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FINITE GROUPS WHOSE ALL PROPER SUBGROUPS ARE $\mathcal{C}$-GROUPS

Pengfei Guo, Haikou, Jianjun Liu, Chongqing

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Abstract. A group G is said to be a $\mathcal{C}$-group if for every divisor d of the order of G, there exists a subgroup H of G of order d such that H is normal or abnormal in G. We give a complete classification of those groups which are not $\mathcal{C}$-groups but all of whose proper subgroups are $\mathcal{C}$-groups.

Keywords: normal subgroup; abnormal subgroup; minimal non-$\mathcal{C}$-group

MSC 2010: 20D10, 20E34

1. Introduction

In this paper, only finite groups are considered and our notation is standard.

Let $\mathfrak{F}$ be a class of groups. A group G is called a minimal non-$\mathfrak{F}$-group or $\mathfrak{F}$-critical group if G does not belong to $\mathfrak{F}$, but all proper subgroups belong to $\mathfrak{F}$. It seems clear that a detailed knowledge of minimal non-$\mathfrak{F}$-groups can give some insight into what makes a group belong to $\mathfrak{F}$. Moreover, arguments by induction or a minimal counterexample where one wants to prove that a group belongs to $\mathfrak{F}$ can benefit from a detailed description of the minimal non-$\mathfrak{F}$-groups. Many scholars have introduced in the past finite groups with this property for some particular classes. For example, Miller and Moreno in [7] considered the minimal non-abelian groups,

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Schmidt in [9] analysed the minimal non-nilpotent groups and Doerk in [3] studied the minimal non-supersolvable groups. Some related topics can be found in [1], [2].

Recall that a subgroup H of a group G is said to be abnormal in G if $g \in \langle H, H^g \rangle$ for all g in G. Recently, Liu, Li and He in [6] called a group G a $\mathcal{C}$-group if for each divisor d of the order of G, G contains a subgroup H of order d such that H is either normal or abnormal in G, and gave the structure of this kind of groups. In this paper, we will give the classification of minimal non-$\mathcal{C}$-groups.

2. Preliminaries

In this section we show some lemmas which are required in Section 3.

Lemma 2.1. *Let H be a subgroup of a group G. Then the following statements are true:*

(a) *Suppose that $H \leqslant K \leqslant G$. If H is abnormal in G, then H is abnormal in K and K is abnormal in G.*

(b) *If H is abnormal in G, then H is self-normalizing in G.*

(c) *Let $N \trianglelefteq G$ and $N \leqslant H$. Then H is abnormal in G if and only if H/N is abnormal in G/N.*

Proof. Statements (a) and (b) hold by [4], Chapter 1, 6.20. We can obtain that Statement (c) follows by a routine check. □

Lemma 2.2 ([6], Theorem 3.1). *The following statements for a group G are equivalent:*

(a) *G is a $\mathcal{C}$-group.*

(b) *Either G is nilpotent or G satisfies the following three conditions:*

  (b1) *G is supersolvable,*

  (b2) *$G/F(G)$ is cyclic of order p, where p is the smallest prime divisor of the order of G, and*

  (b3) *$G/O_p(G)$ is a Frobenius group whose Frobenius complement $P/O_p(G)$ is cyclic of order p, where P is a Sylow p-subgroup of G.*

Lemma 2.3 ([8], 13.4.3). *Let α be a power automorphism of an abelian group A. If A is a p-group of finite exponent, then there is a positive integer l such that $a^\alpha = a^l$ for all a in A. If α is nontrivial and has order prime to p, then α is fixed-point-free.*

Lemma 2.4 ([3]). *Let G be a minimal non-supersolvable group. Then:*

(1) *G is solvable.*

(2) *G has a unique normal Sylow p-subgroup P.*

(3) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $P/\Phi(P)$ is non-cyclic.*
(4) *If $p \neq 2$, then the exponent of P is p.*
(5) *If P is non-abelian and $p = 2$, then the exponent of P is 4.*
(6) *If P is abelian, then the exponent of P is p.*

Lemma 2.5. *If G is a minimal non-$\mathcal{C}$-group, then G is solvable and $|\pi(G)| \in \{2, 3\}$.*

Proof. Since every proper subgroup of G is a $\mathcal{C}$-group, G is supersolvable or minimal non-supersolvable by Lemma 2.2, which implies that G is solvable by Lemma 2.4.

