest path not containing v. Therefore, all the longest paths share no common vertex.

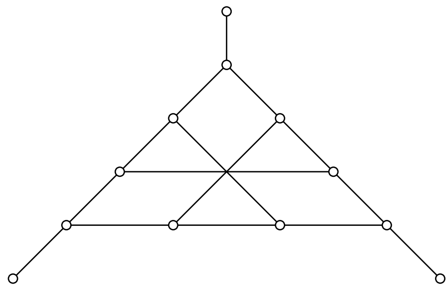

Figure 1. A counterexample to Gallai's conjecture on 12 vertices.

Although Gallai's conjecture has been disproved, finding classes of graphs that support this conjecture is also very meaningful. An obvious such example is the class of trees. In 1990, Klavžar and Petkovšek in [7] proved that Conjecture 1.1 holds on split graphs, and every connected graph such that each block is Hamiltonian-connected, almost Hamiltonian-connected or a cycle. As a corollary, Gallai's conjecture is true for the class of cacti. In 2004, Balister, Györi, Lehel, and Schelp in [1] showed that circular arc graphs support Conjecture 1.1. In 2013, Rezende, Fernandes, Martin and Wakabayashi in [5] proved that Conjecture 1.1 also holds on outer-planar graphs and 2-trees. In 2015, Chen in [3] proved that Gallai's conjecture is true for graphs with small matching number. In 2017, Chen et al. in [4] proved that Gallai's conjecture is true for all series-Parallel graphs (K_4-minor-free graphs).

In this paper, we prove the following statement:

Theorem 1.1. *If G is a connected graph with $\alpha'(G) \leqslant 5$, then all the longest paths of G have a common vertex.*

Theorem 1.1 implies that the following conjecture is true, which was verified by Brinkmann and Van Cleemput, see [2], by using computers.

Conjecture 1.2 (Zamfirescu [11]). A smallest counterexample to Gallai's conjecture is a graph on 12 vertices.

2. Proof of Theorem 1.1

We prove by contradiction. Let G be a counterexample. Since G is connected, if G has no cycle, then G is a tree, and therefore all the longest paths of G have a common vertex (a center vertex of G). So G has a cycle. Let $C = v_1v_2\dots v_rv_1$, $r \geqslant 3$ be a longest cycle of G, and $P = x_0x_1\dots x_s$ be a longest path of G. We write $C[v_i, v_j]$ for the longer subpath of C between v_i and v_j (if there are two different longest paths between v_i and v_j in C, we choose one for $C[v_i, v_j]$ arbitrarily), and $P[x_m, x_n]$ for the subpath of P between x_m and x_n, $1 \leqslant i, j \leqslant r$ and $0 \leqslant m, n \leqslant s$. Since $\alpha'(G) \leqslant 5$, we have that $r \leqslant 11$ and $s \leqslant 10$. If C is a Hamilton cycle, then every longest path of G is a Hamilton path, therefore all the longest paths of G have a common vertex. Thus, C is not a Hamilton cycle. Let $R = G - C$ and $u \in V(R)$.

Claim 2.1. $s \geqslant r$.

Proof. Since G is connected and C is not a Hamilton cycle, there is a vertex $y \in V(R)$ such that $yv_i \in E(G)$, where $v_i \in V(C)$. Then $yv_iv_{i+1}\dots v_{i-1}$ is a path of length r. Since P is a longest path of G, $s \geqslant r$. □

Claim 2.2. *If there is a vertex $v \in V(G)$ such that $N_G(v) = \{v_1, v_2\}$, then every longest path of G containing v must also contain v_1 and v_2.*

Proof. Let Q be a longest path of G such that $v \in V(Q)$. If $v_1 \notin V(Q)$ or $v_2 \notin V(Q)$, then v is an end-vertex of Q, since $N_G(v) = \{v_1, v_2\}$. But now $Q \cup vv_1$ or $Q \cup vv_2$ is a path longer than Q, a contradiction. □

Claim 2.3. *If there is a vertex $v \in V(G) \setminus V(P)$ such that $vx_i \in E(G)$, $1 \leqslant i \leqslant s-1$, then $vx_{i-1}, vx_{i+1} \notin E(G)$.*

Proof. If $vx_{i-1} \in E(G)$ or $vx_{i+1} \in E(G)$, then $(P - x_ix_{i-1}) \cup x_{i-1}vx_i$ or $(P - x_ix_{i+1}) \cup x_{i+1}vx_i$ is a path longer than P, a contradiction. □

Claim 2.4. *If $Q_1 = y_0y_1\dots y_s$ and $Q_2 = z_0z_1\dots z_s$ are two longest paths of G, then $Q_1 \cap Q_2 \neq \emptyset$.*

Proof. If $Q_1 \cap Q_2 = \emptyset$, then since G is connected, there is a path W connecting Q_1 and Q_2. Suppose that W connects $y_j \in Q_1$ and $z_l \in Q_2$, $1 \leqslant j, l \leqslant s-1$, and $Q_1[y_0, y_j]$ is a longer part of Q_1, and $Q_2[z_0, z_l]$ is a longer part of Q_2. Now $Q_1[y_0, y_j] \cup W[y_j, z_l] \cup Q_2[z_l, z_0]$ is a path longer than Q_1, a contradiction. □

By Claim 2.1 and $s \leqslant 10$ we have $r \leqslant 10$. Now we distinguish several cases in the following subsections.

2.1. Proof of the case $r = 3$. If $r = 3$, then by Claim 2.1 we have $s \geqslant 3$.

Claim 2.5. $|V(P) \cap V(C)| \geqslant 1$.

Proof. If $|V(P) \cap V(C)| = 0$, then since G is connected, there is a path W connecting P and C. Suppose that W connects v_i and x_j. We assume that $s \geqslant 6$, for otherwise either $v_{i+1}v_{i-1}v_iWx_jPx_s$ or $v_{i+1}v_{i-1}v_iWx_jPx_0$ is a path longer than P, a contradiction. If $s \geqslant 9$, then x_0x_1, x_2x_3, x_4x_5, x_6x_7, x_8x_9, v_1v_2 are 6 independent edges, a contradiction. Thus $s \leqslant 8$. If $x_j \notin \{x_3, x_{s-3}\}$ or $v_ix_j \notin E(G)$, then $s = 8$ and $x_j = x_4$, for otherwise either $v_{i+1}v_{i-1}v_iWx_jPx_s$ or $v_{i+1}v_{i-1}v_iWx_jPx_0$ is a path longer than P, a contradiction. But now x_0x_1, x_2x_3, x_5x_6, x_7x_8, $v_{i-1}v_{i+1}$ and an edge in x_4Wv_i are 6 independent edges, a contradiction. Thus $x_j \in \{x_3, x_{s-3}\}$ and $v_ix_j \in E(G)$.

Now we can check that for each $v \in \{v_{i-1}, v_{i+1}\}$, the followings hold:

(i) v is not adjacent to any vertex in $V(P)$ (since $r = 3$);

(ii) v is not adjacent to any vertex in $V(G) \setminus (V(P) \cup V(C))$ (since if there is a vertex $z \in V(G) \setminus (V(P) \cup V(C))$ such that $zv \in E(G)$, then $zv_{i+1}v_{i-1}v_ix_jPx_s$ or $zv_{i+1}v_{i-1}v_ix_jPx_0$ or $zv_{i-1}v_{i+1}v_ix_jPx_s$ or $zv_{i-1}v_{i+1}v_ix_jPx_0$ is a path longer than P, a contradiction);

(iii) $d_G(v) = 2$ (since (i) and (ii)).

By Claim 2.2, every longest path containing $v_{i+1}(v_{i-1})$ must also contain v_i. Now we prove that if a longest path Q contains v_i, then Q contains x_j. By Claim 2.4, $P \cap Q \neq \emptyset$. If Q does not contain x_j, then there exists a vertex $x_t \neq x_j$ such that v_iQx_t is a segment of Q and $v_iQ \cap P = \emptyset$. By Claim 2.3, $x_t \notin \{x_{j-1}, x_{j+1}\}$. But now $v_iQx_tPx_jv_i$ is a cycle of length at least 4, a contradiction. Thus, every longest path of G containing v_i must also contain x_j. Therefore all the longest paths of G contain x_j. Since G is a counterexample, $|V(P) \cap V(C)| \geqslant 1$. $\square$

Claim 2.6. *For any longest path Q of G, $|V(Q) \cap V(C)| = 1$ or $|V(Q) \cap V(C)| = 3$.*

Proof. By Claim 2.5, $|V(Q) \cap V(C)| \geqslant 1$. If $|V(Q) \cap V(C)| = 2$, then without loss of generality, suppose that $v_1, v_2 \in V(Q)$. Now $v_1v_2 \in E(Q)$, since if $v_1v_2 \notin E(Q)$, then $v_1v_3v_2Q[v_2, v_1]v_1$ is a cycle longer than C, a contradiction. But now $(Q - v_1v_2) \cup v_1v_3v_2$ is a path longer than Q, a contradiction. $\square$

Claim 2.7. *If there is a longest path $Q = y_0y_1 \ldots y_s$ of G such that $|V(Q) \cap V(C)| = 1$, then all the longest paths of G share a common vertex.*

Proof. Suppose that $V(Q) \cap V(C) = \{y_j\}$. Without loss of generality, suppose that $v_1 = y_j$. Let Q_1 be a longest path of G containing v_i ($i \in \{2, 3\}$). By Claim 2.4, $Q \cap Q_1 \neq \emptyset$. If Q_1 does not contain y_j, then there exists a vertex $y_t \neq y_j$ such that

$v_iQ_1y_t$ is a segment of Q_1 and $v_iQ_1 \cap Q = \emptyset$. By Claim 2.3, $y_t \notin \{y_{j-1}, y_{j+1}\}$. But now $v_iQ_1y_tQy_jv_i$ is a cycle of length at least 4, a contradiction. Thus, every longest path of G containing v_2 (v_3) must also contain $v_1 = y_j$. Therefore all the longest paths of G contain y_j. □

Since G is a counterexample, by Claims 2.6 and 2.7, for every longest path Q of G, $V(C) \subset V(Q)$. But now all the longest paths of G contain $V(C)$, a contradiction.

2.2. Proof of the case $r = 4$. If $r = 4$, then by Claim 2.1, $s \geqslant 4$.

Claim 2.8. *For any longest path Q of G, $|V(Q) \cap V(C)| \geqslant 2$.*

Proof. Let $Q = y_0y_1 \dots y_s$ be a longest path of G. If $|V(Q) \cap V(C)| = 0$, then $s \leqslant 6$, for otherwise y_0y_1, y_2y_3, y_4y_5, y_6y_7 and v_1v_2, v_3v_4 are 6 independent edges, a contradiction. But now we could find a path longer than Q, a contradiction.

If $|V(Q) \cap V(C)| = 1$, then $s \geqslant 6$, for otherwise we could find a path longer than Q, a contradiction. Suppose that $V(Q) \cap V(C) = \{y_j\}$. Without loss of generality, assume $y_j = v_1$. If $s \geqslant 9$, then y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_2v_3 are 6 independent edges, a contradiction. Thus $6 \leqslant s \leqslant 8$.

We can check that for each vertex $v \in \{v_2, v_3, v_4\}$ the following holds:

(i) v is not adjacent to y_{j-1}, y_{j+1} (since otherwise $(Q - v_1y_{j-1}) \cup v_1C[v_1, v]vy_{j-1}$ or $(Q - v_1y_{j+1}) \cup v_1C[v_1, v]vy_{j+1}$ is a path longer than Q);

(ii) v is not adjacent to any vertex in $Q[y_0, y_{j-2}] \cup Q[y_{j+2}, y_s]$ (since $r = 4$).

Let Q_1 be a longest path of G containing v_i, $i \in \{2, 3, 4\}$. By Claim 2.4, $Q \cap Q_1 \neq \emptyset$. If Q_1 does not contain v_1, then there exists a vertex $y_t \neq y_j(v_1)$ such that $v_iQ_1y_t$ is a segment of Q_1 and $v_iQ_1 \cap Q = \emptyset$. By (i) and (ii) we could obtain a cycle of length at least 5, a contradiction.

