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RINGS CONSISTING ENTIRELY OF CERTAIN ELEMENTS

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Abstract. We completely determine when a ring consists entirely of weak idempotents, units and nilpotents. We prove that such ring is exactly isomorphic to one of the following: a Boolean ring; $\mathbb{Z}_3 \oplus \mathbb{Z}_3$; $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring; local ring with nil Jacobson radical; $M_2(\mathbb{Z}_2)$ or $M_2(\mathbb{Z}_3)$; or the ring of a Morita context with zero pairings where the underlying rings are $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Keywords: idempotent; nilpotent; Boolean ring; local ring; Morita context

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Throughout, all rings are associative with an identity. Idempotents, units and nilpotents play important roles in ring theory, cf. [2], [3], [4], [5], [6], [9], [10]. In [8], Immormino determined when a ring consists entirely of idempotents, units, and nilpotent elements. An element a in a ring is called weak idempotent if a or $-a$ is an idempotent. Clearly, every idempotent in a ring is a weak idempotent, but the converse is not true. The motivation of this paper is to investigate when a ring consists entirely of weak idempotents, units, and nilpotent elements. We prove that a ring consisting entirely of such elements is isomorphic to one of the following: a Boolean ring; $\mathbb{Z}_3 \oplus \mathbb{Z}_3$; $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring; local ring with nil Jacobson radical; $M_2(\mathbb{Z}_2)$ or $M_2(\mathbb{Z}_3)$; or the ring of a Morita context with zero pairings where the underlying rings are $\mathbb{Z}_2$ or $\mathbb{Z}_3$. The structure of such rings is thereby completely determined.

We shall use $M_n(R)$ and $T_n(R)$ to denote the ring of all $n \times n$ full matrices and triangular matrices over R, respectively. $J(R)$ stands for the Jacobson radical of R. $\mathrm{Id}(R) = \{e \in R\colon\ e^2 = e \in R\}$, $-\mathrm{Id}(R) = \{e \in R\colon\ e^2 = -e \in R\}$, $U(R)$ is the set of all units in R, and $N(R)$ is the set of all nilpotents in R.

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We begin with a generalization of [1], Corollary 1.13 which is for a commutative ring.

Lemma 1. *Let R be a ring. Then $R = U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R)$ if and only if R is isomorphic to one of the following:*

(1) *a Boolean ring;*

(2) *a division ring;*

(3) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*;*

(4) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean ring.*

Proof. $\Rightarrow$: It is easy to check that R is reduced; hence, it is abelian.

Case I. R is indecomposable. Then R is a division ring.

Case II. R is decomposable. Then $R = A \oplus B$ where $A, B \neq 0$. If $0 \neq x \in A$, then $(x, 0) \in R$ is a weak idempotent. Hence, $x \in R$ is weak idempotent. Hence, $A = \mathrm{Id}(A) \cup -\mathrm{Id}(A)$. Likewise, $B = \mathrm{Id}(B) \cup -\mathrm{Id}(B)$. In view of [1], Theorem 1.12, A and B are isomorphic to one of the following:

(1) $\mathbb{Z}_3$;

(2) a Boolean ring;

(3) $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring.

Thus, R is isomorphic to one of the following:

(a) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;

(b) $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring;

(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ where B is a Boolean ring;

(d) a Boolean ring.

Case (c). $(1, -1, 0) \notin U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R)$, an absurd. Therefore we conclude that R is one of cases (a), (b) and (d), as desired.

$\Leftarrow$: (1) $R = \mathrm{Id}(R)$.

(2) $R = U(R) \cup \mathrm{Id}(R)$.

(3) $U(R) = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$, $\mathrm{Id}(R) = \{(0,0), (0,1), (1,0)\}$ and $-\mathrm{Id}(R) = \{(0,0), (0,-1), (-1,0), (-1,-1)\}$. Thus, $R = U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R)$.