Let $\{P_1, P_2, \ldots, P_n\}$ be a Sylow system of G with $p_1 < p_2 < \ldots < p_n$, where p_i is a prime dividing $|P_i|$. If $n = 1$, then G is nilpotent and Lemma 2.2 means that G is a $\mathcal{C}$-group, and this contradiction forces $n \geqslant 2$. Suppose that $n \geqslant 4$. We can see that $P_2P_3P_4 \ldots P_n$, $P_1P_3P_4 \ldots P_n$, $P_1P_2P_4 \ldots P_n$ and $P_1P_2P_3P_5 \ldots P_n$ are $\mathcal{C}$-groups, and so they are supersolvable by Lemma 2.2 again. This implies that G contains four supersolvable subgroups. Applying a theorem of Doerk (see [3], Satz 4) we can see that G is supersolvable.

By hypothesis, $P_2P_3P_4 \ldots P_n$, $P_1P_2P_4 \ldots P_n$ and $P_1P_2P_3$ are $\mathcal{C}$-groups, so $P_2 \times P_3 \times P_4 \times \ldots \times P_n$ is a nilpotent normal subgroup of G by Lemma 2.2. Hence $F(G) = O_{p_1}(G)P_2P_3P_4 \ldots P_n$ and $G/F(G)$ is cyclic of order p_1. Set $\overline{G} = G/O_{p_1}(G)$. Then $\overline{G} = \overline{P_1} \ltimes F(\overline{G})$. By hypothesis and Lemma 2.2, we have that $\overline{G}$ is not a Frobenius group and so there exists a p_1'-element $\overline{x}$ in $F(\overline{G})$ such that $C_{\overline{G}}(\overline{x}) \nsubseteq F(\overline{G})$, then $\overline{G} = C_{\overline{G}}(\overline{x})F(\overline{G})$ and thus $C_{\overline{G}}(\overline{x})$ contains a Sylow p_1-subgroup $\overline{P_1}$ of $\overline{G}$. Thus $[\overline{P_1}, \overline{x}] = 1$. By hypothesis, P_1P_i is a $\mathcal{C}$-group for $2 \leqslant i \leqslant n$. Suppose that P_1 is normal in P_1P_i for all $i \in \{2, \ldots, n\}$. Then P_1P_i is nilpotent for all $i \in \{2, \ldots, n\}$ and so P_1 is normal in G. This would imply that G is nilpotent, against the hypothesis. Therefore there exists an $i \in \{2, \ldots, n\}$ such that P_1 is abnormal in P_1P_i. Hence P_1 is abnormal in H for every Hall subgroup H of G such that $P_1P_i \leqslant H < G$ by Lemma 2.1. Consequently, P_1 is abnormal in G. On the other hand, by Lemma 2.1 and the above argument, we can see that $\overline{P}_1$ is abnormal in $\overline{G}$, a contradiction. Hence $|\pi(G)| \in \{2, 3\}$. $\square$

Lemma 2.6 ([5]). *Suppose that a p'-group H acts on a p-group G. Let*

$$\Omega(G) = \begin{cases} \Omega_1(G), & p > 2, \\ \Omega_2(G), & p = 2. \end{cases}$$

If H acts trivially on $\Omega(G)$, then H acts trivially on G as well.

3. Main results

In this section, we classify all minimal non-$\mathcal{C}$-groups. Our first result is about $|\pi(G)| = 2$.