Thus, every longest path of G containing v_i, $i \in \{2, 3, 4\}$ must also contain v_1. Therefore all the longest paths of G contain v_1. Since G is a counterexample, $|V(Q) \cap V(C)| \geqslant 2$. □

Claim 2.9. *For any longest path Q of G, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

Proof. If not, then there exists a longest path Q_1 and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. By Claim 2.8, $v_{i+1}, v_{i-2} \in V(Q_1)$. If $v_{i+1}v_{i-2} \in E(Q_1)$, then $(Q_1 - v_{i+1}v_{i-2}) \cup v_{i+1}v_iv_{i-1}v_{i-2}$ is a path longer than Q_1, a contradiction. Thus $v_{i+1}v_{i-2} \notin E(Q_1)$. But now $Q_1[v_{i+1}, v_{i-2}]v_{i-2}v_{i-1}v_iv_{i+1}$ is a cycle longer than C, a contradiction. □

Claim 2.10. *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of G contain v_{i-1} and v_{i+1}.*

Proof. If not, then there is a longest path Q_1 such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. By Claim 2.9, $v_i \in V(Q_1)$. We assume that v_i is not the end-vertex of Q_1, since otherwise adding v_{i-1} or v_{i+1} to Q_1 results in a longer path, a contradiction. But now $d_G(v_i) \geqslant 3$, a contradiction. □

Claim 2.11. *If there is a longest path $Q = y_0y_1 \dots y_s$ of G such that $|V(Q) \cap V(C)| \leqslant 3$, then for each vertex $v_i \in V(C) \setminus V(Q)$, $d_Q(v_i) = 2$, and all the longest paths of G share a common vertex.*

Proof. Without loss of generality, suppose that $v_1 \notin V(Q)$. By Claim 2.9, v_2, $v_4 \in V(Q)$. Since $r = 4$, $v_2w_1v_4$ is a subpath of Q in G (w_1 may be a vertex of $V(C)$). Now v_2 and v_4 are not end-vertices of Q, since otherwise adding v_1 to Q results in a longer path, a contradiction. Suppose $v_2 = y_k, v_4 = y_j$, $1 \leqslant k < j \leqslant s - 1$.

We can check that for each vertex $v \in \{v_1, w_1\}$ the following assertions hold:

(i) v_1 is not adjacent to w_1 (by Claim 2.3);

(ii) v is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+1}, y_s]$ (since $r = 4$);

(iii) $d_Q(v) = 2$ (since (i) and (ii)).

If $d_G(v) = 2$ ($v \in \{v_1, w_1\}$), then by Claim 2.10, all the longest paths of G contain v_2 and v_4. Since G is a counterexample, $d_G(v) \geqslant 3$. Thus, there is a vertex $v' \in V(G) \setminus V(Q)$ such that $vv' \in E(G)$. If there is a vertex $v'' \in V(G) \setminus V(Q)$ such that $v'v'' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[v_4, y_s]$ have lengths at least 3, for otherwise either $v''v'v_1v_2Qy_s$ ($v''v'w_1v_2v_1v_4Qy_s$) or $v''v'v_1v_4Qy_0$ ($v''v'w_1v_4v_1v_2Qy_0$) is a path longer than Q, a contradiction. But now y_0y_1, v_2y_{k-1}, w_1w_1', v_4y_{j+1}, $y_{s-1}y_s$, $v_1'v_1''$ are 6 independent edges, a contradiction. Thus $N_G(v') = N_Q(v') \cup \{v\}$.

Since $N_Q(v) \subseteq \{v_2, v_4\}$, $N_Q(v') \subseteq \{v_2, v_4\} \cup \{v \in Q\}$. By Claim 2.3, $v'v_2, v'v_4 \notin E(G)$. Thus $d_G(v') = 1$.

Now if there is a longest path Q_1 not containing v_2, then by Claim 2.9, $v_1, w_1 \in V(Q_1)$. Since $d_Q(v_1) = d_Q(w_1) = 2$, there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1v_1v_4$ and $u_2w_1v_4$ are two subpaths of Q_1. Since $d_G(u_1) = d_G(u_2) = 1$, $Q_1 = u_1v_1v_4w_1u_2$. But now $u_1v_1v_2w_1v_4Qy_s$ is a path longer than Q_1, a contradiction. Thus, all the longest paths of G contain v_2. □

Since G is a counterexample, by Claim 2.11, for every longest path Q of G, $V(C) \subset V(Q)$. But now all the longest paths of G contain $V(C)$, a contradiction.

2.3. Proof of the case $r = 5$. If $r = 5$, then by Claim 2.1 we have that $s \geqslant 5$.

Claim 2.12. *For any longest path Q of G, $|V(Q) \cap V(C)| \geqslant 3$.*

P r o o f. Let $Q = y_0y_1 \dots y_s$ be a longest path of G. If $|V(Q) \cap V(C)| \leqslant 1$, then $s \leqslant 6$, for otherwise y_0y_1, y_2y_3, y_4y_5, y_6y_7 and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. But now, we could find a path longer than Q, a contradiction. If $|V(Q) \cap V(C)| = 2$, then at least two vertices of $V(C) \setminus V(Q)$ are consecutive in C. Suppose v_iWv_j is a maximum consecutive segment in $C \setminus V(Q)$, and $N_{C\cap Q}(v_i) = \{\overline{v}_i\}$, $N_{C\cap Q}(v_j) = \{\overline{v}_j\}$. Since Q is a longest path of G, the length of $Q[\overline{v}_i, \overline{v}_j]$ is at least 3. But now $v_iWv_j\overline{v}_jQ[\overline{v}_j, \overline{v}_i]\overline{v}_iv_i$ is a cycle of length at least 6, a contradiction. Thus $|V(Q) \cap V(C)| \geqslant 3$. $\square$

Claim 2.13. *For any longest path Q of G, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

P r o o f. We could obtain this result by the proof of Claim 2.12. $\square$

Claim 2.14. *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of G contain v_{i-1} and v_{i+1}.*

P r o o f. Similar to the proof of Claim 2.10, we could obtain this result. $\square$

Claim 2.15. *If there is a longest path $Q = y_0y_1 \dots y_s$ of G such that $|V(Q) \cap V(C)| \leqslant 4$, then for each vertex $v_i \in V(C) \setminus V(Q)$, $d_Q(v_i) = 2$, and all the longest paths of G share a common vertex.*

P r o o f. Without loss of generality, suppose that $v_1 \notin V(Q)$. By Claim 2.13, $v_2, v_5 \in V(Q)$. Since $r = 5$, $v_2w_1v_5$ or $v_2w_1w_2v_5$ is a subpath of Q (w_1, w_2 may be vertices of $V(C)$). Now v_2 and v_5 are not end-vertices of Q, since otherwise adding v_1 to Q results in a longer path, a contradiction. Suppose that $v_2 = y_k$, $v_5 = y_j$, $1 \leqslant k < j \leqslant s - 1$.

We can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) v_1 is not adjacent to w_1, y_{k-1}, y_{j+1} (if $v_2w_1v_5$ is a segment of Q) and w_1, w_2, y_{k-1}, y_{j+1} (if $v_2w_1w_2v_5$ is a segment of Q) (by Claim 2.3);

(ii) v_1 is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+2}, y_s]$ (since $r = 5$);

(iii) $d_Q(v_1) = 2$ (since (i) and (ii)).

If $d_G(v_1) = 2$, then by Claim 2.14, all the longest paths of G contain v_2 and v_5. If $d_G(v_1) \geqslant 3$, then there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1v_1' \in E(G)$. If there exists a vertex $v_1'' \in V(G) \setminus V(Q)$ such that $v_1'v_1'' \in E(G)$, then $s = 8$ and $Q = y_0y_1y_2v_2w_1v_5y_6y_7y_8$, for otherwise the 5 independent edges in Q together with v_1v_1' are 6 independent edges, a contradiction.

We can check that for the vertex w_1, the following assertions hold:

(i) w_1 is not adjacent to $\{v_1', v_1''\}$ (by Claim 2.3);

(ii) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+1}, y_8]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}] \cup Q[y_{j+1}, y_8]$ such that $w_1 z \in E(G)$, then $y_8 Q w_1 z Q v_2 v_1 v_1' v_1''$ or $y_0 Q w_1 z Q v_5 v_1 v_1' v_1''$ is a path of length at least 9, a contradiction);

(iii) w_1 is not adjacent to any vertex in $V(G) \setminus V(Q)$ (since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $z w_1 \in E(G)$, then $y_0 y_1$, $y_2 v_2$, $w_1 z$, $v_5 y_6$, $y_7 y_8$, $v_1' v_1''$ are 6 independent edges);

(iv) $d_G(w_1) = 2$ (since (i), (ii) and (iii)).

If there is a longest path Q_1 not containing v_2, then by Claim 2.13, $v_1 \in V(Q_1)$. By Claim 2.2, $w_1 \notin V(Q_1)$. If $v_5 \in V(Q_1)$, then $C_1 = v_1 Q_1[v_1, v_5] v_5 w_1 v_2 v_1$ is a cycle of length at least 6, a contradiction. Thus $v_5 \notin V(Q_1)$. By Claim 2.13, $v_4 \in V(Q_1)$. But now $C_2 = v_1 Q_1[v_1, v_4] v_4 v_5 w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. Thus, all the longest paths of G contain v_2.

Since G is a counterexample, for each vertex $w \in V(G) \setminus V(Q)$ such that $v_1 w \in E(G)$ we must have $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of G contain v_2. If there exists a longest path Q_2 not containing v_2, then by Claim 2.13, $v_1 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_5]$ such that $V(Q[w_1, u]) \cap V(Q_2) = \{u\}$, for otherwise by Claim 2.13, $v_4 \in V(Q_2)$ and $C_3 = v_1 Q_2[v_1, v_4] v_4 v_5 Q w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. But now $C_4 = v_1 Q_2[v_1, u] u Q w_1 v_2 v_1$ is a cycle of length at least 6, a contradiction. Thus $w_1 \in V(Q_2)$. By the above, $y_{k-1} \notin V(Q_2)$ and $N_G(y_{k-1}) \cap (V(G) \setminus V(Q_2)) = \{v_2\}$. Thus $y_{k-2} \in V(Q_2)$. But now $C_5 = v_1 Q_2[v_1, y_{k-2}] y_{k-2} y_{k-1} v_2 v_1$ is a cycle of length at least 6, a contradiction. $\square$

Since G is a counterexample, by Claim 2.15, for every longest path Q of G, $V(C) \subset V(Q)$. But now all the longest paths of G contain $V(C)$, a contradiction.

2.4. Proof of the case $r = 6$. If $r = 6$, then by Claim 2.1 we have that $s \geqslant 6$.

Claim 2.16. *For any longest path Q of G, $|V(Q) \cap V(C)| \geqslant 3$.*

Proof. Let $Q = y_0 y_1 \dots y_s$ be a longest path of G. Similar to the proof of Claim 2.12, we could obtain that $|V(Q) \cap V(C)| \geqslant 2$.

If $|V(Q) \cap V(C)| = 2$, then at least two vertices of $V(C) \setminus V(Q)$ are consecutive in C. Suppose $v_i W v_j$ is a maximum consecutive segment in $C \setminus V(Q)$, and $N_{C\cap Q}(v_i) = \{\overline{v}_i\}$, $N_{C\cap Q}(v_j) = \{\overline{v}_j\}$. If the length of $W[v_i, v_j]$ is at least 2, then as Q is a longest path of G, the length of $Q[\overline{v}_i, \overline{v}_j]$ is at least 4. But now $v_i W v_j \overline{v}_j Q[\overline{v}_j, \overline{v}_i] \overline{v}_i v_i$ is a cycle of length at least 8, a contradiction. If the length of $W[v_i, v_j]$ is at most 1, then as $r = 6$, there are two independent edges in $C \setminus V(Q)$. We assume that $s \leqslant 6$, since otherwise y_0y_1, y_2y_3, y_4y_5, y_6y_7 and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. But now, we could find a path longer than Q, a contradiction. $\square$

Claim 2.17. *For any longest path Q of G, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

Proof. If not, then there exists a longest path Q_1 and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. By the proof of Claim 2.16, $v_{i+1}, v_{i-2} \in V(Q_1)$. Now v_{i+1}, v_{i-2} are not end-vertices of Q_1, since otherwise adding v_i, v_{i-1} to Q_1 results in a longer path, a contradiction. Suppose that $v_{i+1} = y_k$, $v_{i-2} = y_j$, $k < j$. Since Q_1 is a longest path of G and $r = 6$, $v_{i+1}w_1w_2v_{i-2}$ is a segment of Q_1 (w_1, w_2 may be vertices of $V(C)$). If $s \geqslant 9$, then y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_iv_{i-1} are 6 independent edges, a contradiction. Thus $s \leqslant 8$.

We can check that for each vertex $v \in \{v_i, v_{i-1}\}$, the following assertions hold:

(i) v is not adjacent to w_1, w_2, y_{k-1}, y_{j+1} (by Claim 2.3);

(ii) v is not adjacent to any vertex in $Q_1[y_0, y_{k-2}] \cup Q_1[y_{j+2}, y_s]$ (since $r = 6$);

(iii) $N_{Q_1}(v) \subseteq \{v_{i+1}, v_{i-2}\}$ (since (i) and (ii)).

