ON THE WEIGHTED ESTIMATE OF THE BERGMAN PROJECTION

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Received October 27, 2016. First published February 8, 2018.

Abstract. We present a proof of the weighted estimate of the Bergman projection that does not use extrapolation results. This estimate is extended to product domains using an adapted definition of Békollé-Bonami weights in this setting. An application to bounded Toeplitz products is also given.

Keywords: Bergman space; reproducing kernel; Toeplitz operator; Békollé-Bonami weight

MSC 2010: 47B38, 30H20, 47B35, 42C40, 42A61

1. Introduction and statement of the results

The upper-half plane is the set $\mathcal{H} := \{z = x + \mathrm{i}y \in \mathbb{C} \colon\ y > 0\}$. For $\alpha > -1$ and $1 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathcal{H})$ consists of all analytic functions f on $\mathcal{H}$ such that

$$\|f\|^p_{p,\alpha} := \int_{\mathcal{H}} |f(x+\mathrm{i}y)|^p y^\alpha \,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{1.1}$$

The Bergman space $A^2_\alpha(\mathcal{H})$ $(-1 < \alpha < \infty)$ is a reproducing kernel Hilbert space with kernel $K^\alpha_w(z) = K^\alpha(z,w) = (z - \overline{w})^{-(2+\alpha)}$. That is, for any $f \in A^2_\alpha(\mathcal{H})$, the following representation holds:

$$f(w) = P_\alpha f(w) = \langle f, K^\alpha_w\rangle_\alpha = \int_{\mathcal{H}} f(z) K^\alpha(w,z)\,\mathrm{d}V_\alpha(z), \tag{1.2}$$

where for simplicity, we write $\mathrm{d}V_\alpha(x+\mathrm{i}y) = y^\alpha\,\mathrm{d}x\,\mathrm{d}y$. The positive Bergman operator P^+_α is defined by replacing K^α_w by $|K^\alpha_w|$ in the definition of P_α. Note that the boundedness of P^+_α implies the boundedness of P_α.

DOI: 10.21136/CMJ.2018.0531-16

Let I be an interval in $\mathbb{R}$, we denote by Q_I the set

$$Q_I = \{z = x + \mathrm{i}y \in \mathbb{C} \colon\ x \in I,\ 0 < y < |I|\}.$$

Let ω be a positive locally integrable function defined on $\mathcal{H}$ and $\alpha > -1$. We say ω is a Békollé-Bonami weight (or ω belongs to the class $B_{p,\alpha}(\mathcal{H})$) if

$$[\omega]_{B_{p,\alpha}} := \sup_{\substack{I \subset \mathbb{R} \\ I \text{ interval}}} \left(\frac{1}{|I|^{2+\alpha}} \int_{Q_I} \omega(z)\, \mathrm{d}V_\alpha(z)\right) \left(\frac{1}{|I|^{2+\alpha}} \int_{Q_I} \omega(z)^{1-q}\, \mathrm{d}V_\alpha(z)\right)^{p-1} < \infty,$$

$pq = p + q$.

In [2], [3], Békollé and Bonami proved that the Bergman projection P_α is bounded on $L^p(\mathcal{H}, \omega\, \mathrm{d}V_\alpha)$ if and only if the weight ω is in the class $B_{p,\alpha}(\mathcal{H})$. Pretty recently, Pott and Reguera in [13] have been able to obtain the weighted norm estimate of this operator in terms of the characteristic $[\omega]_{B_{p,\alpha}}$. More precisely, they proved the following.

