

## PROPER CONNECTION NUMBER OF BIPARTITE GRAPHS

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*Abstract.* An edge-colored graph  $G$  is proper connected if every pair of vertices is connected by a proper path. The proper connection number of a connected graph  $G$ , denoted by  $pc(G)$ , is the smallest number of colors that are needed to color the edges of  $G$  in order to make it proper connected. In this paper, we obtain the sharp upper bound for  $pc(G)$  of a general bipartite graph  $G$  and a series of extremal graphs. Additionally, we give a proper 2-coloring for a connected bipartite graph  $G$  having  $\delta(G) \geq 2$  and a dominating cycle or a dominating complete bipartite subgraph, which implies  $pc(G) = 2$ . Furthermore, we get that the proper connection number of connected bipartite graphs with  $\delta \geq 2$  and  $\text{diam}(G) \leq 4$  is two.

*Keywords:* proper coloring; proper connection number; bipartite graph; dominating set

*MSC 2010:* 05C15, 05C69, 05C75

## 1. INTRODUCTION

All graphs in this paper are finite, connected and simple. We follow the terminology and notation of Bondy and Murty [2]. An  $k$ -edge-coloring of a graph is an assignment of  $k$  colors to the edges of  $G$ . An edge coloring is *proper* if adjacent edges receive distinct colors. The minimum number of colors needed in a proper edge coloring of the graph  $G$  is referred to as *the edge chromatic number* of  $G$  and denoted by  $\chi'(G)$ . Except the classical vertex coloring and edge coloring, there are many kinds of colorings being studied, such as list coloring, star coloring and acyclic coloring. In addition, rainbow connection and rainbow vertex-connection

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are the new ones. For more details we refer to a survey paper [5] and a book [6].

Inspired by proper coloring and rainbow coloring in graphs, Borozan et al. [3] introduced the concept of proper coloring of a graph  $G$ . Let  $G$  be an edge-colored graph. A path  $P$  in  $G$  is called a *proper path* if no two adjacent edges of  $P$  are colored the same. An edge-coloring  $c$  is a *proper-path coloring* of a connected graph  $G$  if every pair of distinct vertices  $u, v$  of  $G$  are connected by a proper  $(u, v)$ -path in  $G$ . If  $k$  colors are used, then  $c$  is referred to as a  *$k$ -proper coloring*. The minimum number of colors needed to produce a proper coloring of  $G$  is called the *proper connection number* of  $G$ , denoted by  $\text{pc}(G)$ . More generally, a graph  $G$  is said to be  *$k$ -proper connected* if any two vertices are connected by  $k$  internally pairwise vertex-disjoint proper paths. The  *$k$ -proper connection number* of a  $k$ -connected graph  $G$ , denoted by  $\text{pc}_k(G)$ , is the smallest number of colors that are needed in order to make  $G$   $k$ -proper connected. From the definition, it is easy to get that  $1 \leq \text{pc}(G) \leq \min\{\chi'(G), \text{rc}(G)\} \leq m$ , where  $m$  is the number of edges of  $G$ . Recently, a lot of results have been obtained with respect to several aspects, such as connectivity, minimum degree, complements, operations on graphs and so on. For details we refer to [1], [3] and a dynamic survey paper [4].

Borozan et al. in [3] showed that the proper connection number of a complete bipartite graph is two but the proper connection number of a general bipartite graph is still unknown. Also in [3], Borozan et al. proved that  $\text{pc}(G) = 2$  for a 2-connected bipartite graph  $G$  by induction. However, how to give a proper 2-coloring for a 2-connected bipartite graph? These two problems are interesting and meaningful. We focus on these two problems in this paper.

This paper is organized as follows. In Section 2, we introduce some basic definitions and useful lemmas on the proper connection number of a graph. In Section 3, the sharp upper bound of  $\text{pc}(G)$  for a general bipartite graph  $G$  together with a series of extremal graphs are given. Section 4 gives a proper 2-coloring for the connected bipartite graph  $G$  such that  $\delta(G) \geq 2$  and  $G$  has a dominating cycle or a dominating complete bipartite subgraph. Furthermore, the proper connection number of a connected bipartite graph  $G$  with  $\delta \geq 2$  and  $\text{diam}(G) \leq 4$  are obtained.

## 2. PRELIMINARIES

In this section, we introduce some definitions and present several useful lemmas about the proper coloring of graphs. We begin with some basic conceptions.

**Definition 2.1.** Given a colored path  $P = v_1v_2 \dots v_{s-1}v_s$  between two vertices  $v_1$  and  $v_s$ , we denote by  $\text{star}(P)$  the color of the first edge in the path, i.e.  $c(v_1v_2)$ , and by  $\text{end}(P)$  the last color, i.e.  $c(v_{s-1}v_s)$ .

**Definition 2.2.** Let  $G = (V, E)$  be a connected graph. A  $(v, S)$ -path, where  $v \in V$ ,  $S \subseteq V$ , is a path which starts at  $v$  and ends at one vertex in  $S$ . An  $(S_1, S_2)$ -path, where  $S_1, S_2 \subseteq V$ , is a path which starts at one vertex in  $S_1$  and ends at one vertex in  $S_2$ .

**Definition 2.3.** Let  $G = (V, E)$  be a connected graph. The *distance between two vertices*  $u$  and  $v$  in  $G$ , denoted by  $d(u, v)$ , is the length of a shortest path between them in  $G$ . The *diameter* of  $G$  is the maximum distance between two vertices of  $G$ . The *distance between a vertex  $v$  and a set  $S \subseteq V$*  is  $d(v, S) := \min_{x \in S} d(v, x)$ . The  *$k$ -step neighborhood* of a set  $S \subseteq V$  is  $N^k(S) := \{x \in V : d(x, S) = k\}$ ,  $k \in \{0, 1, 2, \dots\}$ . The *degree* of a vertex  $v$  is  $\deg(v) := |N^1(v)|$ . The *minimum degree* of  $G$  is  $\delta(G) := \min_{x \in V} \deg(x)$ . A vertex is called *pendant* if its degree is 1 and *isolated* if its degree is 0. We may use  $N^k(v)$  in place of  $N^k(\{v\})$ .

**Definition 2.4.** Given a graph  $G$ , a set  $D \subseteq V(G)$  is called a  *$k$ -step dominating set* of  $G$  if every vertex in  $G$  is at a distance at most  $k$  from  $D$ . Further, if  $D$  induces a connected subgraph of  $G$ , then it is called a *connected  $k$ -step dominating set* of  $G$ .

**Definition 2.5.** A two-step dominating set  $D$  of vertices in a graph  $G$  is called a *two-way two-step dominating set* if

- (i) every pendant vertex of  $G$  is included in  $D$ ,
- (ii) every vertex in  $N^2(D)$  has at least two neighbours in  $N^1(D)$ .

Further, if  $G[D]$  is connected,  $D$  is called a *connected two-way two-step dominating set* of  $G$ .

For other notation and terminology, we refer to [2]. Next we state some known results on the proper coloring, which will be useful in the sequel.

**Lemma 2.1** ([3]). *If  $\text{pc}(G) = 2$ , then  $\text{pc}(G \cup v) = 2$  as long as  $d(v) \geq 2$ .*

As the general case of Lemma 2.1, we give the following proposition.