If there is a vertex $v_i' \in V(G) \setminus V(Q_1)$ such that $v_iv_i' \in E(G)$, then as $s \leqslant 8$, $Q_1 = y_0y_1v_{i+1}w_1w_2v_{i-2}y_6y_7y_8$. We assume that $N_G(v_i') = N_{Q_1}(v_i') \cup \{v_i\}$. Since if there is a vertex $v_i'' \in V(G) \setminus V(Q_1)$ such that $v_i'v_i'' \in E(G)$, then $y_0y_1v_{i+1}w_1w_2v_{i-2}v_{i-1}v_iv_i'v_i''$ is a path of length 9, and if $v_i'v_{i-1} \in E(G)$, then $y_0y_1v_{i+1}v_iv_i'v_{i-1}v_{i-2}y_6y_7y_8$ is a path of length 9, a contradiction. Furthermore, we assume that $N_G(v_i') \subseteq \{v_i, v_{i-2}\}$. Since by (iii), we could obtain that $N_{Q_1}(v_i') \subseteq \{v_{i+1}, v_{i-2}\}$, and if $v_i'v_{i+1} \in E(G)$, then $y_0y_1v_{i+1}v_i'v_iv_{i-1}v_{i-2}y_6y_7y_8$ is a path of length 9, a contradiction. Now $v_{i-1}v_{i+1} \notin E(G)$ and there is no vertex $v_{i-1}' \in V(G) \setminus V(Q_1)$ such that $v_{i-1}v_{i-1}' \in E(G)$. Since otherwise $v_i'v_iv_{i-1}v_{i+1}Qy_8$ or $v_{i-1}'v_{i-1}v_iv_{i+1}Qy_8$ is a path of length 9, a contradiction. Thus $d_G(v_{i-1}) = 2$.

Now if there is a longest path Q_2 not containing v_{i-2}, then by Claim 2.2, $v_{i-1} \notin V(Q_2)$. By the proof of Claim 2.16, $v_i, v_{i-3} \in V(Q_2)$. We can check that $N_{Q_2}(v_{i-2}) \subseteq \{v_i, v_{i-3}\}$. Now $w_2 \notin V(Q_2)$ or $y_6 \notin V(Q_2)$. Without loss of generality, suppose that $w_2 \notin V(Q_2)$. As above, we could prove that $N_G(w_2) \subseteq \{v_{i-2}, v_i\}$. By (i), $v_iw_2 \notin E(G)$. Now $d_G(w_2) = 1$, a contradiction to that $Q_1 = y_0y_1v_{i+1}w_1w_2v_{i-2}y_6y_7y_8$. Therefore all the longest paths of G contain v_{i-2}.

Since G is a counterexample, $N_G(v_i) \subseteq \{v_{i+1}, v_{i-2}, v_{i-1}\}$ and $N_G(v_{i-1}) \subseteq \{v_{i+1}, v_{i-2}, v_i\}$. If $v_iv_{i-2} \in E(G)$, then all the longest paths of G contain v_{i-2}. Since if there is a longest path Q_3 not containing v_{i-2}, then as $d_G(v_{i-2}) \geqslant 4$, $v_{i-1} \in V(Q_3)$. Now v_{i-1}, v_i (if they exist) are not end-vertices of Q_3, otherwise adding v_{i-2} to Q_3 results in a longer path, a contradiction. But now $v_{i+1}v_{i-1}v_iv_{i+1}$ is a segment of Q_3, a contradiction. Since G is a counterexample, $d_G(v_i) = 2$. Similarly, $d_G(v_{i-1}) = 2$. Now all the longest paths of G contain v_{i-2}. Since if there is a longest path Q_4 not containing v_{i-2}, then by Claim 2.2, $v_i, v_{i-1} \notin V(Q_4)$. But now by the proof of Claim 2.16, we could obtain that Q_4 is a path of length at least 10, a contradiction. Since G is a counterexample, for any longest path Q of G, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$. □

Claim 2.18. *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of G contain v_{i-1} and v_{i+1}.*

Proof. Similar to the proof of Claim 2.10, we could obtain this result. □

Claim 2.19. *If there is a longest path $Q = y_0y_1\dots y_s$ of G such that $|V(Q) \cap V(C)| \leqslant 5$, then for each vertex $v \in V(C) \setminus V(Q)$, $d_Q(v) = 2$.*

Proof. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.17, v_2, $v_6 \in V(Q)$. Since $r = 6$, $v_2w_1v_6$ or $v_2w_1w_2v_6$ or $v_2w_1w_2w_3v_6$ is a subpath of Q in G (w_1, w_2, w_3 may be vertices of $V(C)$). Now v_2 and v_6 are not end-vertices of Q, since otherwise adding v_1 to Q results in a longer path, a contradiction. Suppose $v_2 = y_k$, $v_6 = y_j$, $1 \leqslant k < j \leqslant s-1$.

Case 2.4.1. $v_2w_1v_6$ is a subpath of Q. In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) v_1 is not adjacent to w_1, y_{k-1}, y_{j+1} (by Claim 2.3);

(ii) v_1 is not adjacent to any vertex in $Q[y_0, y_{k-3}] \cup Q[y_{j+3}, y_s]$ (since $r = 6$);

(iii) $N_Q(v_1) \subseteq \{v_2, v_6, y_{k-2}, y_{j+2}\}$ (since (i) and (ii)).

Claim 2.20. *$v_1y_{j+2} \notin E(G)$.*

Proof. If $v_1y_{j+2} \in E(G)$, then since $r = 6$, $v_1y_{k-2} \notin E(G)$. Now we can check that for each vertex $v \in \{w_1, y_{j+1}\}$, the following assertions hold:

(i) v is not adjacent to any vertex in $Q[y_0, y_{k-1}]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}]$ such that $vz \in E(G)$, then $zQv_2v_1y_{j+2}Qw_1z$ or $zQy_jv_1y_{j+2}y_{j+1}z$ is a cycle of length at least 7);

(ii) v is not adjacent to any vertex in $Q[y_{j+3}, y_s]$ (since if there exists a vertex $z \in Q[y_{j+3}, y_s]$ such that $vz \in E(G)$, then $zQy_jv_1v_2w_1z$ or $zQy_{j+2}v_1v_2Qy_{j+1}z$ is a cycle of length at least 7);

(iii) w_1 is not adjacent to y_{j+1} (since otherwise $y_0Qw_1y_{j+1}y_jv_1y_{j+2}Qy_s$ is a path longer than Q);

(iv) $N_Q(v) \subseteq \{v_2, v_6, y_{j+2}\}$ (since (i), (ii) and (iii)).

If there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1 w_1' \in E(G)$, then $w_1' w_1 v_2 v_1 v_6 Q y_s$ is a path of length at least 7 and there is no vertex $w_1'' \in V(G) \setminus V(Q)$ such that $w_1' w_1'' \in E(G)$. Since if there exists such vertex, then $Q[y_0, v_6]$ has length at least 5, and $y_0 y_1 Q v_2 v_1 y_{j+2} y_{j+1} v_6 w_1 w_1' w_1''$ is a path of length at least 9, and we could find 6 independent edges, a contradiction. Thus $N_G(w_1') = N_Q(w_1')$. Since $N_Q(w_1) \subseteq \{v_2, v_6, y_{j+2}\}$, $N_Q(w_1') \subseteq \{w_1, v_2, v_6, y_{j+2}\}$. By Claim 2.3, $w_1' v_2, w_1' v_6 \notin E(G)$. If $w_1' y_{j+2} \in E(G)$, then $w_1' w_1 v_2 v_1 v_6 y_{j+1} y_{j+2} w_1'$ is a cycle of length 7, a contradiction. Thus $d_G(w_1') = 1$. Similarly, we could obtain that for any vertex $y_{j+1}' \in V(G) \setminus V(Q)$ such that $y_{j+1} y_{j+1}' \in E(G)$, $d_G(y_{j+1}') = 1$.

If there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1' v_1 \in E(G)$, then $s \geqslant 8$, for otherwise either $v_1' v_1 v_2 Q y_s$ or $v_1' v_1 y_{j+2} Q y_0$ is a path longer than Q, a contradiction. Furthermore $s = 8$, for otherwise $y_0 y_1$, $y_2 y_3$, $y_4 y_5$, $y_6 y_7$, $y_8 y_9$, $v_1 v_1'$ are 6 independent edges, a contradiction. We assume that $N_G(v_1') = N_Q(v_1') \cup \{v_1\}$. Since if there exists a vertex $v_1'' \in V(G) \setminus V(Q)$ such that $v_1' v_1'' \in E(G)$, then $v_1'' v_1' v_1 v_2 Q y_s$ is a path of length at least 10, a contradiction. As $N_Q(v_1) = \{v_2, v_6, y_{j+2}\}$, $N_Q(v_1') \subseteq \{v_2, v_6, y_{j+2}\}$. By Claim 2.3, $v_1' v_2, v_1' v_6, v_1' y_{j+2} \notin E(G)$. Thus $d_G(v_1') = 1$.

If there is a longest path Q_1 not containing v_6, then by Claim 2.17, $v_1, w_1, y_{j+1} \in V(Q_1)$. Now v_1, w_1, y_{j+1} are not end-vertices of Q_1, otherwise adding v_6 to Q_1 results in a longer path, a contradiction. If $v_2 v_1 y_{j+2}$ is a subpath of Q_1, then there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2$, $u_2 y_{j+1} y_{j+2}$ or $u_1 y_{j+1} v_2$, $u_2 w_1 y_{j+2}$ are two subpaths of Q_1, since otherwise $v_2 v_1 y_{j+2} w_1 v_2$ or $v_2 v_1 y_{j+2} y_{j+1} v_2$ is a subpath of Q_1, a contradiction. But now $Q_1 = u_1 w_1 v_2 v_1 y_{j+2} y_{j+1} u_2$ or $Q_1 = u_1 y_{j+1} v_2 v_1 y_{j+2} w_1 u_2$ is a path of length 6, a contradiction. Since $N_Q(v_1) = \{v_2, v_6, y_{j+2}\}$, there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1' v_1 v_2$ or $v_1' v_1 y_{j+2}$ is a segment of Q_1. Without loss of generality, suppose that $v_1' v_1 v_2$ is a segment of Q_1. Now there is no vertex $u_1 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2$ or $u_1 y_{j+1} v_2$ is a subpath of Q_1, for otherwise $Q_1 = v_1' v_1 v_2 w_1 u_1$ or $Q_1 = v_1' v_1 v_2 y_{j+1} u_1$, a contradiction. If there is a vertex $u_1 \in V(G) \setminus V(Q)$ such that $u_1 w_1 y_{j+2}$ is a subpath of Q_1, then $v_2 y_{j+1} y_{j+2}$ is a subpath of Q_1, for otherwise $Q_1 = u_1 w_1 y_{j+2} y_{j+1} u_2$ ($u_2 \in V(G) \setminus V(Q)$), a contradiction. But now $Q_1 = v_1' v_1 v_2 y_{j+1} y_{j+2} w_1 u_1$, a contradiction. Since $N_Q(w_1) \subseteq \{v_2, v_6, y_{j+2}\}$, $v_2 w_1 y_{j+2}$ is a subpath of Q_1. Similarly, we could prove that $v_2 y_{j+1} y_{j+2}$ is a subpath of Q_1. But now $v_2 w_1 y_{j+2} y_{j+1} v_2$ is a subpath of Q_1, a contradiction. Thus all the longest paths of G contain v_6. Since G is a counterexample, $v_1 y_{j+2} \notin E(G)$. $\square$

Similarly, we could obtain that $v_1 y_{k-2} \notin E(G)$. Therefore $d_Q(v_1) = 2$.

Case 2.4.2. $v_2 w_1 w_2 v_6$ **is a subpath of** Q**.** In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) v is not adjacent to any vertex in $\{w_1, w_2, y_{k-1}, y_{j+1}\}$ (by Claim 2.3);

(ii) v is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+2}, y_s]$ (since $r = 6$);
(iii) $d_Q(v) = 2$ (since (i) and (ii)).

Case 2.4.3. $v_2w_1w_2w_3v_6$ is a subpath of Q. In this case, similar to the proof of Case 2.4.1 we could obtain that $d_Q(v_1) = 2$. $\square$

Claim 2.21. *If there is a longest path $Q = y_0y_1 \ldots y_s$ of G such that $|V(Q) \cap V(C)| \leqslant 5$, then all the longest paths of G have a common vertex.*

Proof. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.19, $d_Q(v_1) = 2$. If $d_G(v_1) = 2$, then by Claim 2.18, all the longest paths of G share a common vertex.

