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A NEW PROOF OF THE q-DIXON IDENTITY

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Abstract. We give a new and elementary proof of Jackson's terminating q-analogue of Dixon's identity by using recurrences and induction.

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1. Introduction

Jackson's terminating q-analogue of Dixon's identity [2], [8]:

$$\sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}\begin{bmatrix}a+b\\a+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a\\c+k\end{bmatrix}=\begin{bmatrix}a+b+c\\a+b\end{bmatrix}\begin{bmatrix}a+b\\a\end{bmatrix}, \tag{1.1}$$

where the q-binomial coefficients are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)(1-q^2)\dots(1-q^k)(1-q)(1-q^2)\dots(1-q^{n-k})} & \text{if } 0\leqslant k\leqslant n,\\ 0 & \text{otherwise,}\end{cases}$$

is an important identity in combinatorics and number theory. Note that Dixon's identity (see [8], [12], page 43, equation (IV), or [9], page 11, equation (2.6)) is the $q=1$ case of (1.1). Several short proofs of the Dixon or q-Dixon identity can be found in [4], [5], [6], [7]. The q-Dixon identity can also be deduced from the q-Pfaff-Saalschütz identity (see [7], [13]).

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Recently, Mikić [10], [11] gave an elementary proof of Dixon's identity and some other binomial coefficient identities by using recurrences and induction. The aim of this note is to give a new proof of (1.1) by generalizing the argument of [10], [11].

2. Proof of (1.1)

For any integer n let $[n]=(1-q^n)/(1-q)$. Denote the left-hand side of (1.1) by $S(a,b,c)$. We introduce two auxiliary sums as follows:

$$P(a,b,c):=\sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}[a-k][a+k]\begin{bmatrix}a+b\\a+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a\\c+k\end{bmatrix}, \tag{2.1}$$

$$Q(a,b,c):=\sum_{k=-a}^{a}(-1)^k q^{3(k^2+k)/2}[b-k][b+k]\begin{bmatrix}a+b\\a+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a\\c+k\end{bmatrix}. \tag{2.2}$$

It is easy to see that $[k]\begin{bmatrix}n\\k\end{bmatrix}=[n]\begin{bmatrix}n-1\\k-1\end{bmatrix}$, and so for $a,b,c\geqslant 1$,

$$\begin{aligned}P(a,b,c)&=[a+b][a+c]\sum_{k=-a+1}^{a-1}(-1)^k q^{(3k^2+k)/2}\begin{bmatrix}a-1+b\\a-1+k\end{bmatrix}\begin{bmatrix}b+c\\b+k\end{bmatrix}\begin{bmatrix}c+a-1\\c+k\end{bmatrix}\\&=[a+b][a+c]S(a-1,b,c).\end{aligned} \tag{2.3}$$

Similarly, we have

$$Q(a,b,c)=[a+b][b+c]S(a,b-1,c). \tag{2.4}$$

It follows from (2.1) and (2.2) that

$$P(a,b,c)-Q(a,b,c)q^{a-b}=[a+b][a-b]S(a,b,c). \tag{2.5}$$

If $a\neq b$, then from (2.3)–(2.5) we deduce that

$$S(a,b,c)=\frac{1}{[a-b]}([a+c]S(a-1,b,c)-[b+c]S(a,b-1,c)q^{a-b}). \tag{2.6}$$

We need to consider the case when $a=b=c$, separately. Noticing the well known relations (see, for example [1], equations (3.3.3) and (3.3.4))

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n-1\\k\end{bmatrix}q^k+\begin{bmatrix}n-1\\k-1\end{bmatrix}=\begin{bmatrix}n-1\\k\end{bmatrix}+\begin{bmatrix}n-1\\k-1\end{bmatrix}q^{n-k},$$

we have

$$
\begin{aligned}
(2.7)\qquad S(a,a,a) &= \sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}\left(\begin{bmatrix}2a-1\\a+k\end{bmatrix}q^{a+k}+\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}\right)\\
&\quad\times\left(\begin{bmatrix}2a-1\\a+k\end{bmatrix}+\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}q^{a-k}\right)^2\\
&= \sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}\left(\begin{bmatrix}2a-1\\a+k\end{bmatrix}^3 q^{a+k}+\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}^3 q^{2a-2k}\right.\\
&\quad\left.+\begin{bmatrix}2a\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}(1+q^{a-k}+q^{2a})\right).
\end{aligned}
$$

By the symmetry of q-binomial coefficients, it is clear that

$$
\sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}\begin{bmatrix}2a-1\\a+k\end{bmatrix}^3 q^k=\sum_{k=-a}^{a-1}(-1)^k q^{(3k^2+3k)/2}\begin{bmatrix}2a-1\\a+k\end{bmatrix}^3=0,
$$

$$
\sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}^3 q^{-2k}=\sum_{k=-a+1}^{a}(-1)^k q^{(3k^2-3k)/2}\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}^3=0,
$$

and

$$
\begin{aligned}
&\sum_{k=-a}^{a}(-1)^k q^{(3k^2+k)/2}\begin{bmatrix}2a\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}q^{a-k}\\
&\qquad=\sum_{k=1-a}^{a-1}(-1)^k q^{3k^2-k/2}\begin{bmatrix}2a\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k\end{bmatrix}\begin{bmatrix}2a-1\\a+k-1\end{bmatrix}q^a=q^aS(a,a,a-1).
\end{aligned}
$$

Therefore the identity (2.7) implies that

$$
(2.8)\qquad S(a,a,a)=(1+q^a+q^{2a})S(a,a,a-1).
$$

We now give a proof of (1.1) by induction on $a+b+c$. It is clear that (1.1) is true for $a=b=c=1$. Assume that (1.1) holds for all non-negative integers a, b and c with $a+b+c=n$. Let a, b and c be non-negative integers satisfying $a+b+c=n+1$. We consider three cases:

▷ If at least one of the numbers a, b and c is equal to 0, then (1.1) is obviously true.
▷ If $a=b=c$, then by the induction hypothesis we have

$$
S(a,a,a-1)=\begin{bmatrix}3a-1\\2a\end{bmatrix}\begin{bmatrix}2a\\a\end{bmatrix}.
$$

Therefore by (2.8) we obtain

$$S(a,a,a) = (1+q^a+q^{2a}) \begin{bmatrix} 3a-1 \\ 2a \end{bmatrix} \begin{bmatrix} 2a \\ a \end{bmatrix} = \begin{bmatrix} 3a \\ 2a \end{bmatrix} \begin{bmatrix} 2a \\ a \end{bmatrix}.$$

▷ If $a \neq b$, then by (2.6) and the induction hypothesis we get

$$\begin{aligned} S(a,b,c) &= \frac{[a+c]}{[a-b]} \begin{bmatrix} a+b+c-1 \\ a+b-1 \end{bmatrix} \begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix} \\ &\quad - \frac{[b+c]}{[a-b]} \begin{bmatrix} a+b+c-1 \\ a+b-1 \end{bmatrix} \begin{bmatrix} a+b-1 \\ a \end{bmatrix} q^{a-b} \\ &= \begin{bmatrix} a+b+c \\ a+b \end{bmatrix} \begin{bmatrix} a+b \\ a \end{bmatrix} \end{aligned}$$

as desired. If $a = b$, then $a \neq c$, and we can proceed similarly as before by noticing the symmetry of a, b and c in $S(a,b,c)$.

Hence, (1.1) holds for $a+b+c = n+1$, and by induction, it holds for all non-negative integers a, b and c.

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