Theorem 3.1. *Let G be a minimal non-$\mathcal{C}$-group with $|\pi(G)| = 2$. Then G is exactly of one of the following types:*

(I) *$G = \langle x, y \colon x^p = y^{q^n} = 1,\ y^{-1}xy = x^i\rangle$, where $n \geqslant 2$, $i^q \not\equiv 1 \pmod p$ and $i^{q^2} \equiv 1 \pmod p$ with $1 < i < p$.*

(II) *$G = P \rtimes Q$, where $P = \langle a, b\rangle$ is an elementary abelian p-group of order p^2, $Q = \langle y\rangle$ is cyclic of order $q^n > 1$. Define $[a, y] = 1$, $b^y = b^i$, where i is a primitive qth root of unity modulo p with $1 < i < p$.*

(III) *$G = P \rtimes Q$, where $Q = \langle y\rangle$ is cyclic of order $q^n > 1$, with $q \nmid p - 1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle y^q\rangle$ in Q. (In this type, the restriction $p > q$ is not necessary.)*

(IV) *$G = P \rtimes Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle y\rangle$ is cyclic of order $q^n > 1$, y induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and y centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leqslant p^m$. (In this type, the restriction $p > q$ is not necessary.)*

(V) *$G = P \rtimes Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle y\rangle$ is cyclic of order q^n, q is the highest power of q dividing $p - 1$ and $n > 1$. Define $a_j^y = a_{j+1}$ for $0 \leqslant j < q - 1$ and $a_{q-1}^y = a_0^i$, where i is a primitive qth root of unity modulo p.*

P r o o f. Let $G = PQ$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$. We can distinguish two cases:

Case 1. G is supersolvable and $p > q$.

(*1.1*) Assume that P and Q are cyclic.

Let $P = \langle x\rangle$ and $Q = \langle y\rangle$ with $|x| = p^m$ and $|y| = q^n$. Applying a result in [8], 10.1.10, we conclude that $y^{-1}xy = x^i$ with $i^{q^n} \equiv 1 \pmod{p^m}$, $1 < i < p^m$ and $(p^m, q^n(i-1)) = 1$. If $\langle y^q\rangle$ is normal in G, then $F(G) = P \times \langle y^q\rangle$. Set $\overline{G} = G/O_q(G)$, then $\overline{G} = \overline{P} \rtimes \overline{Q}$. It follows that $\overline{Q}$ induces a power automorphism α of order q in $\overline{P}$. By Lemma 2.3, α is fixed-point-free, and so $G/O_q(G)$ is a Frobenius group. Thus, G is a $\mathcal{C}$-group, a contradiction. Hence $\langle y^q\rangle$ is not normal in G. By hypothesis, $P\langle y^q\rangle$ is a $\mathcal{C}$-group. We can see that $\langle y^{q^2}\rangle$ is normal in $P\langle y^q\rangle$ due to Lemma 2.2. Thus $(y^q)^{-1}xy^q = x^{i^q} \neq x$, $(y^{q^2})^{-1}xy^{q^2} = x^{i^{q^2}} = x$, so that $i^q \not\equiv 1 \pmod{p^m}$, $i^{q^2} \equiv 1 \pmod{p^m}$. Surely, y^q induces a power automorphism of order q in P, and every proper subgroup of $\langle y^q\rangle$ is normal in G. If $x^p \neq 1$, then by Lemma 2.3,

$\langle x^p\rangle\langle y^q\rangle \neq \langle x^p\rangle \times \langle y^q\rangle$. By hypothesis and Lemma 2.2, $\langle y^q\rangle$ is normal in $\langle x^p\rangle\langle y\rangle$ and so $\langle x^p\rangle\langle y^q\rangle = \langle x^p\rangle \times \langle y^q\rangle$, a contradiction. Therefore, G is of type (I).

(1.2) Assume that P is non-cyclic and Q is cyclic.