If $d_G(v_1) \geqslant 3$, then similar to the proof of Claim 2.15 in the third, forth, fifth paragraphs, we could obtain that for each vertex $w \in V(G) \setminus V(Q)$ such that $v_1w \in E(G)$ we must have $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of G contain v_2. If there exists a longest path Q_2 not containing v_2, then by Claim 2.17, $v_1 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_6]$ such that $u \in V(Q_2)$, for otherwise by Claim 2.17, $v_5 \in V(Q_2)$ and $C_3 = v_1Q_2[v_1, v_5]v_5v_6Qw_1v_2v_1$ is a cycle of length at least 8, a contradiction. But now as $r = 6$, $C_4 = v_1Q_2[v_1, u]uw_1v_2v_1$ is a cycle of length 6. Now for C_4, $v_2, w_1 \in V(C_4) \setminus V(Q_2)$, a contradiction to Claim 2.17. Thus $w_1 \in V(Q_2)$. By Claim 2.19, $y_{k-1} \notin V(Q_2)$. By the above, $N_G(y_{k-1}) \cap (V(G) \setminus V(Q_2)) = \{v_2\}$. Thus $y_{k-2} \in V(Q_2)$. But now $C_5 = v_1Q_2[v_1, y_{k-2}]y_{k-2}y_{k-1}v_2v_1$ is a cycle of length 6, a contradiction to Claim 2.17. $\square$

Since G is a counterexample, by Claim 2.21, for every longest path Q of G, $V(C) \subset V(Q)$. But now all the longest paths of G contain $V(C)$, a contradiction.

2.5. Proof of the case $r = 7$. If $r = 7$, then by Claim 2.1 we have that $s \geqslant 7$.

Claim 2.22. *For any longest path Q of G, $|V(Q) \cap V(C)| \geqslant 3$.*

Proof. Let $Q = y_0y_1 \ldots y_s$ be a longest path of G. If $|V(Q) \cap V(C)| \leqslant 2$, then y_0y_1, y_2y_3, y_4y_5, y_6y_7 and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. $\square$

Claim 2.23. *For any longest path Q of G, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

Proof. If not, then there exists a longest path Q_1 and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. Now similar to the proof of Claim 2.17 in the first paragraph, we could obtain that $v_{i+1}w_1w_2v_{i-2}$ or $v_{i+1}w_1w_2w_3v_{i-2}$ is a segment of Q_1 (w_1, w_2, w_3 may be vertices of $V(C)$) and $7 \leqslant s \leqslant 8$.

We can check that for each vertex $v \in \{v_i, v_{i-1}\}$ the following assertions hold:

(i) v is not adjacent to $w_1, w_2, y_{k-1}, y_{j+1}$ (if $v_{i+1}w_1w_2v_{i-2}$ is a segment of Q_1) or $w_1, w_2, w_3, y_{k-1}, y_{j+1}$ (if $v_{i+1}w_1w_2w_3v_{i-2}$ is a segment of Q_1) (by Claim 2.3);

(ii) v is not adjacent to any vertex in $Q_1[y_0, y_{k-3}] \cup Q_1[y_{j+3}, y_s]$ (since $r = 7$);

(iii) v is not adjacent to any vertex in $\{y_{k-2}, y_{j+2}\}$ (since otherwise $v_iv_{i-1}y_{k-2}Q_1y_s$ or $v_{i-1}v_iy_{k-2}Q_1y_s$ or $v_iv_{i-1}y_{j+2}Q_1y_0$ or $v_{i-1}v_iy_{j+2}Q_1y_0$ is a path of length at least 9, a contradiction);

(iv) $N_{Q_1}(v) \subseteq \{v_{i+1}, v_{i-2}\}$ (since (i), (ii) and (iii)).

Now similar to the proof of Claim 2.17 in the third, forth and fifth paragraphs, we could obtain that for any longest path Q of G, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$. $\square$

Claim 2.24. *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of G contain v_{i-1} and v_{i+1}.*

Proof. We could obtain this result similarly as in the proof of Claim 2.10. $\square$

Claim 2.25. *If there is a longest path $Q = y_0y_1 \dots y_s$ of G such that $|V(Q) \cap V(C)| \leqslant 6$, then for each vertex $v \in V(C) \setminus V(Q)$, $d_Q(v) = 2$.*

Proof. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.23, v_2, $v_7 \in V(Q)$. Since $r = 7$, $v_2w_1v_7$ or $v_2w_1w_2v_7$ or $v_2w_1w_2w_3v_7$ or $v_2w_1w_2w_3w_4v_7$ is a subpath of Q (w_1, w_2, w_3, w_4 may be vertices of $V(C)$). Now v_2 and v_7 are not end-vertices of Q, since otherwise adding v_1 to Q results in a longer path, a contradiction. Suppose $v_2 = y_k, v_7 = y_j$, $1 \leqslant k < j \leqslant s-1$.

Case 2.5.1. $v_2w_1v_7$ **is a subpath of** Q**.** In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) v_1 is not adjacent to w_1, y_{k-1}, y_{j+1} (by Claim 2.3);

(ii) v_1 is not adjacent to any vertex in $Q[y_0, y_{k-4}] \cup Q[y_{j+4}, y_s]$ (since $r = 7$);

(iii) $N_Q(v_1) \subseteq \{v_2, v_7, y_{k-2}, y_{k-3}, y_{j+2}, y_{j+3}\}$ (since (i) and (ii)).

Claim 2.26. $v_1y_{j+3} \notin E(G)$.

Proof. If $v_1y_{j+3} \in E(G)$, then by Claim 2.3, $v_1y_{j+2} \notin E(G)$. Furthermore, as $r = 7$, $v_1y_{k-2}, v_1y_{k-3} \notin E(G)$. We assume that $d_G(v_1) = 3$. Since if there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1v_1' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2, for otherwise $v_1'v_1v_2Qy_s$ or $v_1'v_1y_{j+3}Qy_0$ is a path longer than Q, a contradiction. But now $s \geqslant 9$ and y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_1v_1' are 6 independent edges, a contradiction.

Now we can check that for the vertex w_1, the following assertions hold:

(i) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+4}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}] \cup Q[y_{j+4}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+3}Qw_1z$ or $zQv_7v_1v_2w_1z$ is a cycle of length at least 8, a contradiction);

(ii) w_1 is not adjacent to any vertex in $\{y_{j+1}, y_{j+2}\}$. Since otherwise

$$y_0Qv_2v_1v_7w_1y_{j+1}Qy_s \quad \text{or} \quad y_0Qv_2w_1y_{j+2}y_{j+1}y_jv_1y_{j+3}Qy_s$$

is a path longer than Q;

(iii) w_1 is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_1z \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2, for otherwise $zw_1v_2v_1v_7Qy_s$ or $zw_1Qy_{j+3}v_1v_2Qy_0$ is a path longer than Q, a contradiction. But now y_0y_1, v_2v_1, w_1z, v_7y_{j+1}, $y_{j+2}y_{j+3}$, $y_{s-1}y_s$ are 6 independent edges.);

(iv) $N_G(w_1) \subseteq \{v_2, v_7, y_{j+3}\}$ (since (i), (ii) and (iii)).

Now we prove that all the longest paths of G contain v_7. If there exists a longest path Q_1 not containing v_7, then by Claim 2.23, $w_1, v_1 \in V(Q_1)$. Now w_1, v_1 are not end-vertices of Q_1, since otherwise adding v_7 to Q_1 results in a longer path, a contradiction. But now $v_2v_1y_{j+3}w_1v_2$ is a subpath of Q_1, a contradiction. Since G is a counterexample, $v_1y_{j+3} \notin E(G)$. □

Similarly, we could obtain that $v_1y_{k-3} \notin E(G)$.

Claim 2.27. $v_1y_{j+2} \notin E(G)$.

Proof. If $v_1y_{j+2} \in E(G)$, then as $r = 7$, $v_1y_{k-2} \notin E(G)$. We can check that for the vertex w_1, the following assertions hold:

(i) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+4}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-2}] \cup Q[y_{j+4}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+2}Qw_1z$ or $zQv_7v_1v_2w_1z$ is a cycle of length at least 8, a contradiction);

(ii) w_1 is not adjacent to any vertex in $\{y_{k-1}, y_{j+1}, y_{j+3}\}$ (since otherwise $y_0Qy_{k-1}w_1v_2v_1v_7Qy_s$ or $y_0Qv_2v_1v_7w_1y_{j+1}Qy_s$ or $y_0Qv_2v_1y_{j+2}Qw_1y_{j+3}Qy_s$ is a path longer than Q, a contradiction);

(iii) $N_Q(w_1) \subseteq \{v_2, v_7, y_{j+2}\}$ (since (i) and (ii)).

Similarly, we could prove that $N_Q(y_{j+1}) \subseteq \{v_2, v_7, y_{j+2}\}$.

If there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1w_1' \in E(G)$, then $w_1'w_1v_2v_1v_7Qy_s$ is a path of length at least 7. If there is a vertex $w_1'' \in V(G) \setminus V(Q)$ such that $w_1'w_1'' \in E(G)$, then $Q[y_0, v_7]$ has length at least 5, for otherwise $w_1''w_1'w_1v_2v_1v_7Qy_s$ is a path longer than Q, a contradiction. But now

$y_0Qv_2v_1y_{j+2}y_{j+1}v_7w_1w_1'w_1''$ is a path of length at least 10, and we could find 6 independent edges, a contradiction. Thus $N_G(w_1') = N_Q(w_1')$. By (iii), $N_Q(w_1') \subseteq \{w_1, v_2, v_7, y_{j+2}\}$. By Claim 2.3, $w_1'v_2, w_1'v_7 \notin E(G)$. If $w_1'y_{j+2} \in E(G)$, then $y_0Qv_2v_1v_7w_1w_1'y_{j+2}Qy_s$ is a path longer than Q, a contradiction. Thus $d_G(w_1') = 1$. Similarly, we could prove that $d_G(y_{j+1}') = 1$ holds for any vertex $y_{j+1}' \in V(G) \setminus V(Q)$ such that $y_{j+1}y_{j+1}' \in E(G)$.

Now if there is a longest path Q_2 not containing v_7, then by Claim 2.23, $v_1, v_6 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then by the proof of Claim 2.16, $y_{j+1}, v_2 \in V(Q_2)$. By the proof of Claim 2.23 we could check that $N_{Q_2}(v_7) \subseteq \{v_2, y_{j+1}\}$. But now $v_1 \notin V(Q_2)$, a contradiction. Thus $w_1 \in V(Q_2)$. Similarly, we could obtain that $y_{j+1} \in V(Q_2)$. Now v_1, w_1, y_{j+1} are not end-vertices of Q_2, since otherwise adding v_7 to Q_2 results in a longer path, a contradiction. If $v_2v_1y_{j+2}$ is a subpath of Q_2, then there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1w_1v_2$, $u_2y_{j+1}y_{j+2}$ or $u_1w_1y_{j+2}$, $u_2y_{j+1}v_2$ are two subpaths of Q_2, for otherwise $y_{j+2}v_1v_2w_1y_{j+2}$ or $y_{j+2}v_1v_2y_{j+1}y_{j+2}$ is a subpath of Q_2, a contradiction. But now $Q_2 = u_1w_1v_2v_1y_{j+2}y_{j+1}u_2$ or $Q_2 = u_2y_{j+1}v_2v_1y_{j+2}w_1u_1$, a contradiction. Thus, there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1'v_1v_2$ or $v_1'v_1y_{j+2}$ is a segment of Q_2. Without loss of generality, suppose that $v_1'v_1v_2$ is a segment of Q_2. Now both $Q[y_0, v_2]$ and $Q[y_{j+2}, y_s]$ have lengths at least 2, for otherwise either $v_1'v_1v_2Qy_s$ or $v_1'v_1y_{j+2}Qy_0$ is a path longer than Q, a contradiction. Furthermore, both $Q[y_0, v_2]$ and $Q[y_{j+2}, y_s]$ have lengths exactly 2, for otherwise $s \geqslant 9$ and y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_1v_1' are 6 independent edges, a contradiction. We assume that $N_G(v_1') = N_Q(v_1') \cup \{v_1\}$. Since if there exists a vertex $v_1'' \in V(G) \setminus V(Q)$ such that $v_1'v_1'' \in E(G)$, then $v_1''v_1'v_1v_2Qy_s$ is a path of length at least 9, a contradiction. Now as $N_Q(v_1) = \{v_2, v_7, y_{j+2}\}$, $N_Q(v_1') \subseteq \{v_2, v_7, y_{j+2}\}$. By Claim 2.3, $v_1'v_2, v_1'v_7, v_1'y_{j+2} \notin E(G)$. Thus $d_G(v_1') = 1$.