(4) $\mathrm{Id}(R) = \{(0,x), (1,x) \colon x \in B\}$ and $-\mathrm{Id}(R) = \{(0,x), (-1,x) \colon x \in B\}$. Therefore $R = \mathrm{Id}(R) \cup -\mathrm{Id}(R)$, as desired. □

Lemma 2. *Let R be a decomposable ring. Then R consists entirely of weak idempotents, units, and nilpotents if and only if R is isomorphic to one of the following:*

(1) *a Boolean ring;*

(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*;*

(3) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean ring.*

P r o o f. $\Rightarrow$: Write $R = A \oplus B$ with $A, B \neq 0$. Then A and B are rings that consist entirely of weak idempotents, units, and nilpotents. If $0 \neq x \in N(A)$, then $(x, 1) \notin \mathrm{Id}(R) \cup -\mathrm{Id}(R) \cup U(R) \cup N(R)$. This shows that $A = U(A) \cup \mathrm{Id}(A) \cup -\mathrm{Id}(A)$. Likewise, $B = U(B) \cup \mathrm{Id}(B) \cup -\mathrm{Id}(B)$. In light of Lemma 1, R is one of the following:

(a) a Boolean ring;
(b) $B \oplus D$ where B is a Boolean ring and D is a division ring;
(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ where B is a Boolean ring;
(d) $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring;
(e) $D \oplus D'$ where D and D' are division rings;
(f) $\mathbb{Z}_3 \oplus B \oplus D$ where B is a Boolean ring and D is a division ring;
(g) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus D$, where D is division ring;
(h) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$;
(i) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ where B is a Boolean ring;
(j) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Case (b). If $0, \pm 1 \neq x \in D$, then $(0, x) \notin U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R)$. This forces $D \cong \mathbb{Z}_2, \mathbb{Z}_3$. Hence, (b) forces R being in (1) or (3). Case (c) does not occur. Case (e) forces $D, D' \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Hence, R is in (1)–(3). Case (f) does not occur except $D \cong \mathbb{Z}_2$. Thus, R is in (1)–(3). Cases (g)–(j) do not occur as $(1, -1, 0), (1, -1, 0, 0) \notin I(R) \cup -\mathrm{Id}(R) \cup N(R)$, as desired.

$\Leftarrow$: This is obvious. $\square$

Theorem 3. *Let R be an abelian ring. Then R consists entirely of weak idempotents, units, and nilpotents if and only if R is isomorphic to one of the following:*

(1) $\mathbb{Z}_3$;
(2) *a Boolean ring;*
(3) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;
(4) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean ring;*
(5) *local ring with nil Jacobson radical.*

P r o o f. $\Rightarrow$: Case I. R is indecomposable. Then $R = U(R) \cup N(R)$. This shows that R is local. Let $x \in J(R)$, then $x \in N(R)$, and so $J(R)$ is nil.

Case II. R is decomposable. In view of Lemma 2, R is isomorphic to one of the following:

(1) a Boolean ring;
(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;
(3) $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring.

This shows that R is isomorphic to one of (1)–(5), as desired.

$\Leftarrow$: This is obvious. $\square$

Lemma 4. *Let R be a ring that consists entirely of weak idempotents, units and nilpotents. Then eRe is a division ring for any noncentral idempotent $e \in R$.*

Proof. Let $e \in R$ be a noncentral idempotent, and let $f = 1 - e$. Then $R \cong \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$. The subring $\begin{pmatrix} eRe & 0 \\ 0 & fRf \end{pmatrix}$ consists entirely of weak idempotents, units and nilpotents. That is, $eRe \oplus fRf$ consists entirely of weak idempotents, units and nilpotents. Set $A = eRe$ and $B = fRf$. Similarly to Lemma 2, $A = U(A) \cup \mathrm{Id}(A) \cup -\mathrm{Id}(A)$. In view of Lemma 1, A is isomorphic to one of the following:

(1) $\mathbb{Z}_3$;
(2) a Boolean ring;
(3) a division ring;
(4) $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring.

That is, A is a division ring or a ring in which every element is weak idempotent. Suppose that eRe is not a division ring. Then eRe must contain a nontrivial idempotent, say $a \in R$. Let $b = e - a$. Let $x \in eRf$ and $y \in fRe$. Choose

$$X_1 = \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} b & x \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 0 \\ y & 0 \end{pmatrix}.$$

Then X_1, X_2, Y_1, Y_2 are not invertible. As $a, b \in eRe$ are nontrivial idempotents, we see that X_1, X_2, Y_1, Y_2 are all not nilpotent matrices. This shows that X_1 and X_2 are both weak idempotents. It follows that $X_1 = \pm X_2^2$ or $X_2^2 = \pm X_2$. As $x \in eRf$, $y \in fRe$, we have $ex = x$ and $fy = y$.

Case I. $X_1 = X_1^2$, $X_2 = X_2^2$. Then $ax = x$, $bx = x$, and so $x = ex = 2x$; hence, $x = 0$.