Since $P \trianglelefteq G$, there exists a chief series

$$1 \trianglelefteq \ldots \trianglelefteq R \trianglelefteq P \trianglelefteq \ldots \trianglelefteq G$$

of G. By Maschke's theorem [8], Theorem 8.1.2, there exists a subgroup N of P such that $P/\Phi(P) = R/\Phi(P) \times N/\Phi(P)$, where $|N/\Phi(P)| = p$ and $N/\Phi(P) \trianglelefteq G/\Phi(P)$. Thus, $N \trianglelefteq G$, $N \ntrianglelefteq R$ and $1 \trianglelefteq N \trianglelefteq P \trianglelefteq G$ is a normal series of G. Applying Schreier's refinement theorem (see [8], Theorem 3.1.2) we obtain that P has another maximal subgroup K such that K is normal in G, and so P has at least two maximal subgroups R and K which are normal in G. Let $Q = \langle y\rangle$. Then both $R\langle y\rangle$ and $K\langle y\rangle$ are $\mathcal{C}$-groups. If $[R, y] = [K, y] = 1$, then G is nilpotent, a contradiction. If $\langle y\rangle$ is abnormal in both $R\langle y\rangle$ and $K\langle y\rangle$, hence $\langle y\rangle$ is abnormal in G and we can conclude that G is a $\mathcal{C}$-group by Lemma 2.2, a contradiction. We may assume without loss of generality that $[R, y] = 1$ and $\langle y\rangle$ is abnormal in $K\langle y\rangle$. This implies that $R \cap K = 1$ and so P is an elementary abelian p-group of order p^2. Set $R = \langle a\rangle$ and $K = \langle b\rangle$. Then $P = \langle a, b\rangle$, $[a, y] = 1$, $b^y = b^i$ and $b^{y^q} = b$. Hence G is of type (II).

(1.3) Assume that P is cyclic and Q is non-cyclic.

Let $P = \langle a\rangle$ and $|a| = p^m$. For two arbitrarily chosen maximal subgroups Q_1 and Q_2 of Q, PQ_1 and PQ_2 are $\mathcal{C}$-groups by hypothesis. It is clear that $O_q(PQ_1)$ $\mathrm{char} PQ_1 \trianglelefteq G$ and $O_q(PQ_2)$ $\mathrm{char} PQ_2 \trianglelefteq G$, so $O_q(PQ_1) \trianglelefteq G$ and $O_q(PQ_2) \trianglelefteq G$. Suppose that $O_q(PQ_1) \neq O_q(PQ_2)$, then $O_q(G) = O_q(PQ_1)O_q(PQ_2)$. If $m = 1$, then $G/O_q(G)$ is a Frobenius group whose Frobenius complement $Q/O_q(G)$ is cyclic of order q. By Lemma 2.2, G is a $\mathcal{C}$-group, a contradiction. So we may assume that $m \geqslant 2$. By hypothesis, $G/O_q(G)$ is not a Frobenius group but $\langle a^{p^{m-1}}\rangle Q$ is a $\mathcal{C}$-group. Hence $\langle a^{p^{m-1}}\rangle O_q(G)/O_q(G) \cdot \langle y\rangle O_q(G)/O_q(G)$ is nilpotent for some $y \in Q \setminus O_q(G)$. Furthermore, $[a^{p^{m-1}}, y] = 1$. Applying Lemma 2.6, we see that G is nilpotent, a contradiction. This means that $O_q(PQ_1) = O_q(PQ_2)$ and it is contained in every maximal subgroup of Q. Thus, $O_q(G) = \Phi(Q)$ is the 2-maximal subgroup of Q, and let $Q = \langle x, y, \Phi(Q)\rangle$. If $a^p \neq 1$, then $\langle a^{p^{m-1}}\rangle Q$ is a $\mathcal{C}$-group, and Q has a maximal subgroup, say $\langle x, \Phi(Q)\rangle$, such that $\langle x, \Phi(Q)\rangle \trianglelefteq \langle P, Q\rangle = G$ by Lemma 2.2 and Lemma 2.6. This contradiction implies $a^p = 1$. If $C_G(P) = P \times \Phi(Q)$, then $G/C_G(P)$ is an elementary abelian q-group of order q^2. However, $G/C_G(P) \lesssim \mathrm{Aut}(P)$, and $\mathrm{Aut}(P)$ is cyclic, a contradiction. Therefore, Q has an element, say x, which is contained in $C_G(P)$, and Q has a maximal subgroup $\langle x, \Phi(Q)\rangle$ which is normal in G, a contradiction.

