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THE METHOD OF LINES FOR HYPERBOLIC STOCHASTIC FUNCTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We apply an approximation by means of the method of lines for hyperbolic stochastic functional partial differential equations driven by one-dimensional Brownian motion. We study the stability with respect to small L^2-perturbations.

Keywords: stochastic partial differential equation; stability of the method of lines; white noise; Volterra stochastic equation

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1. Introduction

The numerical approximation of stochastic partial differential equations (SPDE's) encounters all difficulties that arise in the numerical solution of deterministic PDE's as well as technicalities caused by the nature of the driving white noise process. A well known procedure used to approximate SPDE's is the method of lines. It is a semidiscrete numerical method where the derivatives $\frac{\partial}{\partial x}u(t,x)$ with respect to the space variable are approximated by the finite difference quotients

$$\frac{u(t,x)-u(t,x-h)}{h} \quad \text{or} \quad \frac{u(t,x+h)-u(t,x)}{h},$$

where $h>0$ is a discretization step. By approximating the derivatives we reduce the given PDE to a system of ordinary differential equations. Once spatial derivatives are discretized, we transfer the large system of SPDE's to an integral fixed-point equations taking values in $\mathbb{R}^d$ or Banach spaces. These integral equations are of Volterra type. The most important property of the scheme is convergence. It means that the solution of the difference scheme approximates the solution of the corresponding partial differential equation and the approximation improves as the grid spacing h

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tends to zero. The concept of the stability of the semidiscrete numerical scheme is to perturb both initial condition and the right-hand side of the main equation. These perturbations do not amplify as the discretization step decreases. Under these conditions we prove the convergence of the perturbed solution to the exact solution in the appropriate norm as the discretization step h tends to zero. Next, from the Lax equivalence theorem, we obtain the convergence of the scheme. For the proof of convergence of the method of lines scheme we use an integral representation of the first-order hyperbolic equation and its semidiscrete counterpart. The effectiveness of that representation follows from the method of lines analysis in [16] based on the maximum principle. There is no classical maximum principle for stochastic PDE's, instead we use Doob's martingale inequality to estimate classical Itô integrals. The stability of the method of lines for autonomous linear evolution equation is shown in [20].

The method of lines was widely used to discretize deterministic partial differential equations of various types. It was applied to hyperbolic differential equations by Kreiss and Scherer in [14]. In [3] we can find the method of lines for second-order hyperbolic integro-differential equations in $\mathbb{R}^n$ of the form

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - \nabla^2 u(t,x) = f(t,x) + \displaystyle\int_0^t k(t-s)\nabla^2 u(s,x)\,\mathrm{d}s \quad \text{for } (t,x) \in (0,T] \times \mathbb{R}^n, \\ u(0,x) = f_1(x), \quad \dfrac{\partial u}{\partial t}(0,x) = f_2(x) \quad \text{for } x \in \mathbb{R}^n. \end{cases}$$

A spatial approximation for evolution equations can be found in the work of Bátkai, Csomós and Nickel in [4].

Random transport equations are considered by many authors. The first one who introduced a new type of partial differential equation of first order with a random coefficient of the form

$$\frac{\partial u}{\partial t}(t,x;\omega) + \{\dot{B}_t(\omega) + b(t,x)\}\frac{\partial u}{\partial x}(t,x;\omega) = c(t,x)u(t,x;\omega) + d(t,x)$$

on $[0,T] \times \mathbb{R}$ was Funaki [9]. Kim [12] focused on the Cauchy problem for transport equations with random noise.

The stability of difference schemes whose accuracy is of the second-order for hyperbolic-parabolic equations is studied in [2]. A difference scheme for the Cauchy problem for a hyperbolic equation with a self-adjoint operator is introduced in [1]. The stability of the numerical solution of hyperbolic partial differential-difference equations forward in time and backward in space and error analysis can be found in [22].

There are a number of papers which deal with the method of lines applied to stochastic differential equations. In [23] a finite-difference method is employed to approximate a class of stochastic partial differential equations in weighted Sobolev spaces. The spatial discretization in [21] is carried out using a second-order finite volume method and it is tested for a stochastic advection-diffusion problem and a stochastic Burgers equation driven by white noise. The method of lines for linear eliptic and parabolic SPDE's is presented in [17]. The case of fractional heat equations perturbed by multiplicative cylindrical white noise is treated in [7]. The space discretization for stochastic wave equations is performed in [19].

The aim of this paper is to discuss the uniform stability of the space discretization for the initial value problem corresponding to the stochastic partial differential equation which arises from the McKendrick-von Foerster equation of the form

$$\frac{\partial u}{\partial t}(t,x)+c\frac{\partial u}{\partial x}(t,x)=u(t,x)\lambda\biggl(t,x,u(t,x),\int_0^{\infty}u(t,y)\,\mathrm{d}y\biggr)+\int_{-r}^{0}\int_{-\infty}^{\infty}K(t,s,y)u(t+s,y)\,\mathrm{d}y\,\mathrm{d}s\dot{B}_t$$

with the boundary condition $u(t,0)=v(t)$ for $t\in[0,T]$. Note that the nonlocal coefficient of $\dot{B}_t$ is independent of x. We study the following generalization of the above modified McKendrick-von Foerster equation

$$\frac{\partial u}{\partial t}(t,x)+a(t,x)\frac{\partial u}{\partial x}(t,x)=f(t,x,u_{(t,x)})+g(t,u_{(t,0)})\dot{B}_t, \tag{1.1}$$

where $u_{(t,x)}$ is a Hale-type operator, i.e. $u_{(t,x)}(\tau,\theta)=u(t+\tau,x+\theta)$ for $(\tau,\theta)\in[-r,0]\times\mathbb{R}$, $r\geqslant 0$, $(t,x)\in[0,T]\times\mathbb{R}$. It is crucial for the given volatility g to be independent of x, therefore its composition with the Hale operator has to be independent of x. Hence, we can use it at $(t,0)$. An example of a diffusion term that satisfies the assumptions under which we prove our main results is $g(t,u(t,0))=\sup_x u(t,x)$. This example shows that the technique of the paper is subtle and nearly optimal. One can verify that it is not possible to include in g any dependence on x, because this would lead to Volterra-Itô integrals which are no longer martingales.