**Proposition 2.1.** *If  $\text{pc}(G) = k$ ,  $k \geq 2$ , then  $\text{pc}(G \cup v) \leq k$  as long as  $d(v) \geq 2$ .*

**Proof.** Let  $u, w$  be two neighbours of  $v$  in  $G$ . Assume that  $c$  is a proper  $k$ -coloring of  $G$ , then there is a proper  $(u, w)$ -path  $P$  in  $G$ . Now color the edges  $uv$  and  $wv$  with colors in the color set of  $c$  such that  $c(uv) \neq \text{star}(P)$  and  $c(wv) \neq \text{end}(P)$ . Now we will check that this is a proper  $k$ -coloring of the new graph  $G \cup v$ . If  $uw$  is an edge of  $G$ , then  $c(uv) \neq c(uw)$  and  $c(wv) \neq c(uw)$ . Since every vertex has a proper path to  $u$ , every vertex has a proper path to  $v$ . And for the case that  $uw$  is not an edge of  $G$ , every vertex has a proper path to  $v$  through either  $u$  or  $w$  since every vertex has a proper path to any inner vertex of  $P$ . This completes the proof.  $\square$

**Remark 2.1.** The value of  $\text{pc}(G) - \text{pc}(G \cup v)$  can be arbitrarily large. We construct a new graph  $S'_k$  as follows. Take a star  $S_k$ ,  $k \geq 3$ , as the base graph and the vertex  $v_0$  as the center vertex of  $S_k$ . Add a new vertex  $v$  to  $S_k$  so that  $v$  is adjacent to  $k$  (if  $k$  is even) or  $k-1$  (if  $k$  is odd) vertices in  $V(S_k) - \{v_0\}$ . Since  $|N(v)|$  is even, we can divide  $N(v)$  into two parts  $A$  and  $B$  so that  $|A| = |B| = |N(v)|/2$  and  $A \cap B = \emptyset$ . For the case when  $k$  is odd we denote the only vertex in  $V(S_k) - \{v_0\} - N(v)$  by  $u$ . Now we give a 2-proper coloring of the new graph  $S'_k$  as follows. Color all  $(v, A)$ -edges and  $(v_0, B)$ -edges by 1 and all  $(v, B)$ -edges and  $(v_0, A)$ -edges by 2. For the case when  $k$  is even, we have colored all edges in  $S'_k$  while for the case when  $k$  is odd, color the edge  $uv_0$  with 1 or 2. One can check that this is a 2-proper coloring of  $S'_k$ . Then  $\text{pc}(S_k) - \text{pc}(S'_k) = k - 2$  and this difference can be arbitrarily large if  $k$  is large enough.

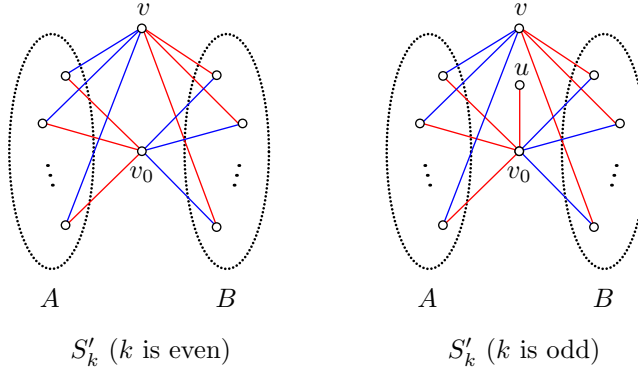


Figure 1. 2-proper colored graphs for Remark 2.1.

### 3. PROPER CONNECTION NUMBER OF BIPARTITE GRAPHS

In this section, we mainly consider the proper connection number of a general bipartite graph. First, we give some known results on the proper connection number of the complete bipartite graphs and the 2-connected (2-edge-connected) bipartite graphs.

**Lemma 3.1** ([3]). *Let  $G = K_{m,n}$ ,  $m \geq n \geq 2k$  for  $k \geq 1$ . Then  $\text{pc}_k(G) = 2$ .*

**Lemma 3.2** ([3]). *Let  $G$  be a graph. If  $G$  is bipartite and 2-connected (2-edge-connected), then  $\text{pc}(G) = 2$  and there exists a 2-coloring of  $G$  that makes it properly connected with the following strong property. For any pair of vertices  $v, w$  there exist two paths  $P_1$  and  $P_2$  between them (not necessarily disjoint) such that  $\text{star}(P_1) \neq \text{star}(P_2)$  and  $\text{end}(P_1) \neq \text{end}(P_2)$ .*

The above lemma implies that it suffices to consider a connected bipartite graph  $G$  containing bridges. The following theorem gives a sharp upper bound for  $\text{pc}(G)$ .

**Theorem 3.1.** *Let  $G$  be a connected bipartite graph containing bridges. If  $b$  is the maximum number of bridges incident with a single vertex in  $G$ , then  $\text{pc}(G) \leq b + 2$  and this upper bound is sharp.*

**Proof.** Let  $G$  be a connected bipartite graph containing bridges. The block decomposition of  $G$  includes its isolated vertices, bridges and maximal 2-connected subgraphs. In order to prove  $\text{pc}(G) \leq b + 2$ , we only need to give a  $(b + 2)$ -proper coloring  $c$  of  $G$ . The coloring  $c$  is defined as follows. For the bridges which are incident to a single vertex  $v$  we color each bridge incident with  $v$  by a distinct fresh color in  $[b] = \{1, 2, \dots, b\}$ . Since  $b$  is the maximum number of bridges incident with a single vertex, it is enough to color all bridges in  $G$  using  $b$  distinct colors. And for blocks which are maximal 2-connected subgraphs of  $G$  we give each of them a  $\{b + 1, b + 2\}$ -proper coloring having strong property as stated in Lemma 3.2. One can check that the above coloring  $c$  is a  $(b + 2)$ -proper coloring of  $G$ .

Now we will show that this upper bound is sharp. For the case when  $b = 1$ , we find the graph  $G_1$  (depicted in Figure 2) with  $\text{pc}(G_1) = 3$ , which implies the sharpness of the upper bound. Since  $G_1$  is not complete, then  $\text{pc}(G_1) \geq 2$ . Suppose that 2 colors are enough to make  $G_1$  proper connected. Note that there are only paths of length 2 or 4 between  $v_1$  and  $v_2$ ,  $v_1$  and  $v_3$ , also  $v_2$  and  $v_3$ . Hence, for any proper coloring  $c_1$  of  $G_1$ ,  $c_1(u_1v_1)$ ,  $c_1(u_2v_2)$  and  $c_1(u_3v_3)$  are pairwise different, a contradiction. So  $\text{pc}(G_1) = 3$ .

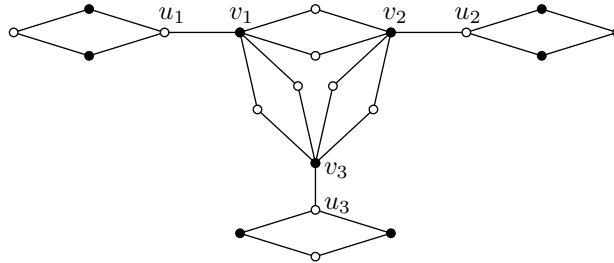


Figure 2. The graph  $G_1$ .

Next we will show that this upper bound is sharp for the case when  $b \geq 2$ . As depicted in Figure 3, the graph  $G_b$  has a cycle  $C$  of length 6. And each vertex of  $C$  is adjacent to  $b$  bridges, which lead to nontrivial blocks. Now we will show the sharpness of this upper bound by proving that  $\text{pc}(G_b) = b + 2$ .