Since $v_1'v_1v_2$ is a subpath of Q_2, there is no vertex $u_1 \in V(G) \setminus V(Q)$ such that $u_1w_1v_2$ or $u_1y_{j+1}v_2$ is a subpath of Q_2, for otherwise $Q_2 = v_1'v_1v_2w_1u_1$ or $Q_2 = v_1'v_1v_2y_{j+1}u_1$, a contradiction. If there is a vertex $u_2 \in V(G) \setminus V(Q)$ such that $u_2w_1y_{j+2}$ is a subpath of Q_2, then $v_2y_{j+1}y_{j+2}$ is a subpath of Q_2, for otherwise $Q_2 = u_2w_1y_{j+2}y_{j+1}u_3$ $(u_3 \in V(G) \setminus V(Q))$, a contradiction. But now $Q_2 = v_1'v_1v_2y_{j+1}y_{j+2}w_1u_2$, a contradiction. Since $N_Q(w_1) \subseteq \{v_2, v_7, y_{j+2}\}$, $v_2w_1y_{j+2}$ is a subpath of Q_2. Similarly, we could prove that $v_2y_{j+1}y_{j+2}$ is a subpath of Q_2. But now $v_2w_1y_{j+2}y_{j+1}v_2$ is a subpath of Q_2, a contradiction. Thus, all the longest paths of G contain v_7. Since G is a counterexample, $v_1y_{j+2} \notin E(G)$. □

Similarly, we could obtain that $v_1y_{k-2} \notin E(G)$. Therefore $d_Q(v_1) = 2$.

Case 2.5.2. $v_2w_1w_2v_7$ **is a subpath of** Q**.** In this case, we can check that for the vertex $v_1 \in V(G) \setminus V(Q)$, the following assertions hold:

(i) v_1 is not adjacent to any vertex in $\{w_1, w_2, y_{k-1}, y_{j+1}\}$ (by Claim 2.3);

(ii) v_1 is not adjacent to any vertex in $Q[y_0, y_{k-3}] \cup Q[y_{j+3}, y_s]$ (since $r = 7$);
(iii) $N_Q(v_1) \subseteq \{v_2, v_7, y_{k-2}, y_{j+2}\}$ (since (i) and (ii)).

If $v_1 y_{k-2} \in E(G)$ or $v_1 y_{j+2} \in E(G)$, then similar to the proof of Case 2.5.1 we could obtain that $d_Q(v_1) = 2$.

Case 2.5.3. $v_2 w_1 w_2 w_3 v_7$ **is a subpath of** Q**.** In this case, similar to the proof of Case 2.5.1 we could obtain that $d_Q(v_1) = 2$.

Case 2.5.4. $v_2 w_1 w_2 w_3 w_4 v_7$ **is a subpath of** Q**.** In this case, similar to the proof of Case 2.5.1 we could obtain that $d_Q(v_1) = 2$. □

Claim 2.28. *If there is a longest path* $Q = y_0 y_1 \ldots y_s$ *of* G *such that* $|V(Q) \cap V(C)| \leqslant 6$, *then all the longest paths of* G *have a common vertex.*

Proof. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.25, $d_Q(v_1) = 2$. If $d_G(v_1) = 2$, then by Claim 2.24, all the longest paths of G share a common vertex.

If $d_G(v_1) \geqslant 3$, then similar to the proof of Claim 2.15 in the third, forth, fifth paragraphs, we could obtain that for any vertex $w \in V(G) \setminus V(Q)$ such that $v_1 w \in E(G)$, $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of G contain v_2. If there exists a longest path Q_2 not containing v_2, then by Claim 2.23, $v_1, v_3 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_7]$ such that $u \in V(Q_2)$, for otherwise by Claim 2.23, $v_6 \in V(Q_2)$. Now $C_2 = v_1 Q_2[v_1, v_6] v_6 v_7 Q w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. Thus $w_1 u \in E(Q)$ and $u \in V(Q_2)$. Now we could check that $N_{Q_2}(v_2) \subseteq \{v_1, u\}$. Thus $u = v_3$. By Claim 2.25, $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now $C_3 = y_{k-2} Q_2[y_{k-2}, u] u w_1 v_2 y_{k-1} y_{k-2}$ is a cycle of length at least 8, a contradiction. Thus $w_1 \in V(Q_2)$. By Claim 2.25, $w_1 = v_3$ and $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now we could check that $N_{Q_2}(v_2) \subseteq \{v_3, y_{k-2}\}$, a contradiction.

Thus, all the longest paths of G contain v_2. □

Since G is a counterexample, by Claim 2.28, for any longest path Q of G, $V(C) \subset V(Q)$. But now all the longest paths of G contain $V(C)$, a contradiction.

2.6. Proof of the case $r = 8$**.** If $r = 8$, then by Claim 2.1 we have that $s \geqslant 8$.

Claim 2.29. *For any longest path* Q *of* G, $|V(Q) \cap V(C)| \geqslant 4$.

Proof. Let $Q = y_0y_1\ldots y_s$ be a longest path of G. If $|V(Q)\cap V(C)| \leqslant 2$, then $y_0y_1, y_2y_3, y_4y_5, y_6y_7$ and two independent edges in $C\setminus V(Q)$ are 6 independent edges, a contradiction. If $|V(Q)\cap V(C)| = 3$, then at least two vertices of $V(C)\setminus V(Q)$ are consecutive in C. Suppose $W[v_i, v_j]$ is a maximum consecutive segment in $C\setminus V(Q)$, and $N_{C\cap Q}(v_i) = \{\overline{v}_i\}$, $N_{C\cap Q}(v_j) = \{\overline{v}_j\}$. If $W[v_i, v_j]$ has length 1, then there are two independent edges in $C \setminus V(Q)$. But now y_0y_1, y_2y_3, y_4y_5, y_6y_7 and the two independent edges are 6 independent edges, a contradiction. If $W[v_i, v_j]$ has length at least 2, then as Q is a longest path of G, the length of $Q[\overline{v}_i, \overline{v}_j]$ is at least 4. Suppose that $\overline{v}_i = y_k$, $\overline{v}_j = y_l$, $k < l$. We assume that both $Q[y_0, y_k]$ and $Q[y_l, y_s]$ have lengths at least 3, for otherwise $v_jWv_i\overline{v}_iQ[\overline{v}_i, y_s]$ or $v_iWv_j\overline{v}_jQ[\overline{v}_j, y_0]$ is a path longer than Q, a contradiction. But now the length of Q is at least 10, and y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9 with an edge in $W[v_i, v_j]$ are 6 independent edges, a contradiction. $\square$

Claim 2.30. *For any longest path Q of G, if $v_i \in V(C)\setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

Proof. If not, then there exists a longest path Q_1 and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. Now similar to the first paragraph of the proof of Claim 2.17 we could obtain that $v_{i+1}w_1w_2v_{i-2}$ or $v_{i+1}w_1w_2w_3v_{i-2}$ is a segment of Q_1 and $s = 8$.

We can check that for each vertex $v \in \{v_i, v_{i-1}\}$, the following assertions hold:

(i) v is not adjacent to w_1, w_2, y_{k-1}, y_{j+1} (if $v_{i+1}w_1w_2v_{i-2}$ is a segment of Q_1) or w_1, w_2, w_3, y_{k-1}, y_{j+1} (if $v_{i+1}w_1w_2w_3v_{i-2}$ is a segment of Q_1) (by Claim 2.3);

(ii) v is not adjacent to any vertex in $Q_1[y_0, y_{k-2}]\cup Q_1[y_{j+2}, y_s]$ (since if there exists a vertex $z \in Q_1[y_0, y_{k-2}] \cup Q_1[y_{j+2}, y_s]$ such that $vz \in E(G)$, then $v_iv_{i-1}zQ_1y_s$ or $v_{i-1}v_izQ_1y_s$ or $v_iv_{i-1}zQ_1y_0$ or $v_{i-1}v_izQ_1y_0$ is a path of length at least 9, a contradiction);

(iii) $N_{Q_1}(v) \subseteq \{v_{i+1}, v_{i-2}\}$ (since (i) and (ii)).

Now similar to the proof of Claim 2.17 in the third, forth and fifth paragraphs we could obtain that for any longest path Q of G, if $v_i \in V(C)\setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$. $\square$

Claim 2.31. *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of G contain v_{i-1} and v_{i+1}.*

Proof. Similar to the proof of Claim 2.10 we could obtain this result. $\square$

Claim 2.32. *If there is a longest path $Q = y_0y_1\ldots y_s$ of G such that $|V(Q)\cap V(C)| \leqslant 7$, then for each vertex $v \in V(C)\setminus V(Q)$, $d_Q(v) = 2$.*

P r o o f. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.31, $v_2, v_8 \in V(Q)$. Since $r = 8$, $v_2w_1v_8$ or $v_2w_1w_2v_8$ or $v_2w_1w_2w_3v_8$ or $v_2w_1w_2w_3w_4v_8$ or $v_2w_1w_2w_3w_4w_5v_8$ is a subpath of Q in G (w_1, w_2, w_3, w_4, w_5 may be vertices of $V(C)$). Now v_2 and v_8 are not end-vertices of Q, since otherwise adding v_1 to Q results in a longer path, a contradiction. Suppose $v_2 = y_k$, $v_8 = y_j$, $1 \leqslant k < j \leqslant s-1$.

Case 2.6.1. $v_2w_1v_8$ **is a subpath of** Q**.** In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:
(i) v_1 is not adjacent to w_1, y_{k-1}, y_{j+1} (by Claim 2.3);
(ii) v_1 is not adjacent to any vertex in $Q[y_0, y_{k-5}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);
(iii) $N_Q(v_1) \subseteq \{v_2, v_8, y_{k-2}, y_{k-3}, y_{k-4}, y_{j+2}, y_{j+3}, y_{j+4}\}$ (since (i) and (ii)).

Claim 2.33. $v_1y_{j+4} \notin E(G)$.

P r o o f. If $v_1y_{j+4} \in E(G)$, then $v_1y_{k-2}, v_1y_{k-3}, v_1y_{k-4} \notin E(G)$, otherwise we could find a cycle longer than C, a contradiction. By Claim 2.3, $v_1y_{j+3} \notin E(G)$. If there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1v_1' \in E(G)$, then $v_1'v_1v_2w_1v_8y_{j+1}y_{j+2}y_{j+3}y_{j+4}y_{j+5}$ is a path of length 9. Since Q is a longest path, $s \geqslant 9$. But now y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_1v_1' are 6 independent edges, a contradiction. Thus $N_G(v_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}\}$.

Now we can check that for the vertex w_1, the following assertions hold:
(i) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+4}Qw_1z$ or $zQy_{j+4}Qv_8v_1v_2w_1z$ is a cycle of length at least 9);
(ii) w_1 is not adjacent to any vertex in $\{y_{j+1}, y_{j+3}\}$. Since otherwise

$$y_0Qv_2v_1v_8w_1y_{j+1}Qy_s \quad \text{or} \quad y_0Qw_1y_{j+3}Qv_8v_1y_{j+4}Qy_s$$

is a path longer than Q;
(iii) w_1 is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_1z \in E(G)$, then $Q[y_0, v_8]$ has length at least 4, otherwise $zw_1v_2v_1v_8Qy_s$ is a path longer than Q. But now y_0y_1, v_2v_1, w_1z, v_8y_{j+1}, $y_{j+2}y_{j+3}$, $y_{j+4}y_{j+5}$ are 6 independent edges.);
(iv) $N_G(w_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}\}$ (since (i), (ii) and (iii)).

If $v_1y_{j+2} \in E(G)$ or $w_1y_{j+2} \in E(G)$, then we can check that for the vertex y_{j+1}, the following assertions hold:
(i) y_{j+1} is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);
(ii) y_{j+1} is not adjacent to y_{j+3} (since otherwise $y_0Qv_2v_1v_8w_1y_{j+2}y_{j+1}y_{j+3}Qy_s$ or $y_0Qy_{j+1}y_{j+3}\ y_{j+2}v_1y_{j+4}Qy_s$ is a path longer than Q, a contradiction);

(iii) y_{j+1} is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $y_{j+1}z \in E(G)$, then $Q[y_0, y_{j+2}]$ has length at least 6, otherwise $zy_{j+1}v_8v_1v_2w_1y_{j+2}Qy_s$ or $zy_{j+1}v_8w_1v_2v_1y_{j+2}Qy_s$ is a path longer than Q. But now y_0y_1, v_2v_1, w_1v_8, $y_{j+1}z$, $y_{j+2}y_{j+3}$, $y_{j+4}y_{j+5}$ are 6 independent edges, a contradiction.);

(iv) $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}\}$ (since (i), (ii) and (iii)).