The main result of the paper, i.e. the stability of the method of lines (Theorem 3.6), is demonstrated by means of appropriate estimates for an infinite-dimensional system of SDE's (Lemma 3.3). For simpler differential operators associated with the method of lines, e.g. defined by

$$(A_hx)_k=\frac{x_{k+1}-x_k}{h},\quad D(A_h)=l^p(\mathbb{Z}),\quad k\in\mathbb{Z},\ x\in l^p(\mathbb{Z}),\ p\geqslant 1,$$

one needs to find its spectrum and show that

$$\sup_{t\in[0,T],\,h\in[0,1]}\|\mathrm{e}^{A_ht}\|<\infty.$$

Equation (1.1) is much more complicated, because the coefficient $a(t,x)$ depends on time and space. Thus, the respective semigroup of operators has no infinitesimal generator, not to mention any spectral properties, inevitable in the frame of Da Prato and Zabczyk [6]. Even if we dropped the time dependence, so $a(t,x)=a(x)$, then spectral properties of the infinitesimal generator $(A_h x)_k = a_k(x_{k+1}-x_k)/h$ would not be obvious. Since the main problem arises in mathematical biology and plays the role of a transport equation describing the kinetics of some densities [18], it should be treated in l^∞ or l^1.

The paper is organized as follows. In Section 2 we introduce basic notations and formulate the problem. The main result concerning the stability of the method of lines (Section 3) bases on several lemmas: the representation, existence and uniqueness, and the estimation of solution. Proofs of these lemmas are presented in Section 4.

2. Formulation of the problem

Fix $T>0$ and $0\leqslant r<\infty$. Denote $\mathbb{R}_+=[0,\infty)$, $\mathbb{Z}_+=\{0,1,\ldots\}$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, $\{B_t\}_{0\leqslant t\leqslant T}$ the standard one-dimensional Brownian motion, $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}=\sigma(B_s, 0\leqslant s\leqslant t)$ its natural filtration which we extend as $\mathcal{F}_t:=\mathcal{F}_0$ for $t\in[-r,0)$. We recall that a standard one-dimensional Brownian motion is a stochastic process $\{B_t\}_{0\leqslant t\leqslant T}$ with the following properties:

(1) $B_0=0$;
(2) With probability 1 the function $t\mapsto B_t$ is continuous in t;
(3) The process $\{B_t\}_{0\leqslant t\leqslant T}$ has stationary, independent increments;
(4) The increment $B_{t+s}-B_s$ has the normal distribution with mean 0 and variance t.

Let $L^2(\Omega)$ be the space of all random variables $Y\colon\Omega\to\mathbb{R}$ such that $\mathbb{E}[Y^2]<\infty$. By $\dot{B}_t$ we denote the time-dependent one-dimensional white noise, a formal or a distributional derivative of Brownian motion. It is a generalized stochastic process constructed as a probability measure on the space of tempered distributions, that is the dual of the Schwartz space of rapidly decreasing smooth real valued functions on $\mathbb{R}$ (see [10]). Let $D_0=[-r,0]\times\mathbb{R}$ and $D_T=[-r,T]\times\mathbb{R}$. By $\mathcal{C}_{D_T}$ we denote the space of these continuous and $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted processes $u\colon D_T\to L^2(\Omega)$ which satisfy

$$\|u\|^2_{\mathcal{C}_{D_T}} := \mathbb{E}\Big[\sup_{t\in[-r,T],x\in\mathbb{R}} |u^2(t,x)|\Big] < \infty. \tag{2.1}$$

Then $\mathcal{C}_{D_T}$ is a Banach space with respect to $\|\cdot\|_{\mathcal{C}_{D_T}}$. For $(t,x) \in [0,T] \times \mathbb{R}$ we define a Hale-type operator of the form

$$u_{(t,x)}(\tau,\theta) = u(t+\tau, x+\theta), \quad (\tau,\theta) \in D_0.$$

Notice that the domain of the Hale-type operator is unbounded. Usually it is considered to be a compact set (see [15]). Suppose that $a\colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $f\colon [0,T] \times \mathbb{R} \times \mathcal{C}(D_0,\mathbb{R}) \to \mathbb{R}$, $g\colon [0,T] \times \mathcal{C}(D_0,\mathbb{R}) \to \mathbb{R}$, $\varphi\colon D_0 \to \mathbb{R}$ are continuous functions and $\mathcal{C}(D_0,\mathbb{R})$ denotes the space of all continuous functions $v\colon D_0 \to \mathbb{R}$. We consider the following initial value problem for the first-order stochastic functional partial differential equation

$$\text{(2.2)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) + a(t,x)\dfrac{\partial u}{\partial x}(t,x) = f(t,x,u_{(t,x)}) + g(t,u_{(t,0)})\dot{B}_t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in [0,T] \times \mathbb{R}, \\ u(t,x) = \varphi(t,x) \quad \text{for } (t,x) \in D_0. \end{cases}$$

The solution to (2.2) is understood as an $\{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}$-adapted continuous process satysfying the integral equation

$$u(t,x) = \varphi(0,\eta^{t,x}(0)) + \int_0^t f(s,\eta^{t,x}(s),u_{(s,\eta^{t,x}(s))})\,\mathrm{d}s + \int_0^t g(s,u_{(s,0)})\,\mathrm{d}B_s,$$

where the last term is the Itô stochastic integral and $\eta^{t,x}(s)$ is the solution of the characteristic equation

$$\eta(s) = x - \int_s^t a(\tau,\eta(\tau))\,\mathrm{d}\tau, \quad 0 \leqslant s \leqslant t \leqslant T.$$

Assumption 2.1. Suppose that the function f is continuous and satisfies the Lipschitz condition with respect to the third (functional) variable, so there exist constants $L_1, C_1 > 0$ such that

$$\text{(2.3)} \qquad |f(t,x,v) - f(t,x,\overline{v})| \leqslant L_1 \sup_{(s,y)\in D_0} |v(s,y) - \overline{v}(s,y)|$$

$$\text{(2.4)} \qquad |f(t,x,0)| \leqslant C_1$$

for all $t \in [0,T]$, $x \in \mathbb{R}$, $v, \overline{v} \in \mathcal{C}(D_0,\mathbb{R})$. Suppose that the function g is continuous and satisfies the Lipschitz condition with respect to the second (functional) variable, so there exist constants $L_2, C_2 > 0$ such that

$$\text{(2.5)} \qquad |g(t,v) - g(t,\overline{v})| \leqslant L_2 \sup_{(s,y)\in D_0} |v(s,y) - \overline{v}(s,y)|$$

$$\text{(2.6)} \qquad |g(t,0)| \leqslant C_2$$

for all $t \in [0,T]$, $v, \overline{v} \in \mathcal{C}(D_0,\mathbb{R})$.