Theorem 1.1 (Pott and Reguera [13])**.** *Let $1 < p, q < \infty$, $p = q(p-1)$ and $-1 < \alpha < \infty$. Suppose that $\omega \in B_{p,\alpha}(\mathcal{H})$. Then P_α is bounded on $L^p(\omega\, \mathrm{d}V_\alpha)$. Moreover,*

$$\|P_\alpha\|_{L^p(\omega\, \mathrm{d}V_\alpha) \to L^p(\omega\, \mathrm{d}V_\alpha)} \leqslant C(p)[\omega]_{B_{p,\alpha}}^{\max\{1, q/p\}}. \tag{1.3}$$

The proof of the above theorem as presented in [13] is as follows: first, the authors proved that the estimate (1.3) holds for $p = 2$ and for all weights ω in the class $B_{2,\alpha}(\mathcal{H})$; secondly, they stated and proved a sharp extrapolation result from which they were able to extend the former estimate to the full range $1 < p < \infty$ and for all $\omega \in B_{p,\alpha}(\mathcal{H})$.

Our main aim in this note is to revisit the proof of the estimate (1.3), provide a kind of simplification by avoiding the use of extrapolation. This kind of direct proof has been already obtained for Calderón-Zygmund operators [11] and it seems natural to also provide a direct proof for the weighted estimate of the Bergman projection.

Let $n \geqslant 2$, the first octant is the set

$$(0, \infty)^n := \{(y_1, \ldots, y_n) \in \mathbb{R}^n \colon\ y_j > 0,\ j = 1, \ldots, n\}$$

and the tube domain over the first octant is

$$\mathcal{H}^n := \mathbb{R}^n + \mathrm{i}(0, \infty)^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon\ \Im z_j > 0,\ j = 1, \ldots, n\}.$$

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, the notation $\alpha < \beta$ or $\alpha = \beta$ means respectively that $\alpha_j < \beta_j$ or $\alpha_j = \beta_j$, $j = 1, \dots, n$. The notation $\alpha > \beta$ is equivalent to $\beta < \alpha$ and $\alpha \leqslant \beta$ is equivalent to $\alpha < \beta$ or $\alpha = \beta$. We also use the notation $\mathbf{0} = (0, \dots, 0)$, $-\mathbf{1} = (-1, \dots, -1)$, and $\underline{\infty} = (\infty, \dots, \infty)$.

For $\alpha > -\mathbf{1}$ and $1 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathcal{H}^n)$ consists of all analytic functions f on $\mathcal{H}^n$ such that

$$\|f\|^p_{p,\alpha} := \int_{\mathcal{H}^n} |f(z_1, \dots, z_n)|^p \, \mathrm{d}V_{\alpha_1}(z_1) \dots \mathrm{d}V_{\alpha_n}(z_n) < \infty. \tag{1.4}$$

We will be also using the notation

$$\mathrm{d}V_\alpha(z) = \mathrm{d}V_{\alpha_1}(z_1) \dots \mathrm{d}V_{\alpha_n}(z_n) := y_1^{\alpha_1} \dots y_n^{\alpha_n} \, \mathrm{d}x_1 \dots \mathrm{d}x_n \, \mathrm{d}y_1 \dots \mathrm{d}y_n.$$

The Bergman space $A^2_\alpha(\mathcal{H}^n)$, $-\mathbf{1} < \alpha < \underline{\infty}$, is a reproducing kernel Hilbert space with kernel

$$K^\alpha_w(z) = K^{\alpha_1}(z_1, w_1) \dots K^{\alpha_n}(z_n, w_n) = \frac{1}{(z_1 - \overline{w_1})^{2+\alpha_1}} \cdots \frac{1}{(z_n - \overline{w_n})^{2+\alpha_n}}.$$

That is, for any $f \in A^2_\alpha(\mathcal{H}^n)$, the following representation holds:

$$f(w) = P_\alpha f(w) = \langle f, K^\alpha_w \rangle_\alpha = \int_{\mathcal{H}^n} f(z) K^\alpha(w, z) \, \mathrm{d}V_\alpha(z). \tag{1.5}$$

Let us introduce the class $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$, $1 < p < \infty$, $\mathbb{R}^n \ni \alpha = (\alpha_1, \dots, \alpha_n) > -\mathbf{1}$. We say a positive locally integrable function ω on $\mathcal{H}^n$ belongs to $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$ if there is a constant $C > 0$ such that for any $k \in \{1, \dots, n\}$,

$$\sup_{\xi = (\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n) \in \mathcal{H}^{n-1}} [\omega(\xi_1, \dots, \xi_{k-1}, \cdot, \xi_{k+1}, \dots, \xi_n)]_{B_{p,\alpha_k}(\mathcal{H})} \leqslant C. \tag{1.6}$$

We denote by $[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)}$ the infimum of the constants C in (1.6).