*Claim 1.* The sharp upper bound of  $\text{pc}(G)$  is greater than  $b$ .

**Proof of Claim 1.** We will prove this claim by showing that  $b$  colors are not enough to make  $G_b$  (see Figure 3) proper connected. Assume that there is a  $[b] = \{1, 2, \dots, b\}$ -proper coloring  $c'$  for  $G_b$ . Obviously, the  $b$  bridges which are adjacent to  $u_i$  must be assigned distinct colors and without loss of generality, we set  $c'(u_1u_{1i}) = c'(u_2u_{2i}) = i$ ,  $1 \leq i \leq b$ . In addition,  $c'(u_1u_6) \neq c'(u_1u_2)$ . Since  $c'(u_1u_6), c'(u_1u_2) \in [b]$  and if  $c'(u_1u_6) = c'(u_1u_2)$ , there is a vertex  $u_{1k}$  such that  $c'(u_1u_{1k}) = c'(u_1u_6) = c'(u_1u_2)$ , which contradicts  $c'$  being a proper coloring of  $G_b$ . So we can assume that  $c'(u_1u_6) = 1$  and  $c'(u_1u_2) = 2$ . But there is no  $(u_{11}, u_{22})$ -proper path under the coloring  $c'$ , a contradiction.

The above claim implies that the sharp upper bound of  $\text{pc}(G)$  is either  $b + 1$  or  $b + 2$ . Suppose that  $G_b$  (see Figure 3) has a  $[b + 1]$ -proper coloring  $c_b$  and assume that  $c_b(u_1u_{1i}) = i$ ,  $1 \leq i \leq b$ .

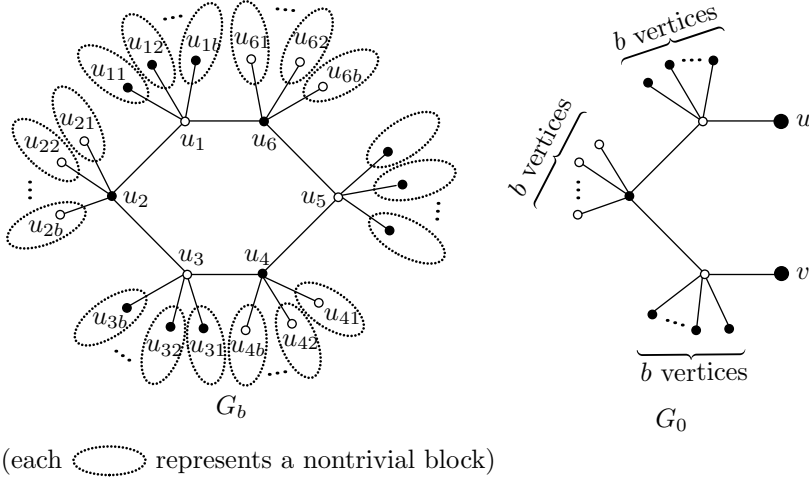


Figure 3. Graphs for the proof of Theorem 3.1.

**Claim 2.** Under the proper coloring  $c_b$ , any two adjacent edges of  $C := u_1u_2u_3u_4u_5u_6u_1$  are assigned different colors.

**Proof of Claim 2.** Suppose there are two adjacent edges on  $C$  with the same color. Without loss of generality, we can assume that  $c_b(u_1u_2) = c_b(u_1u_6)$ . Then  $c_b(u_1u_2) = c_b(u_1u_6) = b + 1$ . If  $b + 1 \notin \{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\}$ , then  $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\} = [b]$ . Since the path  $u_2u_1u_6$  is not proper,  $u_{2i}u_2u_3u_{3i}$ ,  $1 \leq i \leq b$ , must be proper and so  $c_b(u_2u_3) = b + 1$  and  $\{c_b(u_3u_{31}), c_b(u_3u_{32}), \dots, c_b(u_3u_{3b})\} = [b]$ . Additionally,  $u_{3i}u_3u_4$ ,  $1 \leq i \leq b$ , must be proper since  $c_b(u_2u_3) = b + 1 = c_b(u_1u_2)$ . So  $c_b(u_3u_4) = b + 1$  but both  $u_{2i}u_2u_1u_6$  and  $u_{2i}u_2u_3u_4$ ,  $1 \leq i \leq b$ , are not proper, which contradicts  $c_b$  being a proper coloring of  $G_b$ . Hence,  $b + 1 \in \{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\}$  and one of  $[b]$

does not appear in  $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\}$ . Without loss of generality, we assume  $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\} = [b+1] \setminus \{1\}$ . Since  $u_2u_1u_6$  is not proper,  $u_{2i}u_2u_3u_{3i}$ ,  $1 \leq i \leq b$ , must be proper and so  $c_b(u_2u_3) = 1$  and  $\{c_b(u_3u_{31}), c_b(u_3u_{32}), \dots, c_b(u_3u_{3b})\} = [b+1] \setminus \{1\}$ . In addition,  $u_{3i}u_3u_4$ ,  $1 \leq i \leq b$ , must be proper since  $u_3u_2u_1u_6$  is not. So  $c_b(u_3u_4) = 1$  but both  $u_{2i}u_2u_1u_6$  and  $u_{2i}u_2u_3u_4$ ,  $1 \leq i \leq b$  are not proper, which contradicts  $c_b$  being a proper coloring of  $G_b$ . Thus, the result of Claim 2 is obtained.

From Claim 2 we know that for every vertex  $u_j \in V(C)$  there is a bridge  $u_ju_{jk}$  such that either  $c_b(u_ju_{jk}) = c_b(u_ju_{j+1}) \pmod{6}$  or  $c_b(u_ju_{jk}) = c_b(u_ju_{j-1}) \pmod{6}$ . Let us assume  $c_b(u_1u_6) = c_b(u_1u_{11}) = 1$ . It follows that the paths  $u_{11}u_1u_2u_3u_{3i}$ ,  $1 \leq i \leq b$ , and  $u_{11}u_1u_2u_3u_4$  must be proper and  $c_b(u_1u_2) \in [b+1] \setminus \{1\}$ .

If  $c_b(u_1u_2) = 2$  (the cases  $c_b(u_1u_2) = 3, \dots, b$  are all similar), then  $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\} = [b+1] \setminus \{2\}$  since  $u_{11}u_1u_6$  is not a proper path and  $u_{11}u_1u_2u_{2i}$ ,  $1 \leq i \leq b$ , must be proper. Additionally, since  $u_2u_1u_{12}$  is not proper, then  $u_{2i}u_2u_3$ ,  $1 \leq i \leq b$ , must be proper and so  $c_b(u_2u_3) = 2$ . However, this contradicts to the fact that the path  $u_{11}u_1u_2u_3$  must be proper. If  $c_b(u_1u_2) = b+1$ , then  $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\} = [b]$  since  $u_{11}u_1u_6$  is not a proper path and  $u_{11}u_1u_2u_{2i}$ ,  $1 \leq i \leq b$ , must be proper. Take  $c_b(u_2u_3) = 1$  into consideration and for  $c_b(u_2u_3) = 2, \dots, b$ , the analyses are similar. Since  $u_{11}u_1u_2u_3u_{3i}$ ,  $1 \leq i \leq b$ , are all proper paths, then  $\{c_b(u_3u_{31}), c_b(u_3u_{32}), \dots, c_b(u_3u_{3b})\} = [b+1] \setminus \{1\}$ . Without loss of generality, we can assume  $c_b(u_2u_{21}) = 1$  and then  $u_{21}u_2u_3$  is not proper. And this implies that  $u_{3i}u_3u_4$  is proper and  $c_b(u_3u_4) = 1$ . However, this contradicts  $u_{11}u_1u_2u_3u_4$  being a proper path. As a result, the graph  $G_b$  has no proper  $(b+1)$ -coloring and  $\text{pc}(G_b) = b+2$ .  $\square$

From the proof of Theorem 3.1, we can find that the cycle  $C$  in  $G_b$  can be any even cycle. Furthermore, the structure of  $G_0$  (see Figure 3) leads to the sharpness of the above upper bound. With similar analysis for Theorem 3.1, we can obtain the following result.