3. Stability of the method of lines

First we consider the following deterministic initial value problem for the first-order partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + a(t,x)\dfrac{\partial u}{\partial x}(t,x) = \psi(t,x) \quad \text{for } (t,x) \in [0,T]\times\mathbb{R}, \\ u(0,x) = \theta(x) \quad \text{for } x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $a, \psi \in \mathcal{C}([0,T]\times\mathbb{R},\mathbb{R})$, $\theta \in \mathcal{C}(\mathbb{R},\mathbb{R})$. Problem (3.1) is discretized in the spatial variable as follows. We introduce a uniform mesh on $\mathbb{R}$ with the discretization step $h > 0$. The method of lines for (3.1) is of the form

$$\begin{cases} \dfrac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t} + u^{(i)}(t)\dfrac{a^{(i)}(t)}{h} = u^{(i-1)}(t)\dfrac{a^{(i)}(t)}{h} + \psi^{(i)}(t) \quad \text{for } t \in [0,T],\ i \in \mathbb{Z}, \\ u^{(i)}(0) = \theta^{(i)} \quad \text{for } i \in \mathbb{Z}, \end{cases} \tag{3.2}$$

where $a^{(i)}(t)$, $\psi^{(i)}(t)$, $\theta^{(i)}$ are approximations of $a(t,ih)$, $\psi(t,ih)$, $\theta(ih)$, respectively. We assume they are continuous and $a^{(i)}(t) \geqslant 0$. The cases of nonpositive $a^{(i)}(t)$ or arbitrary $a^{(i)}(t)$ are covered in Appendix. The main result is the stability of the method of lines. We shall prove it with the help of several lemmas.

Lemma 3.1. *Suppose that* $a^{(i)}(t) \geqslant 0$ *and*

$$\sup_{t\in[0,T],\, i\in\mathbb{Z}} a^{(i)}(t) < \infty, \qquad \sup_{t\in[0,T],\, i\in\mathbb{Z}} |\psi^{(i)}(t)| < \infty, \quad \sup_{i\in\mathbb{Z}} |\theta^{(i)}| < \infty.$$

Then problem (3.2) *admits a solution* $u^{(i)}(t)$ *satisfying*

$$\sup_{t\in[0,T],\, i\in\mathbb{Z}} |u^{(i)}(t)| < \infty$$

of the form

$$u^{(i)}(t) = \sum_{k=0}^{\infty}\Big(\theta^{(i-k)}\Gamma^{(i,k)}(t,0) + \int_0^t \psi^{(i-k)}(s)\Gamma^{(i,k)}(t,s)\,\mathrm{d}s\Big), \tag{3.3}$$

where $\Gamma^{(i,k)}(t,s) \geqslant 0$ *for* $i \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, $0 \leqslant s \leqslant t \leqslant T$ *is defined by the recurrence*

$$\begin{cases} \Gamma^{(i,0)}(t,s) = \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w}, \\ \Gamma^{(i,k)}(t,s) = \displaystyle\int_s^t \mathrm{e}^{-\int_{s_1}^t (a^{(i)}(w)/h)\,\mathrm{d}w}\,\frac{a^{(i)}(s_1)}{h}\Gamma^{(i-1,k-1)}(s_1,s)\,\mathrm{d}s_1, \end{cases} \tag{3.4}$$

and it satisfies the normalization property

$$\sum_{k=0}^{\infty}\Gamma^{(i,k)}(t,s) = 1. \tag{3.5}$$

The proof of Lemma 3.1 is provided in Section 4. Now let us consider the method of lines for the initial value problem corresponding to the first-order stochastic functional partial differential equation. We denote by J_h the linear interpolating operator given by

$$(J_h u)(t,x) = u^{(i)}(t)\Big(1 - \frac{x-ih}{h}\Big) + u^{(i+1)}(t)\frac{x-ih}{h}, \quad x \in [ih,(i+1)h],$$

which maps discrete functions to continuous functions and whose supremum norm is equal to 1 (see [11]). Then the method of lines for (2.2) is of the form

$$\tag{3.6} \begin{cases} \dfrac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t} + u^{(i)}(t)\dfrac{a^{(i)}(t)}{h} = u^{(i-1)}(t)\dfrac{a^{(i)}(t)}{h} + f(t, ih, (J_h u)_{(t,ih)}) \\ \qquad\qquad\qquad\qquad\qquad + g(t,(J_h u)_{(t,0)})\dot{B}_t \quad \text{for } t \in [0,T],\ i \in \mathbb{Z}, \\ u^{(i)}(t) = \varphi^{(i)}(t) \quad \text{for } t \in [-r,0],\ i \in \mathbb{Z}, \end{cases}$$

where $\varphi^{(i)}(t)$ is the approximation of $\varphi(t,ih)$. By $\mathcal{X}_T$ we denote the space of these continuous and $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted processes $w = (w^{(i)})_{i\in\mathbb{Z}}$, $w^{(i)}\colon [-r,T] \to L^2(\Omega)$ which satisfy

$$\|w\|_t^2 := \mathbb{E}\Big[\sup_{\tilde{t}\in[-r,t],\, i\in\mathbb{Z}} |w^{(i)}(\tilde{t})|^2\Big] < \infty, \quad 0 \leqslant t \leqslant T.$$

If $w\colon [-r,T] \to L^2(\Omega)$ is an $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted continuous process, then

$$\|w\|_t^2 = \mathbb{E}\Big[\sup_{\tilde{t}\in[-r,t]} |w(\tilde{t})|^2\Big], \quad 0 \leqslant t \leqslant T.$$

By $\mathcal{X}_{[T]}$ we denote the space of these continuous and $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted processes $w = (w^{(i)})_{i\in\mathbb{Z}}$, $w^{(i)}\colon [0,T] \to L^2(\Omega)$ which satisfy

$$\|w\|_{[t]}^2 := \mathbb{E}\Big[\sup_{\tilde{t}\in[0,t],\, i\in\mathbb{Z}} |w^{(i)}(\tilde{t})|^2\Big], \quad 0 \leqslant t \leqslant T.$$

If $w\colon [0,T] \to L^2(\Omega)$ is an $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$-adapted continuous process, then

$$\|w\|_{[t]}^2 = \mathbb{E}\Big[\sup_{\tilde{t}\in[0,t]} |w(\tilde{t})|^2\Big], \quad 0 \leqslant t \leqslant T.$$

Using Lemma 3.1 we introduce an integral representation of (3.6) in the following lemma.