Note that the class $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$ is not empty; one easily checks that the weight $\omega = \prod_{j=1}^n \omega_j$ where $\omega_j \in B_{p,\alpha_j}(\mathcal{H})$ belongs to $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$.

From our definition of the class $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$ and the one parameter estimate (1.3), we easily deduce the following result.

Proposition 1.2. *Let $1 < p, q < \infty$, $p = q(p-1)$ and $-\mathbf{1} < \alpha = (\alpha_1, \dots, \alpha_n) < \underline{\infty}$. Suppose that $\omega \in \mathcal{B}_{p,\alpha}(\mathcal{H}^n)$. Then P_α is bounded on $L^p(\mathcal{H}^n, \omega \, \mathrm{d}V_\alpha(z))$. Moreover,*

$$\|P_\alpha\|_{L^p(\mathcal{H}^n, \omega \, \mathrm{d}V_\alpha(z)) \to L^p(\mathcal{H}^n, \omega \, \mathrm{d}V_\alpha(z))} \leqslant C(p) [\omega]^{n \times \max\{1, q/p\}}_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)}. \tag{1.7}$$

In the next section, we provide some useful results needed later. The simplified proof of Theorem 1.1 is given in Section 3. In the last section, we give an application of the estimates (1.3) and (1.7) to the boundedness of the product of Toeplitz operators.

As usual, given two positive quantities A and B, the notation $A \lesssim B$ or $A \gtrsim B$ means that $A \leqslant CB$ or $B \leqslant CA$, respectively, for some absolute positive constant C. The notation $A \backsimeq B$ means that $A \lesssim B$ and $B \lesssim A$. We will use $C(p)$ to say that the constant C depends only on p.

2. Some useful tools

Given an interval $I \subset \mathbb{R}$, the Carleson box associated with I is the subset Q_I of $\mathcal{H}$ defined by

$$Q_I := \{x + \mathrm{i}y \in \mathcal{H} \colon \ x \in I \text{ and } 0 < y < |I|\}.$$

The center of Q_I is the point $w_I := x_I + \mathrm{i}y_I$ such that x_I is the center of the interval I and $y_I = |I|/2$. We have the following result.

Lemma 2.1. *Let I be a subinterval of $\mathbb{R}$ and Q_I the associated Carleson box. Then for any $w \in Q_I$,*

$$|w_I - \overline{w}| \backsimeq y_I$$

where w_I is the center of Q_I.

Proof. Let $w = x + \mathrm{i}y$. On the one hand, we have

$$|w_I - \overline{w}|^2 = (x_I - x)^2 + (y_I + y)^2 > y_I^2.$$

On the other hand, we have

$$|w_I - \overline{w}|^2 = (x_I - x)^2 + (y_I + y)^2 \leqslant 13y_I^2.$$

□

We recall that the normalized reproducing kernel k_w^α, $w = u + \mathrm{i}v$, of $A_\alpha^2(\mathcal{H})$ is given by

$$k_w^\alpha(z) := \frac{K_w^\alpha(z)}{\|K_w^\alpha\|_{2,\alpha}} = \frac{v^{1+\alpha/2}}{(z - \overline{w})^{2+\alpha}}.$$

The following lemma is easy to check.

Lemma 2.2. *Let $1 < p < \infty$, $-1 < \alpha < \infty$. Then*

$$\|k_w^\alpha\|_{p,\alpha} \backsimeq y^{(2+\alpha)(2/p-1)/2}, \quad w = x + \mathrm{i}y \in \mathcal{H}.$$

We have the following estimate.