(*1.4*) Assume that both P and Q are non-cyclic.

We can argue as in (1.2) and (1.3), to conclude easily that P has at least two maximal subgroups R and K which are normal in G, and $O_q(PQ_1) \trianglelefteq G$ and $O_q(PQ_2) \trianglelefteq G$ for two arbitrarily chosen maximal subgroups Q_1 and Q_2 of Q.

If $O_q(PQ_1) \neq O_q(PQ_2)$, then $O_q(G) = O_q(PQ_1)O_q(PQ_2) \lessdot Q$ and $G/F(G)$ is cyclic of order q. Since G is not a $\mathcal{C}$-group, $G/O_q(G)$ is not a Frobenius group by Lemma 2.2. Hence there exist an element a of P and an element y of $Q \setminus O_q(G)$ such that $[\overline{a}, \overline{y}] = 1$, and so $[a, y] = 1$, where $\overline{a} = aO_q(G)$, $\overline{y} = yO_q(G)$. Furthermore, $\langle a\rangle Q$ is nilpotent and we may assume $\langle a\rangle \leqslant R$. If $R \cap K > 1$, then we have that both RQ and KQ are nilpotent by Lemma 2.2. Hence G is nilpotent. This contradiction implies that $P = \langle a, b\rangle$ is an elementary abelian p-group of order p^2, and $Q = \langle y, O_q(G)\rangle$, where $q \mid p-1$ and $1 < O_q(G) \lessdot Q$. Define $[a, y] = 1$, $b^y = b^i$, where i is a primitive qth root of unity modulo p with $1 < i < p$. Clearly, $P\langle y\rangle$ is not a $\mathcal{C}$-group by Lemma 2.2, this possibility does not occur.

We consider the case that $O_q(G) = O_q(PQ_1) = O_q(PQ_2) = \Phi(Q)$ is the 2-maximal subgroup of Q, and let $Q = \langle x, y, \Phi(Q)\rangle$. If RQ is nilpotent, then $[K, Q] \neq 1$ as G is not nilpotent. Since KQ is a $\mathcal{C}$-group, there is a maximal subgroup Q^* of Q such that $[Q^*, K] = 1$ by Lemma 2.2, which implies $Q^* \trianglelefteq G$, a contradiction. Similarly, KQ is not nilpotent either. Hence Q is abnormal in both RQ and KQ. Since RQ is a $\mathcal{C}$-group, $O_q(RQ)$ is a maximal subgroup of Q by Lemma 2.2. We may assume without loss of generality that $O_q(RQ) = \langle y, O_q(G)\rangle$. If $R \cap K \neq 1$, then there is at least a nontrivial element g in $R \cap K$ such that $[g, O_q(RQ)] = 1$. Furthermore, our hypothesis and Lemma 2.2 can be combined to give that $KO_q(RQ)$ is nilpotent. This means that $O_q(RQ)$ is normal in G, a contradiction. Hence P is an elementary abelian p-group of order p^2. Let $P = \langle a\rangle \times \langle b\rangle$. Define $[a, x] = 1$, $[b, y] = 1$, $a^y = a^i$, $b^x = b^j$, $q \mid p-1$, where i, j are two primitive qth roots of unity modulo p with $1 < i, j < p$. Clearly, G contains a subgroup $P\langle x, O_q(G)\rangle$ which is not a $\mathcal{C}$-group by Lemma 2.2, a contradiction.

Case 2. G is not supersolvable.