**Theorem 3.2.** *Let  $G$  be a connected bipartite graph containing bridges and  $b \geq 2$  be an integer. If  $G$  contains  $G_0$  as its induced subgraph and all paths in  $G$  joining any pair of vertices of  $V(G_0)$  except  $\{u, v\}$  appear in  $G_0$ , then  $\text{pc}(G) = b+2$ .*

Additionally, Andrews et al. in [1] gave a lower bound for any nontrivial connected graphs containing bridges, which is stated as follows.

**Theorem 3.3** ([1]). *Let  $G$  be a nontrivial connected graph containing bridges. If  $b$  is the maximum number of bridges incident with a single vertex in  $G$ , then  $\text{pc}(G) \geq b$ .*

Together with the results in Theorem 3.1 and Theorem 3.3, we can directly obtain the following corollary.

**Corollary 3.1.** *Let  $G$  be a connected bipartite graph containing bridges. If  $b$  is the maximum number of bridges incident with a single vertex in  $G$ , then  $\text{pc}(G) \in \{b, b+1, b+2\}$ .*

#### 4. PROPER COLORINGS FOR BIPARTITE GRAPHS

In [7], we found upper bounds of the proper connection number  $\text{pc}(G)$  with the help of two-way dominating sets or two-way two-step dominating sets of a graph  $G$ . This implies that the dominating set is a useful tool to help us to find a proper coloring and determine the proper connection number of a connected graph.

**Theorem 4.1.** *Let  $G = (X, Y)$  be a connected bipartite graph such that  $\delta(G) \geq 2$  and  $G$  has a dominating cycle or a dominating complete bipartite subgraph. Then  $\text{pc}(G) = 2$ .*

**Proof.** In order to prove the theorem, we will distinguish two cases according to different dominating subgraphs of  $G$ .

*Case 1.* Suppose  $C_{2k}$ ,  $k \geq 2$ , be a dominating cycle of  $G$ .

We claim that  $G$  is a 2-connected graph. If not, there exists a cut-vertex, say  $x_0 \in X$ . Then the graph  $G - x_0$  can be composed by at least two bipartite subgraphs  $G_1 = (X_1, Y_1)$ ,  $G_2 = (X_2, Y_2)$ , where  $X_1, X_2 \in X$  and  $Y_1, Y_2 \in Y$ . Since  $G$  has a dominating cycle,  $x_0 \in X \cap V(C_{2k})$  and one of  $X_1$  and  $X_2$  must be empty. Without loss of generality, we assume that  $X_2 = \emptyset$ . Then all the vertices in  $Y_2$  are only adjacent  $x_0$ , which contradicts  $\delta(G) \geq 2$ . Then the graph  $G$  is a 2-connected bipartite graph, and  $\text{pc}(G) = 2$ .

Now we give a proper 2-coloring for a spanning subgraph of  $G$ . Let  $C_{2a}$  be the dominating even cycle of  $G$  and color the edges of  $C_{2a}$  by 1, 2 alternately, so that 121212... For any vertex  $v \in V(G) \setminus V(C_{2a})$  we have  $|N(v) \cap V(C_{2a})| \geq 2$  or  $|N(v) \cap V(C_{2a})| = 1$ . If  $|N(v) \cap V(C_{2a})| \geq 2$ , then take two neighbours of  $v$ , say  $u, w \in C_{2a}$ , and color  $uv$  and  $wv$  so that  $uvwC_{2a}u$  is a proper 2-coloring cycle, where the segment  $wC_{2a}u$  has a greater length than the remaining segment of  $C_{2a}$ . If  $|N(v) \cap V(C_{2a})| = 1$ ,  $v$  must on an open ear of length 3, denoted by  $x_i vv' x_j$  ( $x_i, x_j \in C_{2a}$  and  $v, v' \notin C_{2a}$ ). Color this open ear by 1, 2 alternately as 121212..., so that  $x_i vv' x_j C_{2a} x_i$  is a proper 2-coloring cycle, where the segment  $x_j C_{2a} x_i$  has a smaller length than the remaining segment of  $C_{2a}$ . One can check this is indeed a proper 2-coloring for a spanning subgraph of  $G$ .

*Case 2.* Let  $D = (X, Y)$  be a dominating complete bipartite subgraph of  $G$ .

*Subcase 2.1.*  $|X| = 1$  or  $|Y| = 1$ . Then  $D$  is a star. Without loss of generality, assume the vertex  $v_0$  is the center vertex of  $D$ . Let  $U = N_G^1(v_0)$  and  $V = V(G) \setminus U$ . Obviously,  $v_0 \in V$  and  $N_G^2(v_0) = V \setminus \{v_0\}$ . Since  $G$  is bipartite and  $\delta(G) \geq 2$ ,  $\{v_0\}$  is a connected two-way two-step dominating set of  $G$  and for all  $v \in N_G^1(v_0)$ ,  $N(v) \cap N_G^2(v_0) \neq \emptyset$ . The proof of Theorem 3.1 in [7] implies a proper 2-coloring of a spanning subgraph of  $G$  and so  $\text{pc}(G) = 2$ .

*Subcase 2.2.*  $|X| \geq 2$  or  $|Y| \geq 2$ . The dominating complete bipartite graph  $D$  must be an induced subgraph of  $G$ , since  $G$  is bipartite. Let  $X = \{u_1, u_2, \dots, u_{|X|}\}$ ,  $Y = \{v_1, v_2, \dots, v_{|Y|}\}$  be the bipartition of  $D$ . Set  $D' = V(D) \cup \{w : |N(w) \cap V(D)| \geq 2, w \in V(G) \setminus V(D)\}$ . Then we have  $\text{pc}(G[D']) = 2$  by Lemma 3.1 and Lemma 2.1. Now we give a proper  $\{\text{dark}(1), \text{light}(2)\}$ -coloring  $c$  for  $G[D']$  (see Figure 4) as follows. Set  $c(u_1 v_1) = 1$ ,  $c(u_i v_j) = 1$  and  $c(u_1 v_j) = c(v_1 u_i) = 2$  for all  $2 \leq i \leq |X|$ ,  $2 \leq j \leq |Y|$ . For any vertex  $w \in D' \setminus V(D)$  it is obvious that  $N(w) \cap V(D) \subseteq X$  or  $N(w) \cap V(D) \subseteq Y$  because  $G$  is a bipartite graph. If  $u_1 w \in G$ , then set  $c(u_1 w) = 1$ . Take some  $u_k \in N(w) \cap V(D)$ ,  $2 \leq k \leq |X|$  and set  $c(u_k w) = 2$ . Similarly, if  $v_1 w \in G$ , then set  $c(v_1 w) = 1$ . Take some  $v_l \in N(w) \cap V(D)$ ,  $2 \leq l \leq |Y|$  and set  $c(v_l w) = 2$ . If  $u_1 w \notin G$  and  $v_1 w \notin G$ , then there are  $u_j, u_k \in N(w)$ ,  $2 \leq j \neq k \leq |X|$  or  $v_l, v_i \in N(w)$ ,  $2 \leq l \neq i \leq |Y|$ . In this case, if  $u_j, u_k \in N(w)$ , then color the edges  $u_j w$  and  $u_k w$  so that  $\{c(u_j w), c(u_k w)\} = \{1, 2\}$ ; if  $v_l, v_i \in N(w)$ , then color the edges  $v_l w$  and  $v_i w$  so that  $\{c(v_l w), c(v_i w)\} = \{1, 2\}$ .