Lemma 3.2. *Let $F^{(i)}, G \in \mathcal{X}_{[T]}$, $F = (F^{(i)})_{i\in\mathbb{Z}}$, $a^{(i)} \in \mathcal{C}([0,T],\mathbb{R}_+)$, $\varphi^{(i)} \in \mathcal{C}([-r,0],\mathbb{R})$ and*

$$\sup_{t\in[0,T],\, i\in\mathbb{Z}} a^{(i)}(t) < \infty, \quad \sup_{i\in\mathbb{Z}} |\varphi^{(i)}(0)| < \infty.$$

Then the initial-value problem

$$\text{(3.7)}\qquad \begin{cases} \dfrac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t} + u^{(i)}(t)\dfrac{a^{(i)}(t)}{h} = u^{(i-1)}(t)\dfrac{a^{(i)}(t)}{h} + F^{(i)}(t) + G(t)\dot{B}_t \\ \hfill \text{for } t \in [0,T],\ i \in \mathbb{Z}, \\ u^{(i)}(t) = \varphi^{(i)}(t) \quad \text{for } t \in [-r,0],\ i \in \mathbb{Z} \end{cases}$$

admits a solution $u = (u^{(i)})_{i\in\mathbb{Z}}$, $u^{(i)} \in \mathcal{X}_T$ *of the form*

(3.8)

$$u^{(i)}(t) = \sum_{k=0}^{\infty} \varphi^{(i-k)}(0)\Gamma^{(i,k)}(t,0) + \int_0^t \sum_{k=0}^{\infty} F^{(i-k)}(s)\Gamma^{(i,k)}(t,s)\,\mathrm{d}s + \int_0^t G(s)\,\mathrm{d}B_s$$

for $t \leqslant T$, *where* $\Gamma^{(i,k)}(t,s)$ *are given by* (3.4).

Lemma 3.3. *Let* $F^{(i)}, G \in \mathcal{X}_{[T]}$, $F = (F^{(i)})_{i\in\mathbb{Z}}$, $a^{(i)} \in \mathcal{C}([0,T],\mathbb{R}_+)$, $\varphi^{(i)} \in \mathcal{C}([-r,0],\mathbb{R})$ *and*

$$\sup_{t\in[0,T],\, i\in\mathbb{Z}} a^{(i)}(t) < \infty, \quad \sup_{i\in\mathbb{Z}} |\varphi^{(i)}(0)| < \infty.$$

Then any continuous solution $u = (u^{(i)})_{i\in\mathbb{Z}}$ *of* (3.7) *such that* $u^{(i)} \in \mathcal{X}_T$, *satisfies the estimate*

$$\text{(3.9)}\qquad \|u\|_t^2 \leqslant 3\|u\|_{[0]}^2 + 3t\int_0^t \|F\|_{[s]}^2\,\mathrm{d}s + 12\int_0^t \|G\|_{[s]}^2\,\mathrm{d}s.$$

The proof of Lemma 3.3 is provided in Section 4.

Remark 3.4. Considering (3.7) with the zero initial condition, by the elementary inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$, the estimate (3.9) takes the form

$$\|u\|_t^2 \leqslant 2t\int_0^t \|F\|_{[s]}^2\,\mathrm{d}s + 8\int_0^t \|G\|_{[s]}^2\,\mathrm{d}s.$$

The existence and uniqueness of the solution of Volterra-type integral equations is proved in [5] and [8]. Analogously, by using Picard iterations, we can prove the existence and uniqueness of the solution of (3.6).

Theorem 3.5. *Let Assumption* 2.1 *hold. Suppose that* $a^{(i)} \in \mathcal{C}([0,T],\mathbb{R}_+)$, $\varphi^{(i)} \in \mathcal{C}([-r,0],\mathbb{R})$ *and*

$$\sup_{t\in[0,T],\, i\in\mathbb{Z}} a^{(i)}(t) < \infty, \quad \sup_{i\in\mathbb{Z}} |\varphi^{(i)}(0)| < \infty.$$

Then there exists a solution $u = (u^{(i)})_{i\in\mathbb{Z}}$ *of* (3.6) *such that* $u^{(i)} \in \mathcal{X}_T$.

The proof of Theorem 3.5 is provided in Section 4. Let processes $\phi_1^{(i)}, \phi_2 \in \mathcal{X}_{[T]}$ and $\phi_3^{(i)} \in \mathcal{C}([-r,0],\mathbb{R})$ be deterministic and satisfy $\sup\limits_{i\in\mathbb{Z}} |\phi_3^{(i)}(0)| < \infty$. The perturbed system related to (3.6) is of the form

$$
(3.10)\qquad \begin{cases} \dfrac{\mathrm{d}\bar{u}^{(i)}(t)}{\mathrm{d}t} + \bar{u}^{(i)}(t)\dfrac{a^{(i)}(t)}{h} = \bar{u}^{(i-1)}(t)\dfrac{a^{(i)}(t)}{h} + f(t, ih, (J_h\bar{u})_{(t,ih)}) \\ \qquad\qquad +g(t,(J_h\bar{u})_{(t,0)})\dot{B}_t + \phi_1^{(i)}(t) + \phi_2(t)\dot{B}_t \quad \text{for } t\in[0,T],\ i\in\mathbb{Z}, \\ \bar{u}^{(i)}(t) = \varphi^{(i)}(t) + \phi_3^{(i)}(t) \quad \text{for } t\in[-r,0],\ i\in\mathbb{Z}, \end{cases}
$$

where $\phi_1^{(i)}$ and ϕ_2 are perturbations of the right-hand side of (3.6) and $\phi_3^{(i)}$ are perturbations of the initial conditions. We say that the method of lines (3.6) is stable if

$$\|\bar{u} - u\|_T \to 0 \quad \text{as } h \to 0$$

provided that

$$\|\phi_3\|_0 + \|\phi_1\|_{[T]} + \|\phi_2\|_{[T]} \to 0 \quad \text{as } h\to 0.$$

In the following theorem we show the stability of the method of lines (3.6).

Theorem 3.6. *Let the functions f and g satisfy the assumptions of Lemma* 3.3 *with $F = f$, $G = g$. Then the method of lines* (3.6) *is stable.*

P r o o f. Let u be the solution of (3.6) and $\bar{u}$ be the solution of (3.10). Denote $z^{(i)}(t) = \bar{u}^{(i)}(t) - u^{(i)}(t)$. By Lemma 3.2, $z^{(i)}$ satisfies the integral equation

$$z^{(i)}(t) = \sum_{k=0}^{\infty} \phi_3^{(i-k)}(0)\Gamma^{(i,k)}(t,0) + \int_0^t \sum_{k=0}^{\infty} \tilde{f}_h^{(i-k)}(s)\Gamma^{(i,k)}(t,s)\,\mathrm{d}s + \int_0^t \tilde{g}(s)\,\mathrm{d}B_s$$

for $0 \leqslant t \leqslant T$, where

$$\begin{aligned} \tilde{f}_h^{(i)}(t) &= f(t, ih, (J_h u)_{(t,ih)}) - f(t, ih, (J_h\bar{u})_{(t,ih)}) - \phi_1^{(i)}(t), \\ \tilde{g}(t) &= g(t,(J_h u)_{(t,0)}) - g(t,(J_h\bar{u})_{(t,0)}) - \phi_2(t) \end{aligned}$$

and $\Gamma^{(i,k)}(t,s)$ are given by (3.4). By Lemma 3.3 we obtain the estimate

$$\|z\|_t^2 \leqslant 3\|\phi_3\|_0^2 + 3t\int_0^t \|\tilde{f}_h\|_{[s]}^2\,\mathrm{d}s + 12\int_0^t \|\tilde{g}\|_{[s]}^2\,\mathrm{d}s.$$