In this case, G is a minimal non-supersolvable group and we can assume that $G = PQ$ and $P \trianglelefteq G$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ by Lemma 2.4.

Suppose that M is a maximal subgroup of G containing Q. Set $M = P_3Q$, where P_3 is a Sylow p-subgroup of M. By $[P_3, Q] \leqslant P \cap P_3Q = P_3$, we have $N_G(P_3) \geqslant P_3Q = M$. Since $N_P(P_3) > P_3$, P_3 is normal in G. By Lemma 2.4 and the maximality of M, $P_3 = \Phi(P)$ is the Sylow p-subgroup of M.

(*2.1*) Assume that $Q = \langle y\rangle$ is cyclic.

If G is also a minimal non-nilpotent group, then by [2], Theorem 3, G is either of type (III) or of type (IV).

If G is not a minimal non-nilpotent group and P is abelian, applying [1], Theorems 9 and 10, we assume that $G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle y\rangle$ is cyclic of order q^n, q^f is the highest power of q dividing $p-1$ and $n > f \geqslant 1$. Define $a_j^y = a_{j+1}$ for $0 \leqslant j < q-1$ and $a_{q-1}^y = a_0^i$, where i is a primitive q^fth root of unity modulo p. Since $P\langle y^q\rangle$ is a $\mathcal{C}$-group, y^q induces a fixed-point-free automorphism of order q in P by Lemma 2.2. Hence $a_0^{i^q} = a_0^{y^{q^2}} = a_0$. Thus $i^q \equiv 1 \pmod p$ and $f = 1$, so that G is of type (V).

If G is not a minimal non-nilpotent group and P is non-abelian, by [1], Theorems 9 and 10, we may assume that $G = PQ$ is such that $P = \langle a_0, a_1\rangle$ is an extraspecial group of order p^3 with exponent p, $Q = \langle y\rangle$ is a cyclic group of order 2^n with 2^f the largest power of 2 dividing $p-1$ and $n > f \geqslant 1$, and $a_0^y = a_1$ and $a_1^y = a_0^i x$, where $x \in \langle [a_0, a_1]\rangle$ and i is a primitive 2^fth root of unity modulo p. Since $a_0^{y^2} = a_1^y = a_0^i x \notin \langle a_0\rangle$, $P\langle y^2\rangle$ is a non-nilpotent $\mathcal{C}$-group and $\Phi(P)\langle y\rangle$ is also a $\mathcal{C}$-group. By calculation, G is not a minimal non-$\mathcal{C}$-group.

(*2.2*) Assume that Q is non-cyclic.

Applying [1], Theorems 9 and 10, $p > q$. For two arbitrarily chosen maximal subgroups Q_1 and Q_2 of Q, PQ_1 and PQ_2 are $\mathcal{C}$-groups. By Lemma 2.2, $O_q(PQ_1) \trianglelefteq G$ and $O_q(PQ_2) \trianglelefteq G$. If $O_q(PQ_1) \neq O_q(PQ_2)$, then $O_q(G) = O_q(PQ_1)O_q(PQ_2) \lessdot Q$. Examining the types 6–10 in [1], Theorems 9 and 10, and their proofs, we find none of them is a minimal non-$\mathcal{C}$-group. This implies that $O_q(PQ_1) = O_q(PQ_2)$ is contained in an arbitrary maximal subgroup of Q. Thus, $O_q(G) = \Phi(Q)$ is the 2-maximal subgroup of Q, and let $Q = \langle x, y, \Phi(Q)\rangle$. It is clear that $\Phi(G)_q \leqslant \Phi(Q)$, where $\Phi(G)_q$ is the Sylow q-subgroup of $\Phi(G)$. Examining the types 6–10 in [1], Theorems 9 and 10, and their proofs again, we find none of them coincides with a minimal non-$\mathcal{C}$-group.