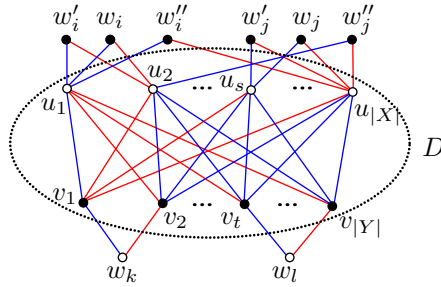


Figure 4. Proper 2-coloring of a subgraph of  $G[D']$ .

Let  $S = V(G) \setminus D' = \{w : |N(w) \cap V(D)| = 1, w \in V(G) \setminus V(D)\}$ . For any  $w \in S$ , if  $w \in N(u_i)$  for some  $i$ ,  $1 \leq i \leq |X|$ , then there is either  $w' \in N(w)$  such that  $w' \in S$  and  $N(w') \cap V(D) \subseteq Y$  (i.e., there is a vertex  $v_j$ ,  $1 \leq j \leq |Y|$  such that  $w' v_j \in E(G)$ ) or  $w' \in D' \setminus V(D)$  and  $N(w') \cap V(D) \subseteq Y$ , since  $G$  is a bipartite graph and  $D$  is a dominating complete bipartite subgraph. Similarly, if  $w \in N(v_k)$  for some  $k$ ,  $1 \leq k \leq |Y|$ , then there is  $w' \in N(w)$  such that  $w' \in S$  and  $N(w') \cap V(D) \subseteq X$  (i.e., there is either a vertex  $u_l$ ,  $1 \leq l \leq |X|$  such that

$w'u_l \in E(G)$  or  $w' \in D' \setminus V(D)$ ) and  $N(w') \cap V(D) \subseteq X$ . For the vertex  $w \in S$  we call the vertex  $w$  of type I in  $S$  if  $N(w) \cap (D' \setminus V(D)) = \emptyset$  (i.e.,  $N(w) \setminus V(D) \subseteq S$ ), the rest elements of  $S$  are called type II ones.

Now we extend the coloring  $c$  of  $G[D']$  to  $G$ . For any vertex  $w \in S$  of type I, there is a 4-cycle  $wu_iv_jw'w$ ,  $1 \leq i \leq |X|$ ,  $1 \leq j \leq |Y|$  in  $G$ . Set  $c(ww') = c(u_iv_j)$  and  $c(wu_i) = c(w'v_j) \in \{1, 2\} \setminus c(u_iv_j)$ . This coloring implies that each vertex  $w \in S$  of type I is contained in a proper 4-cycle except  $w_0 \in S$  in shapes (a), (b), (c), (d), which is depicted in Figure 5. We call all these vertices singular vertices.

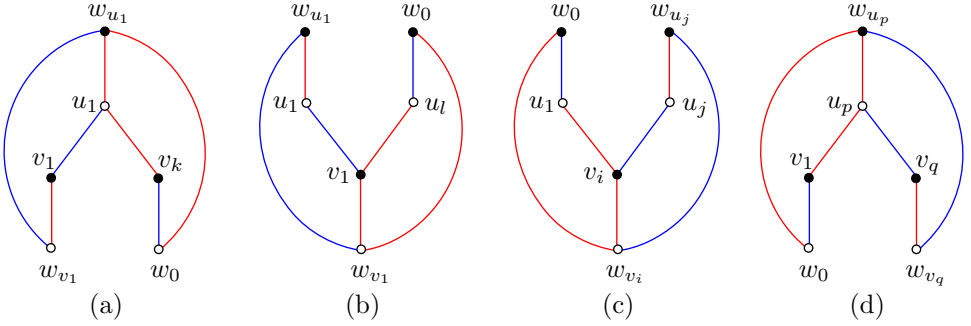


Figure 5. Proper 2-coloring for shapes (a), (b), (c), (d).

For each vertex  $w \in S$  of type II there exists at least one vertex  $w_i^0 \in D' \setminus D$  such that  $ww_i^0 \in G$ . If  $wu_1 \in G$ , then set  $c(wu_1) = 1$  and  $c(ww_i^0) = 2$ . If  $wu_{j_1} \in G$ ,  $2 \leq j_1 \leq |X|$ , then set  $c(wu_{j_1}) = 2$  and  $c(ww_i^0) = 1$ . If  $wv_1 \in G$ , then set  $c(wv_1) = 1$  and  $c(ww_i^0) = 2$ . If  $wv_{j_2} \in G$  ( $2 \leq j_2 \leq |Y|$ ), then set  $c(wv_{j_2}) = 2$  and  $c(ww_i^0) = 1$ . Thus, we give a 2-coloring  $c$  of a spanning subgraph of  $G$  and the above analysis implies that this spanning subgraph is 2-connected. Therefore, the graph  $G$  has a 2-connected bipartite spanning subgraph and Lemma 3.2 implies that  $\text{pc}(G) = 2$ . Now we will prove that the 2-coloring  $c$  defined as above is indeed a proper coloring of  $G$ .

Based on the proper 2-coloring  $c$  of  $G[D']$  as above, vertices of type I in  $S$  can be divided into five classes, as depicted in Figure 6,  $\{w_1, w'_1\}$ ,  $\{w_2, w'_2\}$ ,  $\{w_3, w'_3\}$ ,  $\{w_4, w'_4\}$  and  $\{w''_4, w'''_4\}$ . We only take the first four items into consideration (Figure 6 (0)), since the case of  $\{w''_4, w'''_4\}$  is the same as that of  $\{w_4, w'_4\}$ .

*Case 3.* For any pair of vertices  $w, w'$  of type I in  $S$  there is a proper 2-coloring  $(w, w')$ -path which excludes any  $(D' \setminus V(D), V(D))$ -edges.