Denote $\varepsilon_0 := \|\phi_3\|_0$, $\varepsilon_1 := \|\phi_1\|_{[T]}$ and $\varepsilon_2 := \|\phi_2\|_{[T]}$ and assume that

$$\varepsilon_0 + \varepsilon_1 + \varepsilon_2 \to 0 \quad \text{as } h \to 0.$$

It follows immediately from Assumption 2.1 that

$$\|\tilde{f}_h\|_{[t]}^2 \leqslant 2L_1^2\|z\|_t^2 + 2\varepsilon_1^2, \quad \|\tilde{g}\|_{[t]}^2 \leqslant 2L_2^2\|z\|_t^2 + 2\varepsilon_2^2.$$

To complete the proof, it is sufficient to show that $\|z\|_t \to 0$ on $[0,T]$. It follows from the above estimates and the fact that $(a+b+c)^2 \leqslant 3a^2+3b^2+3c^2$ that

$$\|z\|_t^2 \leqslant 3\varepsilon_0^2 + 3t\int_0^t (2L_1^2\|z\|_s^2 + 2\varepsilon_1^2)\,\mathrm{d}s + 12\int_0^t (2L_2^2\|z\|_s^2 + 2\varepsilon_2^2)\,\mathrm{d}s.$$

Finally, we have

$$\|z\|_t^2 \leqslant 3\varepsilon_0^2 + 6\varepsilon_1^2t^2 + 24\varepsilon_2^2t + (6L_1^2t + 24L_2^2)\int_0^t \|z\|_s^2\,\mathrm{d}s.$$

From Gronwall's lemma we get

$$\|z\|_t^2 \leqslant (3\varepsilon_0^2 + 6\varepsilon_1^2t^2 + 24\varepsilon_2^2t)\mathrm{e}^{t(6L_1^2t+24L_2^2)},$$

and $\|z\|_t \to 0$ for $\varepsilon_0+\varepsilon_1+\varepsilon_2 \to 0$. Thus, the method of lines is stable. $\square$

4. Proofs

In this section, proofs of the lemmas from Section 3 will be presented. First, we prove Lemma 3.1.

P r o o f of Lemma 3.1. From the recurrence equation (3.4) it can be evaluated that $\Gamma^{(i,k)}(t,s) \geqslant 0$. To check formula (3.3) for the solution of (3.2) we begin with the fact that any solution of (3.2) satisfies the integral equation

$$\begin{aligned} u^{(i)}(t) &= \mathrm{e}^{-\int_0^t (a^{(i)}(s)/h)\,\mathrm{d}s}\theta^{(i)} \\ &+ \int_0^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w}\frac{a^{(i)}(s)}{h}u^{(i-1)}(s)\,\mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w}\psi^{(i)}(s)\,\mathrm{d}s \quad \text{for } 0 \leqslant s \leqslant t \leqslant T. \end{aligned} \tag{4.1}$$

By (3.3), the left-hand side (LHS) of (4.1) is given by

$$\mathrm{LHS} = \sum_{k=0}^{\infty}\Big(\theta^{(i-k)}\Gamma^{(i,k)}(t,0) + \int_0^t \psi^{(i-k)}(s)\Gamma^{(i,k)}(t,s)\,\mathrm{d}s\Big).$$

Putting (3.3) for $i-1$ to the right-hand side (RHS) of (4.1), we have

$$\begin{aligned}
\mathrm{RHS} = {} & \mathrm{e}^{-\int_0^t (a^{(i)}(s)/h)\,\mathrm{d}s}\theta^{(i)} \\
& + \int_0^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w} \frac{a^{(i)}(s)}{h} \\
& \times \sum_{k=1}^{\infty} \left(\theta^{(i-k)}\Gamma^{(i-1,k-1)}(s,0) + \int_0^s \psi^{(i-k)}(s_1)\Gamma^{(i-1,k-1)}(s,s_1)\,\mathrm{d}s_1 \right) \mathrm{d}s \\
& + \int_0^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w} \psi^{(i)}(s)\,\mathrm{d}s.
\end{aligned}$$

Changing the order of summation and integration, we have

$$\begin{aligned}
\mathrm{RHS} = {} & \mathrm{e}^{-\int_0^t (a^{(i)}(s)/h)\,\mathrm{d}s}\theta^{(i)} \\
& + \sum_{k=1}^{\infty} \int_0^t \psi^{(i-k)}(s_1) \int_{s_1}^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w} \frac{a^{(i)}(s)}{h} \Gamma^{(i-1,k-1)}(s,s_1)\,\mathrm{d}s\,\mathrm{d}s_1 \\
& + \sum_{k=1}^{\infty} \theta^{(i-k)} \int_0^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w} \frac{a^{(i)}(s)}{h} \Gamma^{(i-1,k-1)}(s,0)\,\mathrm{d}s \\
& + \int_0^t \mathrm{e}^{-\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w} \psi^{(i)}(s)\,\mathrm{d}s.
\end{aligned}$$

Comparing the terms of LHS and RHS, we obtain the recurrence formulas for $\Gamma^{(i,k)}(t,s)$, which completes the proof of the first part. Now we show that

$$\sum_{k=0}^{\infty} \Gamma^{(i,k)}(t,0) = 1 \quad \text{for } 0 \leqslant t \leqslant T.$$

If $\psi^{(i)}(t) = 0$, then

$$u^{(i)}(t) = \sum_{k=0}^{\infty} \theta^{(i-k)}\Gamma^{(i,k)}(t,0)$$

is the only solution of (3.2). Hence

$$u^{(i)}(t) = \sum_{k=0}^{\infty} \Gamma^{(i,k)}(t,0)$$

is the only solution to the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t} + u^{(i)}(t)\dfrac{a^{(i)}(t)}{h} = u^{(i-1)}(t)\dfrac{a^{(i)}(t)}{h} & \text{for } t \in [0,T],\ i \in \mathbb{Z}, \\ u^{(i)}(t) = 1 \quad \text{for } t \in [-r,0],\ i \in \mathbb{Z}. \end{cases}$$