Now we prove that all the longest paths of G contain v_8. If there exists a longest path Q_1 not containing v_8, then by Claim 2.30, Q_1 contains w_1, y_{j+1}, v_1. Now w_1, y_{j+1}, v_1 are not end-vertices of Q_1, since otherwise adding v_8 to Q_1 results in a longer path, a contradiction. Thus $v_2w_1y_{j+2}$ or $v_2w_1y_{j+4}$ or $y_{j+2}w_1y_{j+4}$ is a segment of Q_1. Without loss of generality, suppose that $v_2w_1y_{j+2}$ is a segment of Q_1. Now $v_2y_{j+1}y_{j+4}$ or $y_{j+2}y_{j+1}y_{j+4}$ is a segment of Q_1. Without loss of generality, suppose that $v_2y_{j+1}y_{j+4}$ is a segment of Q_1. Now we assume that $y_{j+2}v_1y_{j+4}$ is a segment of Q_1, since otherwise $v_2v_1y_{j+4}y_{j+1}v_2$ or $v_2v_1y_{j+2}w_1v_2$ is a segment of Q_1, a contradiction. But now $y_{j+4}y_{j+1}v_2w_1y_{j+2}v_1y_{j+4}$ is a segment of Q_1, a contradiction.

Since G is a counterexample, $v_1y_{j+2}, w_1y_{j+2} \notin E(G)$. Now if there exists a longest path Q_2 not containing v_8, then by Claim 2.30, Q_2 contains v_1, w_1. We say that v_1, w_1 are not end-vertices of Q_2, since otherwise adding v_8 to Q_2 results in a longer path, a contradiction. But now $v_2v_1y_{j+4}w_1v_2$ is a segment of Q_2, a contradiction. Thus all the longest paths of G contain v_8. Since G is a counterexample, $v_1y_{j+4} \notin E(G)$. $\square$

Similarly, we could prove that $v_1y_{k-4} \notin E(G)$.

Claim 2.34. $v_1y_{j+3} \notin E(G)$.

Proof. If $v_1y_{j+3} \in E(G)$, then as $r = 8$, $v_1y_{k-2}, v_1y_{k-3} \notin E(G)$. By Claim 2.3, $v_1y_{j+2} \notin E(G)$. We assume that $d_G(v_1) = 3$, since if there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1v_1' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2. But now $s \geqslant 9$ and y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_1v_1' are 6 independent edges, a contradiction.

Now we can check that for the vertex w_1 the following assertions hold:

(i) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+5}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-2}] \cup Q[y_{j+5}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+3}Qv_8w_1z$ or $zQv_8v_1v_2w_1z$ is a cycle of length at least 9);

(ii) w_1 is not adjacent to any vertex in $\{y_{k-1}, y_{j+4}\}$. Since otherwise

$$y_0Qy_{k-1}w_1v_2v_1v_8Qy_s \quad \text{or} \quad y_0Qv_2v_1y_{j+3}Qw_1y_{j+4}Qy_s$$

is a path longer than Q;

(iii) w_1 is not adjacent to any vertex in $\{y_{j+1}, y_{j+2}\}$. Since otherwise

$$y_0Qv_2v_1v_8w_1y_{j+1}Qy_s \quad \text{or} \quad y_0Qw_1y_{j+2}y_{j+1}v_8v_1y_{j+3}Qy_s$$

is a path longer than Q;

(iv) w_1 is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_1z \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2, for otherwise $zw_1v_2v_1v_8Qy_s$ or $zw_1Qy_{j+3}v_1v_2Qy_0$ is a path longer than Q. But now $s \geqslant 9$ and y_0y_1, v_2v_1, w_1z, v_8y_{j+1}, $y_{j+2}y_{j+3}$, $y_{j+4}y_{j+5}$ are 6 independent edges, a contradiction.);

(v) $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}\}$ (since (i), (ii), (iii) and (iv)).

Now we prove that all the longest paths of G contain v_8. If there exists a longest path Q_3 not containing v_8, then by Claim 2.30, $v_1, v_7 \in V(Q_3)$. If $w_1 \in V(Q_3)$, then w_1, v_1 are not end-vertices of Q_3, since otherwise adding v_8 to Q_3 results in a longer path, a contradiction. But now $v_2v_1y_{j+3}w_1v_2$ is a subpath of Q_3, a contradiction. Thus $w_1 \notin V(Q_3)$. By the proof of Claim 2.29, $v_2, y_{j+1} \in V(Q_3)$. We could check that $N_{Q_3}(v_8) \subseteq \{v_2, y_{j+1}\}$, a contradiction to $v_1, v_7 \in V(Q_3)$.

Since G is a counterexample, $v_1y_{j+3} \notin E(G)$. □

Similarly, we could prove that $v_1y_{k-3} \notin E(G)$.

Claim 2.35. $v_1y_{j+2} \notin E(G)$.

Proof. If $v_1y_{j+2} \in E(G)$, then by the proof of the case $v_1y_{j+4} \in E(G)$, $v_1y_{k-2} \notin E(G)$.

Now we can check that for the vertex w_1, the following assertions hold:

(i) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-3}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);

(ii) w_1 is not adjacent to any vertex in $\{y_{j+1}, y_{k-1}, y_{j+3}\}$ (since otherwise $y_0Qv_2v_1v_8w_1y_{j+1}Qy_s$ or $y_0Qy_{k-1}w_1v_2v_1v_8Qy_s$ or $y_0Qv_2v_1y_{j+2}Qw_1y_{j+3}Qy_s$ is a path longer than Q);

(iii) $N_Q(w_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}, y_{k-2}\}$ (since (i) and (ii)).

By symmetry, we could prove that $N_Q(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}, y_{k-2}\}$.

If $w_1y_{k-2} \in E(G)$, then $w_1y_{j+4}, y_{j+1}y_{j+4} \notin E(G)$, for otherwise

$$y_{k-2}y_{k-1}v_2v_1v_8Qy_{j+4}w_1y_{k-2} \quad \text{or} \quad y_{k-2}y_{k-1}v_2v_1y_{j+2}y_{j+3}y_{j+4}y_{j+1}v_8w_1y_{k-2}$$

is a cycle of length at least 10, a contradiction. Now $Q[y_0, v_8]$ has length at least 5, otherwise $y_{k-1}y_{k-2}w_1v_2v_1v_8Qy_s$ is a path longer than Q, a contradiction. Further, if there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1w_1' \in E(G)$, then $Q[y_{j+2}, y_s]$ has length at least 2, otherwise $w_1'w_1v_8y_{j+1}y_{j+2}v_1v_2Qy_0$ is a path longer than Q, a contradiction. But now y_0y_1, $y_{k-1}v_2$, w_1w_1', v_8v_1, $y_{j+1}y_{j+2}$, $y_{j+3}y_{j+4}$ are 6 independent edges, a contradiction. Thus $N_G(w_1) = N_Q(w_1) \subseteq \{v_2, v_8, y_{j+2}, y_{k-2}\}$. Similarly, we could obtain that $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{k-2}\}$. Now if there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1'v_1 \in E(G)$, then $Q[y_0, v_8]$ has length at least 6, otherwise $v_1'v_1v_2y_{k-1}y_{k-2}w_1v_8Qy_s$ is a path longer than Q, a contradiction. But now $s \geqslant 9$ and we could find 6 independent edges, a contradiction. Thus $N_G(v_1) = N_Q(v_1) \subseteq \{v_2, v_8, y_{j+2}\}$.

Now we prove that all the longest paths of G contain v_8. If there is a longest path Q_4 not containing v_8, then by Claim 2.30, $v_1, w_1, y_{j+1} \in V(Q_4)$. We assume that v_1, w_1, y_{j+1} are not end-vertices of Q_4, since otherwise adding v_8 to Q_4 results in a longer path, a contradiction. Since $N_G(v_1) \subseteq \{v_2, v_8, y_{j+2}\}$, $v_2v_1y_{j+2}$ is a subpath of Q_4. Now either $v_2w_1y_{k-2}$ and $y_{j+2}y_{j+1}y_{k-2}$ or $y_{j+2}w_1y_{k-2}$ and $v_2y_{j+1}y_{k-2}$ are two subpaths of Q_4. But now $y_{k-2}w_1v_2v_1y_{j+2}y_{j+1}y_{k-2}$ or $y_{k-2}w_1y_{j+2}v_1v_2y_{j+1}y_{k-2}$ is a subpath of Q_4, a contradiction. Since G is a counterexample, $w_1y_{k-2} \notin E(G)$. Similarly, we could prove that $w_1y_{j+4}, y_{j+1}y_{k-2}, y_{j+1}y_{j+4} \notin E(G)$. Thus $N_Q(v) \subseteq \{v_2, v_8, y_{j+2}\}$, $v \in \{w_1, y_{j+1}\}$.

If there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1w_1' \in E(G)$, then $N_G(w_1') \subseteq \{w_1, y_{j+2}\}$. Since if there exists a vertex $w_1'' \in V(G) \setminus V(Q)$, then both $Q[y_0, v_2]$ and $Q[y_{j+2}, y_s]$ have lengths at least 3, for otherwise $w_1''w_1'w_1v_2v_1v_8Qy_s$ or $w_1''w_1'w_1v_8y_{j+1}y_{j+2}v_1v_2Qy_0$ is a path longer than Q, a contradiction. But now y_0y_1, $y_{k-1}v_2$, w_1w_1', v_8v_1, $y_{j+1}y_{j+2}$, $y_{j+3}y_{j+4}$ are 6 independent edges, a contradiction. Thus $N_G(w_1') = N_Q(w_1')$. Since $N_Q(w_1) \subseteq \{v_2, v_8, y_{j+2}\}$, $N_Q(w_1') \subseteq \{w_1, v_2, v_8, y_{j+2}\}$. By Claim 2.3, $w_1'v_2, w_1'v_8 \notin E(G)$. Thus $N_G(w_1') \subseteq \{w_1, y_{j+2}\}$. Similarly, we could prove that for any vertex $y_{j+1}' \in V(G) \setminus V(Q)$ such that $y_{j+1}y_{j+1}' \in E(G)$, $N_G(y_{j+1}') \subseteq \{y_{j+1}, v_2\}$.

Now we prove that all the longest paths of G contain v_8. If there exists a longest path Q_5 not containing v_8, then by Claim 2.30, $v_1, v_7 \in V(Q_5)$. If $w_1 \notin V(Q_5)$, then by the proof of Claim 2.29, $v_2, y_{j+1} \in V(Q_5)$. We could check that $N_{Q_5}(v_8) \subseteq \{v_2, y_{j+1}\}$, a contradiction to $v_1, v_7 \in V(Q_5)$. Thus $w_1 \in V(Q_5)$. Similarly, we could prove that $y_{j+1} \in V(Q_5)$. Now if $v_2v_1y_{j+2}$ is a subpath of Q_5, then there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1w_1v_2$, $u_2y_{j+1}y_{j+2}$ or $u_1w_1y_{j+2}$, $u_2y_{j+1}v_2$ are two segments of Q_5. But now $Q_5 = u_1w_1v_2v_1y_{j+2}y_{j+1}u_2$ or $Q_5 = u_2y_{j+1}v_2v_1y_{j+2}w_1u_1$, a contradiction. Thus, there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1'v_1v_2$ or $v_1'v_1y_{j+2}$ is a segment of Q_5. Now $Q = y_0y_1v_2w_1v_8y_{j+1}y_{j+2}y_7y_8$, otherwise we could find 6 independent edges, a contradiction. We assume that $w_1'y_{j+2} \notin E(G)$, otherwise $y_0y_1v_2v_1v_8w_1w_1'y_{j+2}y_7y_8$ is a path longer than Q, a contradiction. Thus $d_G(w_1') = 1$. Similarly, $d_G(y_{j+1}') = 1$. Without loss of generality, suppose that $v_1'v_1v_2$ is a segment of Q_5. Since $s = 8$, $N_G(v_1') = N_Q(v_1') \cup \{v_1\}$. Since $N_Q(v_1) \subseteq \{v_2, v_8, y_{j+2}\}$, $N_Q(v_1') \subseteq \{v_2, v_8, y_{j+2}\}$. By Claim 2.3, $v_1'v_2, v_1'v_8, v_1'y_{j+2} \notin$

$E(G)$. Thus $d_G(v_1')=1$. If $v_2w_1y_{j+2}$ or $v_2y_{j+1}y_{j+2}$ is a segment of Q_5, then there exists a vertex $u\in V(G)\setminus V(Q)$ such that $y_{j+2}y_{j+1}u$ or $y_{j+2}w_1u$ is a segment of Q_5. But now $Q_5=v_1'v_1v_2w_1y_{j+2}y_{j+1}u$ or $Q_5=v_1'v_1v_2y_{j+1}y_{j+2}w_1u$, a contradiction. Thus, there are two vertices $u_1,u_2\in V(G)\setminus V(Q)$ such that $u_1w_1v_2$, $u_2y_{j+1}y_{j+2}$ or $u_1w_1y_{j+2}$, $u_2y_{j+1}v_2$ are two segments of Q_5. Now either $Q_5=u_1w_1v_2v_1v_1'$ or $Q_5=u_2y_{j+1}v_2v_1v_1'$, a contradiction.