Lemma 2.3. *Let $1 < p < \infty$, $-1 < \alpha < \infty$. Then there is a constant $C > 0$ such that for any analytic function f on $\mathcal{H}$,*

$$|f(z)|^p \leqslant C\|k_z^\alpha\|_{p,\alpha}^{-p}\|fk_z^\alpha\|_{p,\alpha}^p, \quad z \in \mathcal{H}. \tag{2.1}$$

Proof. Let $\alpha > -1$. If Q_I is centered at $z = x + \mathrm{i}y$, then using the mean value property and the two previous lemmas we obtain that there exists a constant $C > 0$ such that

$$\begin{aligned}
|f(z)|^p &\leqslant \frac{C}{V_\alpha(Q_I)}\int_{Q_I}|f(w)|^p\,\mathrm{d}V_\alpha(w)\\
&\backsimeq \frac{C}{y^{2+\alpha}}\int_{Q_I}|f(w)|^p\,\mathrm{d}V_\alpha(w)\\
&= C\frac{y^{(2+\alpha)(p-2)/2}}{y^{(2+\alpha)p/2}}\int_{Q_I}|f(w)|^p\,\mathrm{d}V_\alpha(w)\\
&\backsimeq C\|k_z^\alpha\|_{p,\alpha}^{-p}\int_{Q_I}|f(w)k_z^\alpha|^p\,\mathrm{d}V_\alpha(w)\\
&\leqslant C\|k_z^\alpha\|_{p,\alpha}^{-p}\|fk_z^\alpha\|_{p,\alpha}^p.
\end{aligned}$$

□

3. Weighted inequalities for the Bergman projection

Let us start by recalling some notions and notations. We consider the system of dyadic grids

$$\mathcal{D}^\beta := \{2^j([0,1) + m + (-1)^j\beta)\colon\ m \in \mathbb{Z},\ j \in \mathbb{Z}\}, \quad \text{for } \beta \in \{0, 1/3\}.$$

For more on this system of dyadic grids and its applications, we refer to [1], [6], [7], [9], [13]. We also consider the following positive operators introduced by Pott and Reguera in [13]:

$$Q_\alpha^\beta f := \sum_{I\in\mathcal{D}^\beta}\Big\langle f, \frac{1_{Q_I}}{|I|^{2+\alpha}}\Big\rangle_\alpha 1_{Q_I}. \tag{3.1}$$

By comparing the positive kernel

$$K_\alpha^+(z,w) = \frac{1}{|z-w|^{2+\alpha}}$$

and the box-type kernel

$$K_\alpha^\beta(z,w) := \sum_{I\in\mathcal{D}^\beta} \frac{1_{Q_I}(z)1_{Q_I}(w)}{|I|^{2+\alpha}},$$

one obtains the following result (see [13] for details).

Proposition 3.1. *There is a constant $C > 0$ such that for any $f \in L^1_{\mathrm{loc}}(\mathcal{H})$, $f \geqslant 0$, and $z \in \mathcal{H}$,*

$$P_\alpha^+ f(z) \leqslant C \sum_{\beta\in\{0,1/3\}} Q_\alpha^\beta f(z). \tag{3.2}$$

We also observe that if we put $\sigma = \omega^{1-q}$ and use the notation $|Q_I|_{\omega,\alpha} = \int_{Q_I} \omega \, \mathrm{d}V_\alpha$ and $|Q_I|_\alpha = |Q_I|_{1,\alpha}$, then

$$[\omega]_{B_{p,\alpha}} = \sup_{I\subset\mathbb{R}} \frac{|Q_I|_{\omega,\alpha}|Q_I|^{p-1}_{\sigma,\alpha}}{|Q_I|^p_\alpha}.$$

P r o o f of Theorem 1.1. Our proof is inspired by the same type of proof for Calderón-Zygmund operators, (see [11]). We start by recalling that given a dyadic grid $\mathcal{D}^\beta$ and a positive weight ω, the dyadic maximal function $M^\beta_{\omega,\alpha}$ is defined for any $f \in L^1_{\mathrm{loc}}(\mathcal{H})$ by

$$M^\beta_{\omega,\alpha} f = \sup_{I\in\mathcal{D}^\beta} \frac{1_{Q_I}}{|Q_I|_{\omega,\alpha}} \int_{Q_I} |f|\omega \, \mathrm{d}V_\alpha.$$