On the other hand, $u^{(i)}(t)=1$ is the solution of this sample problem. Thus

$$\sum_{k=0}^{\infty}\Gamma^{(i,k)}(t,0)=1 \quad \text{for } 0\leqslant t\leqslant T.$$

Similarly, we can prove that

$$\sum_{k=0}^{\infty}\Gamma^{(i,k)}(t,s)=1 \quad \text{for } 0\leqslant s\leqslant t\leqslant T.$$

It is sufficient to consider (3.2) with $\psi=1$ and $\theta=0$. This completes the proof. $\square$

Proof of Lemma 3.3. From Lemma 3.2 we have

$$\begin{aligned}\|u^{(i)}\|_t &\leqslant \Big\|\sum_{k=0}^{\infty}\varphi^{(i-k)}(0)\Gamma^{(i,k)}(t,0)\Big\|_{[t]}+\Big\|\int_0^t\sum_{k=0}^{\infty}F^{(i-k)}(s)\Gamma^{(i,k)}(t,s)\,\mathrm{d}s\Big\|_{[t]}\\ &\quad+\Big\|\int_0^t G(s)\,\mathrm{d}B_s\Big\|_{[t]}=I_1+I_2+I_3.\end{aligned}$$

Since $\varphi^{(i)}(0)$, $\Gamma^{(i,k)}(t,0)$ are deterministic and by (3.5) we obtain the estimate of I_1 of the form

$$\begin{aligned}I_1^2=\Big\|\sum_{k=0}^{\infty}\varphi^{(i-k)}(0)\Gamma^{(i,k)}(t,0)\Big\|_{[t]}^2&=\sup_{\tilde t\in[0,t],\,i\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}\varphi^{(i-k)}(0)\Gamma^{(i,k)}(\tilde t,0)\Big|^2\\ &\leqslant\sup_{l\in\mathbb{Z}}|\varphi^{(l)}(0)|^2=\|u\|_{[0]}^2.\end{aligned}$$

Now we estimate I_2.

$$\begin{aligned}I_2^2&=\Big\|\sum_{k=0}^{\infty}\int_0^{\tilde t}F^{(i-k)}(s)\Gamma^{(i,k)}(\tilde t,s)\,\mathrm{d}s\Big\|_{[t]}^2\\ &=\mathbb{E}\Big[\sup_{\tilde t\in[0,t],\,i\in\mathbb{Z}}\Big|\sum_{k=0}^{\infty}\int_0^{\tilde t}F^{(i-k)}(s)\Gamma^{(i,k)}(\tilde t,s)\,\mathrm{d}s\Big|^2\Big].\end{aligned}$$

Using the Cauchy-Schwarz inequality we have

$$I_2^2\leqslant\mathbb{E}\Big[\sup_{\tilde t\in[0,t],\,i\in\mathbb{Z}}\Big|\int_0^{\tilde t}\sum_{k=0}^{\infty}\Gamma^{(i,k)}(\tilde t,s)\,\mathrm{d}s\times\int_0^{\tilde t}\sum_{k=0}^{\infty}(F^{(i-k)}(s))^2\Gamma^{(i,k)}(\tilde t,s)\,\mathrm{d}s\Big|\Big].$$

By (3.5) we obtain

$$
\begin{aligned}
I_2^2 &\leqslant \mathbb{E}\bigg[\sup_{\tilde t\in[0,t],\, i\in\mathbb{Z}}\bigg|\tilde t\int_0^{\tilde t}\sum_{k=0}^{\infty}(F^{(i-k)}(s))^2\Gamma^{(i,k)}(\tilde t,s)\,\mathrm{d}s\bigg|\bigg]\\
&\leqslant \mathbb{E}\bigg[\sup_{\tilde t\in[0,t],\, i\in\mathbb{Z}}\bigg|\tilde t\int_0^{\tilde t}\sum_{k=0}^{\infty}\Gamma^{(i,k)}(\tilde t,s)\sup_{\tilde s\in[0,t],\, l\in\mathbb{Z}}(F^{(l)}(\tilde s))^2\,\mathrm{d}s\bigg|\bigg]\\
&\leqslant \mathbb{E}\bigg[\sup_{\tilde t\in[0,t],}\bigg|\tilde t\int_0^{\tilde t}\sup_{\tilde s\in[0,t],\, l\in\mathbb{Z}}(F^{(l)}(\tilde s))^2\,\mathrm{d}s\bigg|\bigg]\leqslant t\int_0^t\|F\|_{[s]}^2\,\mathrm{d}s.
\end{aligned}
$$

Applying the Doob martingale inequality (see [13]) and the Itô isometry (see [13]), we have

$$
I_3^2=\mathbb{E}\bigg[\sup_{\tilde t\in[0,t]}\bigg|\int_0^{\tilde t}G(s)\,\mathrm{d}B_s\bigg|^2\bigg]\leqslant 4\mathbb{E}\bigg[\bigg|\int_0^{t}G(s)\,\mathrm{d}B_s\bigg|^2\bigg]\leqslant 4\int_0^t\|G\|_{[s]}^2\,\mathrm{d}s.
$$

Hence

$$
\|u^{(i)}\|_t\leqslant\sqrt{\|u\|_{[0]}^2}+\sqrt{t\int_0^t\|F\|_{[s]}^2\,\mathrm{d}s}+\sqrt{4\int_0^t\|G\|_{[s]}^2\,\mathrm{d}s}.
$$

From the fact that $(a+b+c)^2\leqslant 3a^2+3b^2+3c^2$ we obtain the estimate

$$
\|u^{(i)}\|_t^2\leqslant 3\|u\|_{[0]}^2+3t\int_0^t\|F\|_{[s]}^2\,\mathrm{d}s+12\int_0^t\|G\|_{[s]}^2\,\mathrm{d}s,
$$

which completes the proof. □

P r o o f of Theorem 3.5. In order to prove the existence of a solution we define

$$
\begin{aligned}
u_0^{(i)}(t)&=\varphi^{(i)}(t)\quad\text{for } t\in[-r,0],\ i\in\mathbb{Z},\\
u_0^{(i)}(t)&=\varphi(0)\quad\text{for } t\in[0,T],\ i\in\mathbb{Z}
\end{aligned}
$$

and the Picard iterations

$$
\frac{\mathrm{d}u_{m+1}^{(i)}(t)}{\mathrm{d}t}+a^{(i)}(t)\frac{u_{m+1}^{(i)}(t)-u_{m+1}^{(i-1)}(t)}{h}=f(t,ih,(J_hu_m)_{(t,ih)})+g(t,(J_hu_m)_{(t,0)})\dot B_t
$$