Conversely, it is clear that the groups of types (I) to (V) are minimal non-$\mathcal{C}$-groups. $\square$

The following result classifies all minimal non-$\mathcal{C}$-groups with $|\pi(G)| = 3$.

Theorem 3.2. *Let G be a minimal non-$\mathcal{C}$-group with $|\pi(G)| = 3$. Then G is exactly of one of the following types:*

(I) *$G = \langle a, b\colon a^p = b^{qr^m} = 1, b^{-1}ab = a^i\rangle$ with $r \mid p-1$, $p > q > r$ and $m \geqslant 1$, where $i^q \not\equiv 1 \pmod p$ and $i^r \equiv 1 \pmod p$ with $1 < i < p$.*

(II) *$G = \langle a, b\colon a^p = b^{q^m r} = 1, b^{-1}ab = a^i\rangle$ with $q \mid p-1$, $q > r$ and $m \geqslant 1$, where $i^q \equiv 1 \pmod p$ and $i^r \not\equiv 1 \pmod p$ with $1 < i < p$.*

(III) *$G = \langle a, b\colon a^p = b^{qr} = 1, b^{-1}ab = a^i\rangle$ with $q > r$, where $i^r \not\equiv 1 \pmod p$, $i^q \not\equiv 1 \pmod p$, and $i^{qr} \equiv 1 \pmod p$ with $1 < i < p$.*

(IV) *$G = (P \rtimes Q) \rtimes R$, where R is a cyclic subgroup of order r, normalizing a Sylow q-subgroup $Q = \langle x\rangle$ of G, $Q/\Phi(Q)$ is an irreducible R-module over the field* *of q elements, and P is an irreducible QR-module over the field of p elements, where $q \mid p-1$, $r \mid p-1$ and $r \mid q-1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, coincides with $\langle x^q \rangle$ and centralizes P.*

Proof. Let $G = PQR$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, $R \in \mathrm{Syl}_r(G)$ with $p > q > r$.

We first consider the case that G is supersolvable. In this case, we have that P is normal in G. If P is non-cyclic, then due to our hypothesis P_1QR and P_2QR are $\mathcal{C}$-groups for distinct maximal subgroups P_1, P_2 of P which are normal in G, which implies that Q is normal in G as QR is a $\mathcal{C}$-group. It is clear that R is not normal in G. If R is abnormal in both PR and QR, then R is abnormal in G and so G is a $\mathcal{C}$-group, a contradiction. If $[P,R] = 1$ or $[Q,R] = 1$, then P_1QR and P_2QR are nilpotent by Lemma 2.2; this means that G is nilpotent. This contradiction forces that P is cyclic. Moreover, if $|P| > p$, then $\Phi(P)QR$ is nilpotent by Lemma 2.2 and so G is nilpotent by Lemma 2.6, a contradiction. Hence $|P| = p$.

Assume that $[P,Q] = 1$. Then Q is normal in G. If $[Q,R] = 1$ and $[P,R] \neq 1$, then G is a metacyclic group. In fact, if $|Q| > q$, then PQ_1R is a $\mathcal{C}$-group for no maximal subgroup Q_1 of Q by Lemma 2.2, a contradiction. Thus $|Q| = q$. If R is not cyclic, then Lemma 2.2 and our hypothesis can be combined to give that both PQR_1 and PQR_2 are nilpotent for distinct maximal subgroups R_1 and R_2 of R. This implies that G is nilpotent, our Lemma 2.2 provides a contradiction. Therefore, G is a metacyclic group. Applying a result in [8], 10.1.10, we assume

$$G = \langle a, b\colon\ a^p = b^{qr^m} = 1,\ b^{-1}ab = a^i \rangle$$

where $r \mid p-1$, $i^{qr^m} \equiv 1 \pmod p$ and $(p, qr^m(i-1)) = 1$ with $1 < i < p$. Since $a^{b^r} = a$ and $a^{b^q} \neq a$, we have $i^r \equiv 1 \pmod p$ and $i^q \not\equiv 1 \pmod p$. This is a group of type (I). Similarly, we can see that G is of type (I) if $[P,R] = 1$ and $[Q,R] \neq 1$. If R is abnormal in both PR and QR, then R is abnormal in G. We can show that G is a $\mathcal{C}$-group, a contradiction.