Thus, all the longest paths of G contain v_8. Since G is a counterexample, $v_1y_{j+2}\notin E(G)$. □

Similarly, we could prove that $v_1y_{k-2}\notin E(G)$. Thus $d_Q(v_1)=2$.

Case 2.6.2. $v_2w_1w_2v_8$ **is a subpath of** Q**.** In this case, we can check that for vertex $v_1\in V(G)\setminus V(Q)$ the following assertions hold:

(i) v_1 is not adjacent to any vertex in $\{w_1,w_2,y_{k-1},y_{j+1}\}$ (by Claim 2.3);

(ii) v_1 is not adjacent to any vertex in $Q[y_0,y_{k-4}]\cup Q[y_{j+4},y_s]$ (since $r=8$);

(iii) $N_Q(v_1)\subseteq\{v_2,v_8,y_{j+2},y_{j+3},y_{k-2},y_{k-3}\}$ (since (i) and (ii)).

Claim 2.36. *$v_1y_{j+2}\notin E(G)$ and $v_1y_{k-2}\notin E(G)$.*

Proof. We could obtain the result similarly as in the proof of Case 2.6.1. □

Claim 2.37. *$v_1y_{j+3}\notin E(G)$.*

Proof. If $v_1y_{j+3}\in E(G)$, then as $r=8$, $v_1y_{k-3},v_1y_{k-2}\notin E(G)$. By Claim 2.3, $v_1y_{j+2}\notin E(G)$. If there exists a vertex $v_1'\in V(G)\setminus V(Q)$ such that $v_1v_1'\in E(G)$, then both $Q[y_0,v_2]$ and $Q[y_{j+3},y_s]$ have lengths at least 2. But now y_0y_1, v_2w_1, w_2v_8, $y_{j+1}y_{j+2}$, $y_{j+3}y_{j+4}$, v_1v_1' are 6 independent edges, a contradiction. Thus $d_G(v_1)=3$.

We can check that for vertex w_2 the following assertions hold:

(i) w_2 is not adjacent to any vertex in $\{y_{k-1},y_{j+2}\}$. Since otherwise

$$y_0Qy_{k-1}w_2w_1v_2v_1v_8Qy_s \quad\text{or}\quad y_0Qw_2y_{j+2}y_{j+1}v_8v_1y_{j+3}Qy_s$$

is a path longer than Q;

(ii) w_2 is not adjacent to any vertex in $Q[y_0,y_{k-2}]\cup Q[y_{j+4},y_s]$ (since $r=8$);

(iii) w_2 is not adjacent to any vertex in $V(G)\setminus V(Q)$ (Since if there exists a vertex $z\in V(G)\setminus V(Q)$ such that $w_2z\in E(G)$, then $s\geqslant 9$, for otherwise $zw_2w_1v_2v_1v_8Qy_s$ is a path longer than Q. But now y_0y_1, v_2w_1, w_2z, v_8v_1, $y_{j+1}y_{j+2}$, $y_{j+3}y_{j+4}$ are 6 independent edges, a contradiction.);

(iv) $N_G(w_2)\subseteq\{v_2,v_8,y_{j+3},w_1,y_{j+1}\}$ (since (i), (ii) and (iii)).

Similarly, we could prove that $N_G(y_{j+1})\subseteq\{v_2,v_8,y_{j+2},y_{j+3},w_2\}$.

We can check that for vertex w_1 the following assertions hold:

(i) w_1 is not adjacent to y_{j+1}, y_{j+4} (since otherwise $y_0Qv_2v_1v_8w_2w_1y_{j+1}Qy_s$ or $y_0Qv_2v_1y_{j+3}Qw_1y_{j+4}Qy_s$ is a path longer than Q);

(ii) w_1 is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);

(iii) w_1 is not adjacent to any vertex in $V(G)\backslash V(Q)$ (Since if there exists a vertex $z \in V(G)\backslash V(Q)$ such that $w_1z \in E(G)$, then $s \geqslant 9$, for otherwise $zw_1Qy_{j+3}v_1v_2Qy_0$ is a path longer than Q. But now $y_{k-1}v_2$, w_1z, w_2v_8, $y_{j+1}y_{j+2}$, $y_{j+3}v_1$, $y_{j+4}y_{j+5}$ are 6 independent edges, a contradiction.);

(iv) $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$ (since (i), (ii) and (iii)).

Similarly, we could prove that $N_G(y_{j+2}) \subseteq \{v_2, v_8, y_{j+3}, y_{j+1}, w_1\}$.

Now we prove that all the longest paths of G contain v_8. If there exists a longest path Q_6 not containing v_8, then by Claim 2.30, $v_1, w_2, y_{j+1} \in V(Q_6)$. Now v_1, w_2 and y_{j+1} are not end-vertices of Q_6, since otherwise adding v_8 to Q_6 results in a longer path, a contradiction. Since $d_G(v_1) = 3$, $v_2v_1y_{j+3}$ is a segment of Q_6. If $v_2w_2w_1$ is a segment of Q_6, then since $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+3}, w_2\}$, $y_{j+3}y_{j+1}y_{j+2}$ is a segment of Q_6. Since $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$, $N_G(y_{j+2}) \subseteq \{v_2, v_8, y_{j+3}, w_1, y_{j+1}\}$, $Q_6 = w_1w_2v_2v_1y_{j+3}y_{j+1}y_{j+2}$, a contradiction. If $v_2w_2y_{j+1}$ is a segment of Q_6, then since $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+3}, w_2\}$, $y_{j+1}y_{j+2} \in E(Q_6)$. Now $y_{j+2}w_1 \notin E(Q_6)$, otherwise $y_0Qv_2w_1y_{j+2}y_{j+1}w_2v_8v_1y_{j+3}Qy_s$ is a path longer than Q, a contradiction. If $w_1 \notin V(Q_6)$, then $(Q_6 - v_2w_2) \cup v_2w_1w_2$ is a path longer than Q_6, a contradiction. Thus $w_1 \in V(Q_6)$. Since $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$, $y_{j+3}w_1 \in E(Q_6)$. But now $Q_6 = y_{j+2}y_{j+1}w_2v_2v_1y_{j+3}w_1$, a contradiction. If $v_2w_2y_{j+3}$ is a segment of Q_6, then $y_{j+3}w_2v_2v_1y_{j+3}$ is a segment of Q_6, a contradiction. Thus $w_2v_2 \notin E(Q_6)$. Similarly, $y_{j+1}v_2, w_2y_{j+3}, y_{j+1}y_{j+3} \notin E(Q_6)$. Therefore $w_1w_2y_{j+1}$ and $w_2y_{j+1}y_{j+2}$ are two segments of Q_6. Since $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$, $N_G(y_{j+2}) \subseteq \{v_2, v_8, y_{j+3}, w_1, y_{j+1}\}$, $w_1v_2, y_{j+2}y_{j+3} \in E(Q_6)$ or $w_1y_{j+3}, y_{j+2}v_2 \in E(Q_6)$. But now $v_2v_1y_{j+3}y_{j+2}y_{j+1}w_2w_1v_2$ or $v_2v_1y_{j+3}w_1w_2y_{j+1}y_{j+2}v_2$ is a segment of Q_6, a contradiction. Therefore all the longest paths of G contain v_8. Since G is a counterexample, $v_1y_{j+3} \notin E(G)$. □

Similarly, we could obtain that $v_1y_{k-3} \notin E(G)$. Therefore $d_Q(v_1) = 2$.

Case 2.6.3. $v_2w_1w_2w_3v_8$ is a subpath of Q. In this case, similar to the proof of Case 2.6.1, $d_Q(v_1) = 2$.

Case 2.6.4. $v_2w_1w_2w_3w_4v_8$ is a subpath of Q. In this case, similar to the proof of Case 2.6.1, $d_Q(v_1) = 2$.

Case 2.6.5. $v_2w_1w_2w_3w_4w_5v_8$ is a subpath of Q. In this case, similar to the proofs of Case 2.6.1 and Case 2.6.2, $d_Q(v_1) = 2$. □

Claim 2.38. *If there is a longest path $Q = y_0y_1\ldots y_s$ of G such that $|V(Q) \cap V(C)| \leqslant 7$, then all the longest paths of G have a common vertex.*

P r o o f. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.32, $d_Q(v_1) = 2$. If $d_G(v_1) = 2$, then by Claim 2.31, all the longest paths of G share a common vertex.

If $d_G(v_1) \geqslant 3$, then similar to the proof of Claim 2.15 in the third, forth, fifth paragraphs we could obtain that for any vertex $w \in V(G) \setminus V(Q)$ such that $v_1w \in E(G)$, $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of G contain v_2. If there exists a longest path Q_2 not containing v_2, then by Claim 2.30, $v_1, v_3 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_8]$ such that $u \in V(Q_2)$, for otherwise by Claim 2.30, $v_7 \in V(Q_2)$. By the proof of Claim 2.29, Q_2 is a path of length at least 10, a contradiction. Thus $w_1u \in E(Q)$ and $u \in V(Q_2)$. Now we could check that $N_{Q_2}(v_2) \subseteq \{v_1, u\}$. Thus $u = v_3$. By Claim 2.32, $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now Q_2 is a path of length at least 10, a contradiction. Thus $w_1 \in V(Q_2)$. By Claim 2.32, $w_1 = v_3$ and $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now we could check that $N_{Q_2}(v_2) \subseteq \{v_3, y_{k-2}\}$, a contradiction. □

Since G is a counterexample, by Claim 2.38, for any longest path Q of G, $V(C) \subset V(Q)$. But now all the longest paths of G contain $V(C)$, a contradiction.

2.7. Proof of the case $r = 9$. If $r = 9$, then by Claim 2.1 we have that $s \geqslant 9$. Theorem 2.1 is very important for the following proof.

Theorem 2.1 (Petersen's theorem, [8]). *Every bridgeless cubic graph has a perfect matching.*

Claim 2.39. *If $Q = y_0y_1\ldots y_s$ is a longest path of G, then there is no edge in $G \setminus V(Q)$.*

P r o o f. If there is an edge $e \in G \setminus V(Q)$, then the 5 independent edges in Q together with e are 6 independent edges, a contradiction. □

Claim 2.40. *For any $v \in V(G) \setminus V(Q)$, $d_G(v) = 3$.*

P r o o f. If $d(v) \leqslant 2$, then suppose that $vx \in E(G)$. Now we assume that all the longest paths of G contain x, since if there is a longest path Q_1 not containing x, then by Claim 2.39, $v \in V(Q_1)$. Now v is not the end-vertex of Q_1, since otherwise adding x to Q_1 results in a longer path, a contradiction. But now $d_G(v) \geqslant 3$, a contradiction.

If $d_G(v) \geqslant 4$, then by Claim 2.3 and $r = 9$, $d_G(v) = 4$. Suppose that $N_G(v) = \{u_1, u_2, u_3, u_4\}$ and $u_1 = y_{j_1}$, $u_2 = y_{j_2}$, $u_3 = y_{j_3}$, $u_4 = y_{j_4}$, $1 \leqslant j_1 < j_2 < j_3 < j_4 \leqslant s-1$.

If $vu_1Q[u_1, u_4]u_4v$ is a cycle of length 9, then either $u_1w_1w_2u_2w_3u_3w_4u_4$ or $u_1w_1u_2w_2w_3u_3w_4u_4$ or $u_1w_1u_2w_2u_3w_3w_4u_4$ is a subpath of Q. If $u_1w_1w_2u_2w_3u_3w_4u_4$ is a subpath of Q, then $Q = y_0u_1w_1w_2u_2w_3u_3w_4u_4y_9$, for otherwise we could find 6 independent edges, a contradiction.

We can check that for each vertex $v \in \{w_3, w_4\}$ the following assertions hold:

(i) v is not adjacent to y_0, y_9 (since $r = 9$);

(ii) v is not adjacent to w_1, w_2 (since otherwise we could obtain a path longer than Q);

(iii) v is not adjacent to any vertex in $V(G) \setminus V(Q)$ (since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $vz \in E(G)$, then y_0u_1, w_1w_2, u_2v, w_3z, u_3w_4, u_4y_9 or y_0u_1, w_1w_2, u_2v, w_3u_3, w_4z, u_4y_9 are 6 independent edges, a contradiction);

(iv) $w_3w_4 \notin E(G)$ (since otherwise $y_0Qw_3w_4u_3vu_4Qy_s$ is a path longer than Q, a contradiction);

(v) $N_G(v) \subseteq \{u_1, u_2, u_3, u_4\}$ (since (i), (ii), (iii) and (iv)).