for $t\in[0,T]$, $i\in\mathbb{Z}$, $m\in\mathbb{Z}_+$ with the initial condition

$$
u_{m+1}^{(i)}(t)=\varphi^{(i)}(t)\quad\text{for } t\in[-r,0].
$$

Using (3.9), we can prove by induction on m that the sequence $(u_m^{(i)})_m$ is a Cauchy sequence in the Banach space $\mathcal{X}_T$. Hence it is convergent and its limit is a solution to the Cauchy problem (3.6). Let $\Delta u_m^{(i)}(t) = u_{m+1}^{(i)}(t) - u_m^{(i)}(t)$. For $m = 0$ we have

$$\begin{aligned}\frac{d\Delta u_0^{(i)}(t)}{\mathrm{d}t} &+ a^{(i)}(t)\frac{\Delta u_0^{(i)}(t) - \Delta u_0^{(i-1)}(t)}{h}\\ &= f(t, ih, (J_h u_0)_{(t,ih)}) + g(t, (J_h u_0)_{(t,0)})\dot{B}_t, \quad i \in \mathbb{Z},\ t \in [0,T].\end{aligned}$$

From Remark 3.4 we obtain

$$\|\Delta u_0(t)\|_t^2 \leqslant 2t\int_0^t \|(f_h)_0\|_{[s]}^2\,\mathrm{d}s + 8\int_0^t \|g_0\|_{[s]}^2\,\mathrm{d}s,$$

where

$$(f_h)_0 = f(t, ih, (J_h u_0)_{(t,ih)}), \quad g_0 = g(t, (J_h u_0)_{(t,0)}).$$

We shall estimate $\|(f_h)_0\|_{[t]}$ and $\|g_0\|_{[t]}$. By Assumption 2.1 we have

$$\begin{aligned}\|f(t, ih, (J_h u_0)_{(t,ih)})\|_{[t]}^2 &\leqslant 2L_1^2\|u_0\|_t^2 + 2C_1^2,\\ \|g(t, (J_h u_0)_{(t,0)})\|_{[t]}^2 &\leqslant 2L_2^2\|u_0\|_t^2 + 2C_2^2.\end{aligned}$$

Under the inductive assumption for $m \geqslant 0$, we shall show the assertion for $m+1$.

$$\begin{aligned}\frac{\mathrm{d}\Delta u_{m+1}^{(i)}(t)}{\mathrm{d}t} &+ a^{(i)}(t)\frac{\Delta u_{m+1}^{(i)}(t) - \Delta u_{m+1}^{(i-1)}(t)}{h}\\ &= f(t, ih, (J_h u_m)_{(t,ih)}) - f(t, ih, (J_h u_{m-1})_{(t,ih)})\\ &\quad + g(t, (J_h u_m)_{(t,0)})\dot{B}_t - g(t, (J_h u_{m-1})_{(t,0)})\dot{B}_t\end{aligned}$$

for $i \in \mathbb{Z}$, $t \in [0,T]$. Applying Remark 3.4, we obtain

$$\|\Delta u_{m+1}(t)\|_t^2 \leqslant 2t\int_0^t \|(\Delta f_h)_m\|_{[s]}\,\mathrm{d}s + 8\int_0^t \|\Delta g_m\|_{[s]}\,\mathrm{d}s,$$

where

$$\begin{aligned}(\Delta f_h)_m &= f(t, ih, (J_h u_m)_{(t,ih)}) - f(t, ih, (J_h u_{m-1})_{(t,ih)}),\\ \Delta g_m &= g(t, (J_h u_m)_{(t,0)}) - g(t, (J_h u_{m-1})_{(t,0)}).\end{aligned}$$

Hence, by Assumption 2.1 one can show that

$$\|(\Delta f_h)_m\|_{[t]}^2 \leqslant L_1^2\|\Delta u_m^{(i)}\|_t^2, \quad \|(\Delta g)_m\|_{[t]} \leqslant L_2\|\Delta u_m^{(i)}\|_t.$$

Consequently,

$$\|\Delta u_{m+1}^{(i)}\|_t^2 \leqslant (2tL_1^2 + 8L_2^2) \int_0^t \|\Delta u_m^{(i)}\|_s^2 \,\mathrm{d}s.$$

It follows that

$$\|\Delta u_{m+1}^{(i)}\|_t^2 \leqslant \text{const} \cdot \frac{(2tL_1^2 + 8L_2^2)^m}{m!}.$$

Thus, the series of partial sums

$$u_0^{(i)}(t) + \sum_{k=0}^{n-1} [u_{k+1}^{(i)}(t) - u_k^{(i)}(t)] = u_n^{(i)}(t)$$

is convergent on $[0, T]$. Thus, $u_m^{(i)}(t)$ converges to the unique solution $u^{(i)}(t)$ of equation (3.6). □

5. Appendix

For $a^{(i)}(t) \leqslant 0$ the method of lines for (3.2) is of the form

$$\begin{aligned} &\frac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t} - u^{(i)}(t)\frac{a^{(i)}(t)}{h} = -u^{(i+1)}(t)\frac{a^{(i)}(t)}{h} + \psi^{(i)}(t) \quad \text{for } t \in [0, T],\ i \in \mathbb{Z}, \\ &u^{(i)}(t) = \varphi^{(i)}(t) \quad \text{for } t \in [-r, 0],\ i \in \mathbb{Z}. \end{aligned} \tag{5.1}$$

Any solution $u = (u^{(i)})_{i\in\mathbb{Z}}$ of (5.1) such that $u^{(i)} \in \mathcal{X}_T$ has the integral representation (3.3), where $\Gamma^{(i,k)}(t, s) \geqslant 0$ for $i \in \mathbb{Z}$ and $0 \leqslant s \leqslant t \leqslant T$ is defined by the recurrence

$$\begin{aligned} \Gamma^{(i,0)}(t, s) &= \mathrm{e}^{\int_s^t (a^{(i)}(w)/h)\,\mathrm{d}w}, \\ \Gamma^{(i,k)}(t, s) &= -\int_s^t \mathrm{e}^{\int_{s_1}^t (a^{(i)}(w)/h)\,\mathrm{d}w}\, \frac{a^{(i)}(s_1)}{h} \Gamma^{(i-1,k-1)}(s_1, s)\,\mathrm{d}s_1. \end{aligned}$$