Case II. $X_1 = X_1^2$, $X_2 = -X_2^2$. Then $ax = x$, $bx = -x$, and so $x = ex = 0$.

Case III. $X_1 = -X_1^2$, $X_2 = X_2^2$. Then $ax = -x$, $bx = x$, and so $x = ex = 0$.

Case IV. $X_1 = -X_1^2$, $X_2 = -X_2^2$. Then $a = -a^2$, $ax = -x$, $b = -b^2$ and $bx = -x$. Hence, $(e - a)x = -x$, and so $x = ex = -2x$, hence, $3x = 0$. As $a \in R$ is an idempotent, we see that $a = a^2$, hence, $a = -a$, and so $2a = 0$. It follows that $x = -ax = (2a)x - (3x)a = 0$.

Thus, $x = 0$ in any case. We infer that $eRf = 0$. Likewise, $fRe = 0$. Hence, $e \in R$ is central, an absurd. This completes the proof. □

Lemma 5. *Let R be a ring that consists entirely of weak idempotents, units and nilpotents. Then eRe is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ for any noncentral idempotent $e \in R$.*

Proof. Let $e \in R$ be a noncentral idempotent. In view of Lemma 4, eRe is a division ring. Set $f = 1 - e$. For any $u \in eRe$ we assume that $u \neq 0$, $u \neq e$,

$u \neq -e$, then the matrix

$$X = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$$

is neither a unit, nor a weak idempotent, nor a nilpotent element. This gives a contradiction. Therefore $u = 0$, $u = e$ or $u = -e$, as desired. □

Recall that a ring R is semiprime if it has no nonzero nilpotent ideals. Furthermore, we derive:

Theorem 6. *Let R be a nonabelian ring that consists entirely of units, weak idempotents, and nilpotents. If R is semiprime, then it is isomorphic to $M_2(\mathbb{Z}_2)$ or $M_2(\mathbb{Z}_3)$.*

Proof. Suppose that R is semiprime. In view of Lemma 4, eRe is a division ring for any noncentral idempotent $e \in R$. It follows by [7], Lemma 21 that R is isomorphic to $M_2(D)$ for a division ring D. Choose $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(D)$. Then E_{11} is a noncentral idempotent. According to Lemma 5, $R \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$, as asserted. □

Recall that a ring R is a NJ-ring provided that for any $a \in R$, either $a \in R$ is regular or $1 - a \in R$ is a unit [11]. Clearly, all rings in which every elements consist entirely of units, weak idempotents, and nilpotents are NJ-rings.

Theorem 7. *Let R be a nonabelian ring that consists entirely of weak idempotents, units and nilpotents. If R is not semiprime, then it is isomorphic to the ring of a Morita context with zero pairings where the underlying rings are $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Suppose that R is not semiprime. Clearly, R is a NJ-ring. In view of [11], Theorem 2, R must be a regular ring, a local ring or isomorphic to the ring of a Morita context with zero pairings where the underlying rings are both division rings. If R is regular, it is semiprime, a contradiction. If R is local, it is abelian, a contradiction. Therefore, R is isomorphic to the ring of a Morita context $T = (A, B, M, N, \varphi, \psi)$ with zero pairings φ, ψ where the underlying rings are division rings A and B. Choose $E = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then $E \in T$ is a noncentral idempotent. In light of Lemma 5, $A \cong ETE \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Likewise, $B \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. This completes the proof. □

With these information we completely determine the structure of rings that consist entirely of weak idempotents, units and nilpotents.

Theorem 8. *Let R be a ring. Then R consists entirely of weak idempotents, units and nilpotents if and only if R is isomorphic to one of the following:*

(1) *a Boolean ring;*
(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*;*
(3) $\mathbb{Z}_3 \oplus B$ *where B is a Boolean ring;*
(4) *local ring with a nil Jacobson radical;*
(5) $M_2(\mathbb{Z}_2)$ *or* $M_2(\mathbb{Z}_3)$*;*
(6) *the ring of a Morita context with zero pairings where the underlying rings are $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

P r o o f. $\Rightarrow$: This is obvious by Theorem 3, Theorem 6 and Theorem 7.

$\Leftarrow$: Cases (1)–(4) are easy. Cases (5)–(6) are verified by checking all possible (generalized) matrices over $\mathbb{Z}_2$ and $\mathbb{Z}_3$. $\square$

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