For any pair of vertices  $w, w'$  of type I in  $S$ , if neither  $w$  nor  $w'$  is  $w_0$  in shape (a), (b), (c) or (d), then  $\{w, w'\} \subseteq \{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4\}$ . So it suffices to show that there is a proper path between any pair of vertices in  $\{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4\}$ . One can check that  $w_1u_1w_2, w_1u_1v_1u_2w_3,$

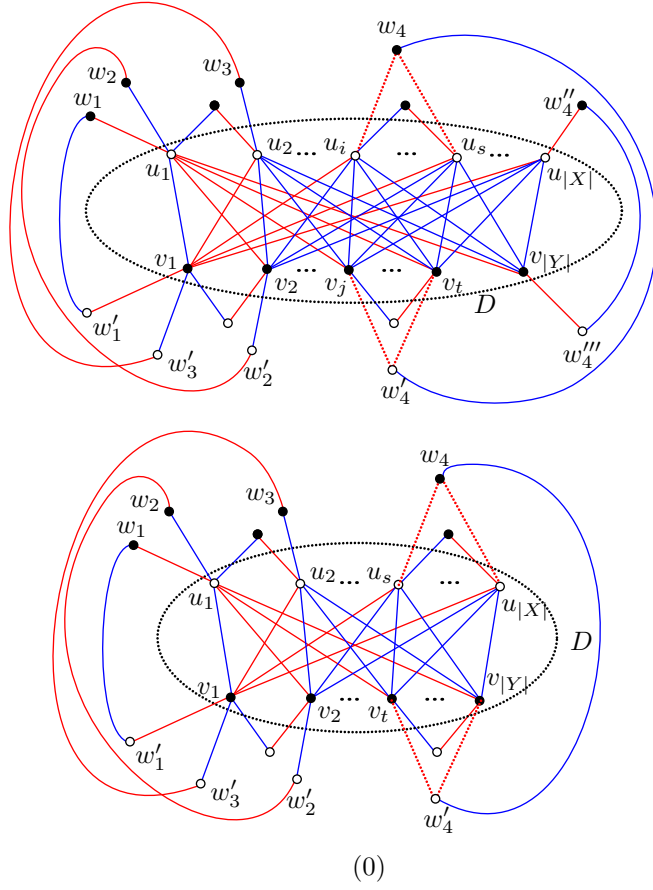


Figure 6. Proper 2-coloring for edges between vertices of type I in  $S$  and  $D'$ .

$w_1 w'_1 v_1 u_1 v_2 u_s w_4$  (or  $w_1 w'_1 v_1 u_1 v_2 u_{|X|} w_4$ ),  $w_1 w'_1$ ,  $w_1 w'_1 v_1 u_1 v_2 w'_2$ ,  $w_1 w'_1 v_1 w'_3$  and  $w_1 u_1 v_1 u_2 v_t w'_4$  (or  $w_1 u_1 v_1 u_2 v_{|Y|} w'_4$ ) are proper  $(w_1, w_2)$ ,  $(w_1, w_3)$ ,  $(w_1, w_4)$ ,  $(w_1, w'_1)$ ,  $(w_1, w'_2)$ ,  $(w_1, w'_3)$  and  $(w_1, w'_4)$ -paths, respectively. One can similarly find proper paths between any other pair of vertices in  $\{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4\}$ .

Now we assume that both  $w$  and  $w'$  are singular vertices in shape (a), (b), (c) or (d). We only need to show that every pair of singular vertices can reach each other through proper paths. If  $w, w'$  both are singular vertices of shape (a), then  $w v_k u_1 v_1 w_{v_1} u_{u_1} w'$  is a proper  $(w, w')$ -path. If  $w$  is a singular vertex of shape (a) and  $w'$  is a singular vertex of shape (b), then  $w v_k u_1 v_1 u_l w'$  is a proper  $(w, w')$ -path. If  $w$  is a singular vertex of shape (a) and  $w'$  is a singular vertex of shape (c), then  $w v_k u_1 w'$  is a proper  $(w, w')$ -path. If  $w$  is a singular vertex of shape (a) and  $w'$  is a singular vertex of shape (d), then  $w w_{u_1} v_{v_1} v_1 w'$  is a proper  $(w, w')$ -path. The cases of other pairs of singular vertices are similar and we omit the details.

Finally, we consider the case that one of  $w, w'$  is a singular vertex while the other is not. Without loss of generality, we assume that  $w$  is a singular vertex in shape (a), (b), (c) or (d). Thus, it suffices to show that there is a proper path between  $w$  and every vertex in  $\{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4\}$ . If  $w$  is a singular vertex of shape (a), then  $wv_k u_1 v_1 w'_1 w_1$ ,  $wv_k u_1 w_2$ ,  $wv_k u_1 v_1 u_2 w_3$ ,  $ww_{u_1} w_{v_1} v_1 u_1 v_2 u_s w_4$  (or  $ww_{u_1} w_{v_1} v_1 u_1 v_2 u_{|X|} w_4$ ),  $wv_k u_1 v_1 w'_1$ ,  $ww_{u_1} w_{v_1} v_1 u_1 v_2 w'_2$ ,  $ww_{u_1} w_{v_1} v_1 w'_3$  and  $ww_{u_1} w_{v_1} v_1 u_1 v_t w'_4$  (or  $ww_{u_1} w_{v_1} v_1 u_1 v_{|Y|} w'_4$ ) are proper  $(w, w_1)$ ,  $(w, w_2)$ ,  $(w, w_3)$ ,  $(w, w_4)$ ,  $(w, w'_1)$ ,  $(w, w'_2)$ ,  $(w, w'_3)$  and  $(w, w'_4)$ -paths, respectively. If  $w$  is a singular vertex of shape (b), then  $wu_l v_1 u_1 w_1$ ,  $ww_{v_1} w_{u_1} u_1 w_2$ ,  $ww_{v_1} w_{u_1} u_1 v_1 u_2 w_3$ ,  $wu_l w_4$  (if  $u_l = u_s$  or  $u_l = u_{|X|}$ ) or  $wu_l v_1 u_1 v_2 u_s$  (or  $u_{|X|}$ )  $w_4$ ,  $wu_l v_1 u_1 w_1 w'_1$ ,  $wu_l v_1 u_1 v_2 w'_2$ ,  $wu_l v_1 w'_3$  and  $ww_{v_1} w_{u_1} u_1 v_1 u_2 v_t$  (or  $v_{|Y|}$ )  $w'_4$  are proper  $(w, w_1)$ ,  $(w, w_2)$ ,  $(w, w_3)$ ,  $(w, w_4)$ ,  $(w, w'_1)$ ,  $(w, w'_2)$ ,  $(w, w'_3)$  and  $(w, w'_4)$ -paths, respectively. For the cases when  $w$  is a singular vertex of shape (c) or (d), one can check it in the similar way.

By checking the proper paths above between any pair of vertices of type I in  $S$ , we obtain a direct observation that there is a proper path under the coloring  $c$  between any pair of vertices of type I in  $S$  such that these proper paths do not contain any  $(D' \setminus V(D), V(D))$ -edges. And then, we complete the proof of Case 3.

Considering the proper 2-coloring  $c$  of  $G[D']$ , we divide vertices of type II in  $S$  into four classes, as depicted in Figure 7,  $\{w_5, w'_5\}$ ,  $\{w_6, w'_6\}$ ,  $\{w_7, w'_7\}$  and  $\{w_8, w'_8\}$ . For  $i \in \{5, 6, 7, 8\}$  (see Figure 7) it is possible that only one of  $\{w_i w_3^0, w_i w_4^0\}$  appears in  $G$ , so we use dotted lines to label the edges  $w_i w_3^0$  and  $w_i w_4^0$ . And we use dotted edges for  $\{w'_i w_5^0, w'_i w_6^0\}$  for the same reason in Figure 7.

*Case 4.* For any pair of vertices  $w, w'$  of type II in  $S$  there is a proper 2-coloring  $(w, w')$ -path such that all edges except the first and the last ones of this path are contained in  $G[D']$ .