Now we prove that all the longest paths of G contain u_3. If there exists a longest path Q_1 not containing u_3, then by Claim 2.39, $w_3, w_4, v \in V(Q_1)$. Now w_3, w_4, v are not end-vertices of Q_1, since otherwise adding u_3 to Q_1 results in a longer path, a contradiction. Suppose that u_1vu_2 is a segment of Q_1, then $u_1w_3u_4$, $u_2w_4u_4$ or $u_2w_3u_4$, $u_1w_4u_4$ are two segments of Q_1. But now $u_4w_3u_1vu_2w_4u_4$ or $u_4w_4u_1vu_2w_3u_4$ is a segment of Q_1, a contradiction. Thus, all the longest paths of G contain u_3. Since G is a counterexample, $u_1w_1w_2u_2w_3u_3w_4u_4$ is not a subpath of Q. Similarly, $u_1w_1u_2w_2u_3w_3w_4u_4$ is not a subpath of Q.

If $u_1w_1u_2w_2w_3u_3w_4u_4$ is a subpath of Q, then $Q = y_0u_1w_1u_2w_2w_3u_3w_4u_4y_9$, for otherwise we could find 6 independent edges, a contradiction.

We can check that for each vertex $v \in \{w_1, w_2\}$ the following assertions hold:

(i) v is not adjacent to y_0, y_9, w_4 (since otherwise we could find a path longer than Q);

(ii) v is not adjacent to any vertex in $V(G) \setminus V(Q)$ (since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $vz \in E(G)$, then $zw_1Qu_4vu_1y_0$ or $zw_2Qu_4vu_2Qy_0$ is a path longer than Q);

(iii) w_1 is not adjacent to $\{w_2, w_3\}$. Since otherwise

$$y_0u_1vu_2w_1w_2Qy_9 \quad \text{or} \quad y_0Qw_1w_3w_2u_2vu_3Qy_9$$

is a path longer than Q, a contradiction;

(iv) $N_G(w_1) \subseteq \{u_1, u_2, u_3, u_4\}$ and $N_G(w_2) \subseteq \{w_3, u_1, u_2, u_3, u_4\}$ (since (i), (ii) and (iii)).

Similarly, we could prove that $N_G(w_3) \subseteq \{w_2, u_1, u_2, u_3, u_4\}$ and $N_G(w_4) \subseteq \{u_1, u_2, u_3, u_4\}$.

Now we prove that all the longest paths of G contain u_2. If there exists a longest path Q_2 not containing u_2, then by Claim 2.39, $w_1, w_2, v \in V(Q_2)$. Now w_1, w_2, v are not end-vertices of Q_2, since otherwise adding u_2 to Q_2 results in a longer path, a contradiction. If u_1vu_3 is a segment of Q_2, then $u_1w_1u_4$ or $u_3w_1u_4$ is a segment of Q_2. Without loss of generality, suppose that $u_1w_1u_4$ is a segment of Q_2. Now $u_4w_2w_3$ or $u_3w_2w_3$ is a segment of Q_2. Without loss of generality, suppose that $u_4w_2w_3$ is a segment of Q_1. Since $N_G(w_3) \subseteq \{w_2, u_1, u_2, u_3, u_4\}$, w_3 is an end-vertex of Q_2. If $w_4 \in V(Q_2)$, then since $N_G(w_4) \subseteq \{u_1, u_2, u_3, u_4\}$, $Q_2 = w_3w_2u_4w_1u_1vu_3w_4$, a contradiction. Thus $w_4 \notin V(Q_2)$. Now u_3 is not an end-vertex of Q_2. If u_3y_0 or $u_3y_s \in E(Q_2)$, then $Q_2 = w_3w_2u_4w_1u_1vu_3y_0$ or $Q_2 = w_3w_2u_4w_1u_1vu_3y_s$, a contradiction. Thus, there exists a vertex $z \in V(G) \setminus V(Q)$ such that $u_3z \in E(Q_2)$. But now $Q_2 = w_3w_2u_4w_1u_1vu_3z$, a contradiction. Thus u_1vu_3 is not a segment of Q_2. Similarly, we could prove that u_1vu_4, u_3vu_4 are not segments of Q_2, a contradiction. Thus, all the longest paths of G contain u_2. Since G is a counterexample, $vu_1Q[u_1, u_4]u_4v$ is a cycle of length 8.

As above, we could prove that $N_G(w_i) \subseteq \{u_1, u_2, u_3, u_4\}$ $(i = 1, 2, 3)$, and all the longest paths of G have a common vertex, a contradiction. Thus $d_G(v) = 3$. □

Claim 2.41. *G has an independent set of 6 edges.*

P r o o f. Since G is a counterexample, for any vertex $v \in V(G)$ there exists a longest path not containing it. By Claim 2.40, $d_G(v) = 3$. Since $s \geqslant 9$, G has at least 12 vertices. Suppose that X is a connected component in $G \setminus V(C)$. If $|X| \geqslant 4$, then there is a path of length at least 11, and therefore G has 6 independent edges. If $|X| = 3$, then since $G[X]$ is connected, there is a spanning path in $G[X]$. Since G is cubic, there is a path of length at least 11 and G has 6 independent edges. If $|X| \leqslant 2$, then since G is a connected cubic graph, the edges connecting X and C are not cut edges. Thus G is a bridgeless cubic graph. By Theorem 2.1, G has 6 independent edges. □

By Claim 2.41, G has 6 independent edges, a contradiction.

2.8. Proof of the case $r = 10$. If $r = 10$, then by Claim 2.1 we have that $s = 10$.

Claim 2.42. *If $Q = y_0y_1 \ldots y_{10}$ is a longest path of G, then there is no edge in $G \setminus V(Q)$.*

P r o o f. If there is an edge $a \in G \setminus V(Q)$, then 5 independent edges in Q together with a are 6 independent edges, a contradiction. □

Claim 2.43. *For any $v \in V(G) \setminus V(Q)$, $d_G(v) = 3$.*

Proof. Since $s = 10$, $N_G(v) \subseteq \{y_1, y_3, y_5, y_7, y_9\}$. If $d_G(v) \leqslant 2$, then suppose that $vx \in E(G)$. Now we claim that all the longest paths of G contain x. Since if there is a longest path Q_1 not containing x, then by Claim 2.42, $v \in Q_1$. We see that v is not the end-vertex of Q_1, since otherwise adding x to Q_1 results in a longer path, a contradiction. But now $d_G(v) \geqslant 3$, a contradiction.

If $d_G(v) = 5$, then $y_iy_j \notin E(G)$, $i, j \in \{0, 2, 4, 6, 8, 10\}$ (without loss of generality, suppose that $i < j$), for otherwise $y_0Qy_iy_jQy_{i+1}vy_{j+1}Qy_s$ is a path longer than Q, a contradiction. Now for the cycle $C_1 = vy_1y_2y_3y_4y_5y_6y_7y_8y_9v$ there is no edge in $G \setminus V(C_1)$ and for each vertex $x \in V(G) \setminus V(C_1)$, $N_G(x) \subseteq \{y_1, y_3, y_5, y_7, y_9\}$. If there is a longest path Q_2 of G not containing y_1, then Q_2 contains at most 9 vertices, a contradiction. Thus, every longest path of G contains y_1. Since G is a counterexample, $d_G(v) \neq 5$.

If $d_G(v) = 4$ and $vy_1, vy_9 \in E(G)$, then $vy_1y_2y_3y_4y_5y_6y_7y_8y_9v$ is a cycle of length 10. If $vy_3 \notin E(G)$, then $y_2y_j \notin E(G)$, $j \in \{6, 8, 0, 10\}$, for otherwise $y_0y_1vy_{j-1}Qy_2y_jQy_s$ or $y_{10}Qy_2y_0y_1v$ or $y_{10}y_2Qy_9vy_1y_0$ is a path longer than Q, a contradiction. Furthermore, $y_iy_j \notin E(G)$, $i, j \in \{0, 4, 6, 8, 10\}$ (without loss of generality, suppose that $i < j$), for otherwise $y_0Qy_iy_jQy_{i+1}vy_{j+1}Qy_s$ is a path longer than Q, a contradiction. Now for the cycle $C_2 = vy_1y_2y_3y_4y_5y_6y_7y_8y_9v$ there is no edge in $G \backslash V(C_2)$ and for each vertex $x \in V(G) \backslash V(C_2)$, $N_G(x) \subseteq \{y_1, y_3, y_5, y_7, y_9\}$. If $y_2y_4 \notin E(G)$, then as above, every longest path of G not containing y_1 contains at most 9 vertices, a contradiction. Since G is a counterexample, $y_2y_4 \in E(G)$. Now if there is a longest path Q_3 of G not containing y_1, then Q_3 contains at most 10 vertices, a contradiction. Thus $vy_3 \in E(G)$. Similarly, we could prove that $vy_5, vy_7 \in E(G)$. But now $d_G(v) = 5$, a contradiction. Thus $vy_1 \notin E(G)$ or $vy_9 \notin E(G)$. Now we could obtain that $y_iy_j \notin E(G)$, $i, j \in \{2, 4, 6, 8\}$ (without loss of generality, suppose that $i < j$), for otherwise $y_0Qy_iy_jQy_{i+1}vy_{j+1}Qy_s$ is a path longer than Q, a contradiction. If $y_0y_i \in E(G)$ or $y_{10}y_i \in E(G)$, $i \in \{0, 4, 6, 8, 10\}$, then $y_2y_1y_0y_{10}Qy_3v$ or $y_{10}Qy_iy_0Qy_{i-1}v$ or $y_{10}y_iQy_9vy_{i-1}Qy_0$ is a path longer than Q, a contradiction. Thus $y_0y_i, y_{10}y_i \notin E(G)$, $i \in \{0, 4, 6, 8, 10\}$. Furthermore, $y_2y_{10} \notin E(G)$, for otherwise $y_0y_1y_2y_{10}Qy_3v$ is a path longer than Q, a contradiction. If $y_4y_1 \in E(G)$, then $C_3 = y_1y_4Qy_9vy_3y_2y_1$ is a cycle of length 10. Now $y_2y_0 \notin E(G)$, for otherwise $y_0y_2y_1y_4y_3vy_5Qy_s$ is a path longer than Q, a contradiction. But now, as above, we could obtain that all the longest paths of G contain y_3, y_5, y_7, y_9, a contradiction. Thus $y_4y_1 \notin E(G)$. Similarly, we could obtain that $y_iy_1 \notin E(G)$, $i \in \{6, 8\}$. Now $N_G(y_i) \subseteq \{y_3, y_5, y_7, y_9\}$, $i \in \{4, 6, 8\}$. If there is a longest path Q_3 of G not containing y_7, then by Claim 2.42, $y_6, y_8, v \in V(Q_3)$. Now y_6, y_8, v are not end-vertices of Q_3. Since otherwise adding y_7 to Q_3 results in a longer path, a contradiction. If y_3vy_5

is a segment of Q_3, then $y_3y_6y_9$, $y_5y_8y_9$ or $y_5y_6y_9$, $y_3y_8y_9$ are two segments of Q_3. But now $y_9y_6y_3vy_5y_8y_9$ or $y_9y_8y_3vy_5y_6y_9$ is a segment of Q_3, a contradiction. Thus, y_3vy_5 is not a segment of Q_3. Similarly, we could obtain that $y_3vy_9, y_5vy_9 \notin E(G)$, a contradiction. Thus $d_G(v) \neq 4$. Therefore, $d_G(v) = 3$. □

Claim 2.44. *G has an independent set of 6 edges.*

Proof. Since G is a counterexample, for any vertex $v \in V(G)$ there exists a longest path not containing it. By Claim 2.43, $d_G(v) = 3$. Since $s \geqslant 10$, G has at least 12 vertices. Suppose that X is a connected component in $G \setminus V(C)$. If $|X| \geqslant 2$, then the 5 independent edges and an edge in X are 6 independent edges, a contradiction. Thus $|X| = 1$. Since G is a connected cubic graph, there are three edges connecting X and C. Thus, the edges connecting X and C are not cut edges. Now G is a bridgeless cubic graph. By Theorem 2.1, G has 6 independent edges. □

By Claim 2.44, G has 6 independent edges, a contradiction. Thus, we complete the proof of Theorem 1.1. □

Proof of Conjecture 1.2. By Theorem 1.1, Gallai's conjecture is true for every connected graph G with $\alpha'(G) \leqslant 5$. Thus, a smallest counterexample to Gallai's conjecture must have at least 6 independent edges. As the graph in Figure 1 has 12 vertices, we complete the proof. □

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