In the case when the coefficient $a^{(i)}(t)$ changes sign we cannot employ the forward and backward finite difference quotients as it leads to the discontinuity of coefficients. Instead, we employ a viscosity term

$$\frac{h}{2}|a|\frac{\partial^2 u}{\partial x^2}$$

to get the following form of the method of lines for (3.2):

$$\begin{aligned} &\frac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t} + a^{(i)}(t)\frac{u^{(i+1)}(t) - u^{(i-1)}(t)}{2h} \\ &\quad - \frac{h}{2}|a^{(i)}(t)|\frac{u^{(i+1)}(t) - 2u^{(i)}(t) + u^{(i-1)}(t)}{h^2} = \psi^{(i)}(t) \quad \text{for } t \in [0, T],\ i \in \mathbb{Z}, \\ &u^{(i)}(t) = \varphi^{(i)}(t) \quad \text{for } t \in [-r, 0],\ i \in \mathbb{Z}. \end{aligned}$$

Hence

$$\begin{aligned}(5.2)\qquad \frac{\mathrm{d}u^{(i)}(t)}{\mathrm{d}t}+\frac{|a^{(i)}(t)|}{h}u^{(i)}(t) &= \frac{|a^{(i)}(t)|-a^{(i)}(t)}{2h}u^{(i+1)}(t)\\ &\quad+\frac{|a^{(i)}(t)|+a^{(i)}(t)}{2h}u^{(i-1)}(t)+\psi^{(i)}(t)\quad \text{for } t\in[0,T],\ i\in\mathbb{Z},\\ u^{(i)}(t) &= \varphi^{(i)}(t)\quad \text{for } t\in[-r,0],\ i\in\mathbb{Z}.\end{aligned}$$

Any solution $u=(u^{(i)})_{i\in\mathbb{Z}}$ of (5.2) such that $u^{(i)}\in\mathcal{X}_T$ has the integral representation (3.3), where

$$\begin{aligned}\Gamma^{(i,0)}(t,s) &= \mathrm{e}^{-\int_s^t(|a^{(i)}(w)|/h)\,\mathrm{d}w},\\ \Gamma^{(i,k)}(t,s) &= \int_s^t \mathrm{e}^{-\int_{s_1}^t(|a^{(i)}(w)|/h)\,\mathrm{d}w}\Big(\frac{|a^{(i)}(s_1)|-a^{(i)}(s_1)}{2h}\Gamma^{(i+1,k-1)}(s_1,s)\\ &\quad+\frac{|a^{(i)}(s_1)|+a^{(i)}(s_1)}{2h}\Gamma^{(i-1,k-1)}(s_1,s)\Big)\,\mathrm{d}s_1.\end{aligned}$$

References

[1] *A. Ashyralyev, M. E. Koksal, R. P. Agarwal*: A difference scheme for Cauchy problem for the hyperbolic equation with self-adjoint operator. Math. Comput. Modelling *52* (2010), 409–424. zbl MR doi

[2] *A. Ashyralyev, H. A. Yurtsever*: The stability of difference schemes of second-order of accuracy for hyperbolic-parabolic equations. Comput. Math. Appl. *52* (2006), 259–268. zbl MR doi

[3] *D. Bahuguna, J. Dabas, R. K. Shukla*: Method of lines to hyperbolic integro-differential equations in $\mathbb{R}^n$. Nonlinear Dyn. Syst. Theory *8* (2008), 317–328. zbl MR

[4] *A. Bátkai, P. Csomós, G. Nickel*: Operator splittings and spatial approximations for evolution equations. J. Evol. Equ. *9* (2009), 613–636. zbl MR doi

[5] *M. A. Berger, V. J. Mizel*: Volterra equations with Ito integrals I. J. Integral Equations *2* (1980), 187–245. zbl MR

[6] *G. Da Prato, J. Zabczyk*: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications 44, Cambridge University Press, Cambridge, 1992. zbl MR doi

[7] *L. Debbi, M. Dozzi*: On a space discretization scheme for the fractional stochastic heat equations. Avaible at `https://arxiv.org/abs/1102.4689v1`.

[8] *A. Friedman*: Stochastic Differential Equations and Applications, Vol. 1. Probability and Mathematical Statistics 28, Academic Press, New York, 1975. zbl MR

[9] *T. Funaki*: Construction of a solution of random transport equation with boundary condition. J. Math. Soc. Japan *31* (1979), 719–744. zbl MR doi

[10] *H. Holden, B. Øksendal, J. Ubøe, T. Zhang*: Stochastic Partial Differential Equations. A Modeling, White Noise Functional Approach. Probability and Its Applications, Birkhäuser, Basel, 1996. zbl MR doi

[11] *Z. Kamont*: Hyperbolic Functional Differential Inequalities and Applications. Mathematics and Its Applications 486, Kluwer Academic Publishers, Dordrecht, 1999. zbl MR doi

[12] *J. U. Kim*: On the Cauchy problem for the transport equation with random noise. J. Funct. Anal. *259* (2010), 3328–3359. zbl MR doi

[13] *F. C. Klebaner*: Introduction to Stochastic Calculus with Applications. Imperial College Press, London, 2005. zbl MR doi

[14] *H.-O. Kreiss, G. Scherer*: Method of lines for hyperbolic differential equations. SIAM J. Numer. Anal. *29* (1992), 640–646. zbl MR doi
[15] *H. Leszczyński*: Quasi-linearisation methods for a nonlinear heat equation with functional dependence. Georgian Math. J. *7* (2000), 97–116. zbl MR
[16] *H. Leszczyński*: Comparison ODE theorems related to the method of lines. J. Appl. Anal. *17* (2011), 137–154. zbl MR doi
[17] *S. McDonald*: Finite difference approximation for linear stochastic partial differential equation with method of lines. MPRA Paper No. 3983. Avaible at `http://mpra.ub.uni-muenchen.de/3983` (2006).
[19] *L. Quer-Sardanyons, M. Sanz-Solé*: Space semi-discretisations for a stochastic wave equation. Potential Anal. *24* (2006), 303–332. zbl MR doi
[20] *S. C. Reddy, L. N. Trefethen*: Stability of the method of lines. Numer. Math. *62* (1992), 235–267. zbl MR doi
[21] *A. Rößler, M. Seaïd, M. Zahri*: Method of lines for stochastic boundary-value problems with additive noise. Appl. Math. Comput. *199* (2008), 301–314. zbl MR doi
[22] *K. K. Sharma, P. Singh*: Hyperbolic partial differential-difference equation in the mathematical modeling of neuronal firing and its numerical solution. Appl. Math. Comput. *201* (2008), 229–238. zbl MR doi
[23] *H. Yoo*: Semi-discretization of stochastic partial differential equations on $\mathbb{R}^1$ by a finite-difference method. Math. Comput. *69* (2000), 653–666. zbl MR doi

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