At first, we illustrate that there is a proper path between each pair of vertices in  $\{w_5, w_6, w_7, w_8, w'_5, w'_6, w'_7, w'_8\}$ . If  $w_5 w_3^0 \in G$ , as depicted in Figure 7 (1),  $w_5 u_1 v_2 u_p w_6$ ,  $w_5 u_1 v_2 u_m w_7$ ,  $w_5 u_1 v_2 u_{|X|} w_8$ ,  $w_5 u_1 v_3 u_3 v_1 w'_5$ ,  $w_5 w_3^0 v_1 u_3 v_g w'_6$ ,  $w_5 w_3^0 v_1 u_3 v_j w'_7$  and  $w_5 w_3^0 v_1 u_3 v_{|Y|} w'_8$  are, respectively, proper  $(w_5, w_6)$ ,  $(w_5, w_7)$ ,  $(w_5, w_8)$ ,  $(w_5, w'_5)$ ,  $(w_5, w'_6)$ ,  $(w_5, w'_7)$  and  $(w_5, w'_8)$ -paths. Otherwise, there is a vertex like  $w_4^0$  (as depicted in Figure 7 (1)) such that  $w_5 w_4^0 \in G$ . Thus,  $w_5 u_1 v_2 u_p w_6$ ,  $w_5 u_1 v_2 u_m w_7$ ,  $w_5 u_1 v_2 u_{|X|} w_8$ ,  $w_5 u_1 v_3 u_3 v_1 w'_5$ ,  $w_5 w_4^0 v_q (=v_g) w'_6$  or  $w_5 w_4^0 v_q u_1 v_1 u_2 v_g w'_6$ ,  $w_5 w_4^0 v_q (=v_j) w'_7$  or  $w_5 w_4^0 v_q u_1 v_1 u_2 v_j w'_7$ , and  $w_5 w_4^0 v_q (=v_{|Y|}) w'_8$  or  $w_5 w_4^0 v_q u_1 v_1 u_2 v_{|Y|} w'_8$  are proper  $(w_5, w_6)$ ,  $(w_5, w_7)$ ,  $(w_5, w_8)$ ,  $(w_5, w'_5)$ ,  $(w_5, w'_6)$ ,  $(w_5, w'_7)$  and  $(w_5, w'_8)$ -paths, respectively. For any other pair of vertices in  $\{w_5, w_6, w_7, w_8, w'_5, w'_6, w'_7, w'_8\}$ , one can find proper paths in the same way.

Additionally, we need to show that for two vertices  $w, w'$  which both are like  $w_i$  or  $w'_i$ ,  $5 \leq i \leq 8$ , there is a proper  $(w, w')$ -path. Take the case when  $w, w'$  are both like  $w_5$  as an example. Suppose  $ww^0, ww'^0 \in G$  and the edge coloring is defined

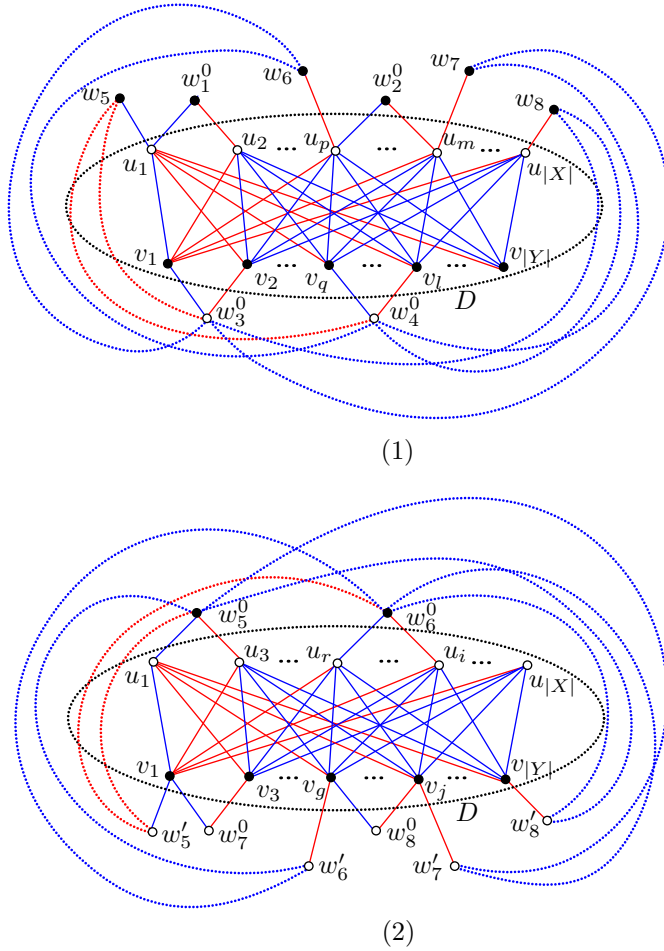


Figure 7. Proper 2-coloring for edges between vertices of type II in  $S$  and  $D'$ .

as  $c$ , where  $w^0, w'^0 \in D' \setminus V(D)$ . Then  $ww^0v_1u_2v_2u_1w'$  or  $ww^0v_2u_1w'$  is a proper  $(w, w')$ -path in  $G$ .

By verifying the proper paths above between any pair of vertices of type II in  $S$ , we obtain a direct observation that there is a proper path under the coloring  $c$  between any pair of vertices of type II in  $S$  such that all edges except the first ones and the last ones in these proper paths are contained in  $G[D']$ . And this implies the result in Case 4.

*Case 5.* For any pair of vertices  $w, w'$  of type I and II in  $S$ , respectively, there is a proper 2-coloring  $(w, w')$ -path which does not contain any  $(D' \setminus V(D), V(D))$ -edge.

As depicted in Figure 8,  $\{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4, w_1^1, w_2^1, w_3^1, w_4^1, w_5^1, w_6^1\}$  are all distinct classes vertices of type I in  $S$ , where  $\{w_1^1\}, \{w_2^1, w_3^1\}, \{w_4^1\}$  and  $\{w_5^1, w_6^1\}$

are, respectively, singular vertices in shapes (c), (b), (d) and (a). In addition,  $\{w_5, w_6, w_7, w_8\}$  are all vertices of distinct classes with type II in  $S$ . Here, we need to check that there is a proper  $(w, w')$ -path for any  $w \in \{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4, w'_1, w'_2, w'_3, w'_4, w'_5, w'_6\}$  and  $w' \in \{w_5, w_6, w_7, w_8\}$ . If  $w = w_1$ , then  $w_1 u_1 w_5$ ,  $w_1 w'_1 v_1 u_1 v_2 u_r w_6$ ,  $w_1 w'_1 v_1 u_1 v_2 u_s w_7$  and  $w_1 w'_1 v_1 u_1 v_2 u_{|X|} w_8$  are proper  $(w, w_5)$ ,  $(w, w_6)$ ,  $(w, w_7)$  and  $(w, w_8)$ -paths, respectively. If  $w = w_4$ , then  $w_4 u_r v_2 u_1 w_5$  (if  $w_4 u_r \in G$ ) or  $w_4 u_s v_2 u_1 w_5$  (if  $w_4 u_s \in G$ ),  $w_4 w'_4 v_g$  (or  $v_t$ )  $u_r w_6$ ,  $w_4 w'_4 v_g$  (or  $v_t$ )  $u_s w_7$  and  $w_4 w'_4 v_g$  (or  $v_t$ )  $u_{|X|} w_8$  are proper  $(w, w_5)$ ,  $(w, w_6)$ ,  $(w, w_7)$  and  $(w, w_8)$ -paths, respectively. If  $w = w'_1$ , then  $w'_1 w_{v_i} w_{u_j} u_j v_i u_1 w_5$ ,  $w'_1 u_1 v_2 u_r w_6$ ,  $w'_1 u_1 v_2 u_s w_7$ ,  $w'_1 u_1 v_2 u_{|X|} w_8$  are proper  $(w, w_5)$ ,  $(w, w_6)$ ,  $(w, w_7)$  and  $(w, w_8)$ -paths, respectively. With similar analysis, we can find proper paths between other pairs of vertices and as a direct observation, these paths do not contain any  $(D' \setminus V(D), V(D))$ -edge.