Assume that $[P,Q] \neq 1$. If Q is not cyclic, then there exist distinct maximal subgroups Q_1 and Q_2 of Q such that they are normal in QR. It is clear that PQ_1R and PQ_2R are $\mathcal{C}$-groups by hypothesis. It follows from Lemma 2.2 that $[P,Q_1] = 1 = [P,Q_2]$ and hence $[P,Q] = 1$, a contradiction. Hence Q is cyclic and let $|Q| = q^m$. If $|R| > r$, then by Lemma 2.2, PQR_1 is a $\mathcal{C}$-group for any maximal subgroup R_1 of R, and so $PQ = P \times Q$, a contradiction. Thus, $|R| = r$. Combining the above arguments we obtain that G is also a metacyclic group. If $[P,R] = 1 = [Q,R]$, then we have that this is a group of type (II). If R commutes with one of P and Q, then it follows that $m = 1$ and hence G is of type (III). If $[P,R] \neq 1$ and $[Q,R] \neq 1$, then we can see that G is a $\mathcal{C}$-group, a contradiction.

Now we consider the case that G is minimal non-supersolvable.

Applying [1], Theorems 9, 10, we may first assume that $P \trianglelefteq G$, R is a cyclic subgroup of order r^{s+t}, with r a prime number and s and t integers such that $s \geqslant 1$ and $t \geqslant 0$, normalizing Q, $Q/\Phi(Q)$ is an irreducible R-module over the field of q elements, whose kernel is the subgroup D of order r^t of R, and P is an irreducible QR-module over the field of p elements, where $q \mid p-1$, $r^s \mid p-1$ and $r \mid q-1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, coincides with $\Phi(Q) \times D$ and centralizes P. If $|R| > r$, then there exists a maximal subgroup R_1 of R such that PQR_1 is a $\mathcal{C}$-group, and so $PQ = P \times Q$ by Lemma 2.2, a contradiction. Hence $|R| = r$. On the other hand, if Q is non-cyclic, then there exist two maximal subgroups Q_1 and Q_2 of Q such that PQ_1R and PQ_2R are $\mathcal{C}$-groups. This induces that Q_1 and Q_2 centralize P by Lemma 2.2, and so PQ is nilpotent, a contradiction. It makes Q cyclic, so G is of type (IV).

We may next assume that R is a cyclic subgroup of order 2^{s+t}, with s and t integers such that $s \geqslant 1$ and $t \geqslant 0$, normalizing a Sylow q-subgroup Q of G, $Q/\Phi(Q)$ is an irreducible R-module over the field of q elements whose kernel is the subgroup D of order 2^t of R, and P is an extraspecial group of order p^3 and exponent p such that $P/\Phi(P)$ is an irreducible QR-module over the field of p elements, where $q \mid p-1$ and $2^s \mid p-1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, is equal to $\Phi(Q) \times D$ and centralizes P. Arguing as above, we easily obtain that Q is cyclic and $|R| = r$. Examining the subgroups PQ and $\Phi(P)QR$ of G, we conclude that at least one of them is not a $\mathcal{C}$-group, a contradiction.

Conversely, it is clear that the groups of types (I) to (IV) are minimal non-$\mathcal{C}$-groups. $\square$

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Authors' addresses: Pengfei Guo, School of Mathematics and Statistics, Hainan Normal University, No. 99 Longkun South Road, Haikou 571158, Hainan, P. R. China, e-mail: `guopf999@163.com`; Jianjun Liu (corresponding author), School of Mathematics and Statistics, Southwest University, No. 2 Tiansheng Road, Beibei 400715, Chongqing, P. R. China, e-mail: `liujj198123@163.com`.