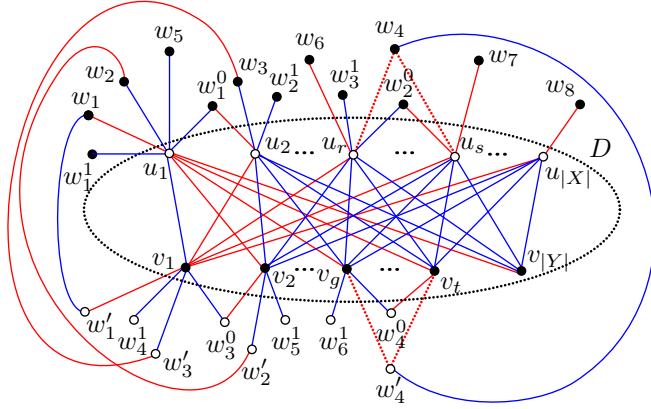


Figure 8. Proper 2-coloring for edges between vertices of types I and II in  $S$ .

*Case 6.* For any pair of vertices  $w \in S$  and  $w' \in D'$ , there is a proper 2-coloring  $(w, w')$ -path such that all edges except the first edge of this proper path are contained in  $G[D']$ .

Now we give proper  $(w, w')$ -paths for any  $w \in S$  of type I and  $w' \in D'$  first. As depicted in Figure 6 (0),  $w_1 u_1$ ,  $w_1 u_1 v_1 u_i$ ,  $2 \leq i \leq |X|$ ,  $i \neq s$  or  $i \neq |X|$ ,  $w_1 u_1 v_1$ ,  $w_1 u_1 v_1 u_2 v_j$ ,  $2 \leq j \leq |Y|$ ,  $w_1 u_1 w'_1$ ,  $w_1 u_1 v_1 u_s w'_2$ ,  $w_1 u_1 v_1 u_2 v_2 w'_3$  and  $w_1 u_1 v_1 u_2 v_{|Y|} w'_4$  are proper  $(w_1, X)$ ,  $(w_1, Y)$  and  $(w_1, D' \setminus V(D))$ -paths, respectively. If  $w = w_4$ , then  $w_4 u_s$  (or  $u_{|X|}$ )  $v_2 u_1$ ,  $w_4 u_s$  (or  $u_{|X|}$ ),  $w_4 u_s$  (or  $u_{|X|}$ )  $v_2 u_1 v_1 u_i$ ,  $2 \leq i \leq |X|$ ,  $w_4 u_s$  (or  $u_{|X|}$ )  $v_2 u_1 v_1$ ,  $w_4 u_s$  (or  $u_{|X|}$ )  $v_j$ ,  $2 \leq j \leq |Y|$ ,  $w_4 u_s$  (or  $u_{|X|}$ )  $v_2 u_1 w'_1$ ,  $w_4 u_s$  (or  $u_{|X|}$ )  $v_2 u_1 v_1 u_s w'_2$ ,  $w_4 u_s$  (or  $u_{|X|}$ )  $v_2 w'_3$  and  $w_4 u_s$  (or  $u_{|X|}$ )  $v_{|Y|} w'_4$  are, respectively, proper  $(w_4, X)$ ,  $(w_4, Y)$  and  $(w_4, D' \setminus V(D))$ -paths. Through the same way we can find proper  $(w_2, D')$ ,  $(w_3, D')$ ,  $(w'_1, D')$ ,  $(w'_2, D')$ ,  $(w'_3, D')$  and  $(w'_4, D')$ -paths and all edges except the first ones of these proper paths are contained in  $G[D']$ . This

implies that we can also find proper  $(w_0, D')$ -paths, when  $w_0$  is a singular vertex of shape (a) (or (b), (c), (d)) in similar way as that for  $w'_2$  (or  $w_3, w_2, w'_3$ ).

Next we will give proper  $(w, w')$ -paths for any  $w \in S$  of type II and  $w' \in D'$ . As depicted in Figure 7 (1),  $w_5u_1, w_5u_1v_2u_i, 2 \leq i \leq |X|, w_5u_1v_2u_2v_1, w_5u_1v_j, 2 \leq j \leq |Y|, w_5u_1v_2u_2w_1^0, w_5u_1v_2u_mw_2^0, w_5u_1v_2u_2v_1w_3^0$  and  $w_5u_1v_qw_4^0$  are proper  $(w_5, X), (w_5, Y)$  and  $(w_5, D' \setminus V(D))$ -paths, respectively. If  $w = w_8$ , then  $w_8u_{|X|}v_2u_1, w_8u_{|X|}, w_8u_{|X|}v_2u_1v_iu_i, 2 \leq i < |X|, w_8u_{|X|}v_2u_1v_1, w_8u_{|X|}v_j, 2 \leq j \leq |Y|, w_8u_{|X|}v_2u_1w_1^0, w_8u_{|X|}v_2u_1v_1u_pw_2^0, w_8u_{|X|}v_2w_3^0$  and  $w_8u_{|X|}v_lw_4^0$  are proper  $(w_8, X), (w_8, Y)$  and  $(w_8, D' \setminus V(D))$ -paths, respectively. Similarly, we can find proper  $(w_6, D'), (w_7, D'), (w'_5, D'), (w'_6, D'), (w'_7, D')$  and  $(w'_8, D')$ -paths and all edges except the first ones of these proper paths are contained in  $G[D']$ , which implies the result in Case 6.

Therefore,  $c$  is indeed a proper 2-coloring of the graph  $G$ . □

**Corollary 4.1.** *Let  $G = (X, Y)$  be a connected noncomplete bipartite graph with minimum degree  $\delta(G) \geq 2$  and  $\text{diam}(G) \leq 4$ . Then  $\text{pc}(G) = 2$ .*

**Proof.** If  $G$  is 2-connected, then Lemma 3.2 implies the result. Otherwise, we can assume that  $v_0 \in X$  be a cut vertex of  $G$  and  $G_1, G_2, \dots, G_t, t \geq 2$  be the connected components of  $G \setminus \{v_0\}$ , where  $G_i = (X_i, Y_i), X_i \subseteq X, Y_i \subseteq Y$ . It follows that  $Y_i \neq \emptyset$ , where  $1 \leq i \leq t$ . Also,  $X_i \neq \emptyset, 1 \leq i \leq t$  since  $\delta(G) \geq 2$ . In addition,  $\text{diam}(G) \leq 4$  implies that  $v_0$  is adjacent to all vertices of  $\bigcup_{i=1}^t Y_i$ . For each vertex  $v \in X_i, N_{G \setminus \{v_0\}}(v) \subseteq Y_i$  and  $|N_{G \setminus \{v_0\}}(v)| = |N_G(v)| \geq 2$ . Hence, the star  $G\left[\{v_0\} \bigcup_{i=1}^t Y_i\right]$  is a dominating set of the bipartite graph  $G$  and Theorem 4.1 suggests that  $\text{pc}(G) = 2$ . □

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