We observe that using for example the techniques in [5], one obtains the following estimate for $1 < p < \infty$:

$$\|M^\beta_{\omega,\alpha} f\|_{L^p(\omega\,\mathrm{d}V_\alpha)} \leqslant C(p)\|f\|_{L^p(\omega\,\mathrm{d}V_\alpha)}. \tag{3.3}$$

We recall that given Q_I, its upper-half is the set

$$T_I := \Big\{x + \mathrm{i}y \in \mathcal{H} \colon \ x \in I, \ \text{and} \ \frac{|I|}{2} < y < |I|\Big\}.$$

It is clear that the family $\{T_I\}_{I\in\mathcal{D}}$ where $\mathcal{D}$ is a dyadic grid in $\mathbb{R}$ provides a tiling of $\mathcal{H}$.

Now observe that to prove Theorem 1.1, it is enough by Proposition 3.1 to prove that the following boundedness holds (with the right estimate of the norm):

$$Q^\beta_\alpha \colon \ L^p(\omega\,\mathrm{d}V_\alpha) \to L^p(\omega\,\mathrm{d}V_\alpha), \quad \beta \in \{0, 1/3\} \tag{3.4}$$

and also observe the usual fact that the latter is equivalent to

$$Q_\alpha^\beta(\sigma\cdot):\ L^p(\sigma\,\mathrm{d}V_\alpha)\to L^p(\omega dV_\alpha),\quad \beta\in\{0,1/3\},\ \sigma=\omega^{1-q}. \tag{3.5}$$

Let $f\in L^p(\sigma\,\mathrm{d}V_\alpha)$ and $g\in L^q(\omega\,\mathrm{d}V_\alpha)$ with $f,g>0$. We aim at estimating

$$\langle Q_\alpha^\beta(\sigma f),g\omega\rangle_\alpha=\int_{\mathcal{H}}Q_\alpha^\beta(\sigma f)g\omega\,\mathrm{d}V_\alpha.$$

We start with the case $p\geqslant 2$. Clearly, using the notation

$$B_{\sigma,\alpha}(f,Q_I)=\frac{1}{|Q_I|_{\sigma,\alpha}}\int_{Q_I}f\sigma\,\mathrm{d}V_\alpha$$

and

$$B_{\omega,\alpha}(g,Q_I)=\frac{1}{|Q_I|_{\omega,\alpha}}\int_{Q_I}g\omega\,\mathrm{d}V_\alpha,$$

we obtain

$$\begin{aligned}
\Pi &:= \langle Q_\alpha^\beta(\sigma f),g\omega\rangle_\alpha\\
&=\sum_{I\in\mathcal{D}^\beta}\langle\sigma f,1_{Q_I}\rangle_\alpha\langle\omega g,1_{Q_I}\rangle_\alpha|Q_I|^{-1-\alpha/2}\\
&=\sum_{I\in\mathcal{D}^\beta}B_{\sigma,\alpha}(f,Q_I)B_{\omega,\alpha}(g,Q_I)\frac{|Q_I|_{\sigma,\alpha}|Q_I|_{\omega,\alpha}}{|Q_I|_\alpha}\\
&\leqslant[\omega]_{B_{p,\alpha}}\sum_{I\in\mathcal{D}^\beta}B_{\sigma,\alpha}(f,Q_I)B_{\omega,\alpha}(g,Q_I)\frac{|Q_I|_{\sigma,\alpha}|Q_I|_{\omega,\alpha}}{|Q_I|_\alpha}\times\frac{|Q_I|_\alpha^p}{|Q_I|_{\sigma,\alpha}^{p-1}|Q_I|_{\omega,\alpha}}\\
&=[\omega]_{B_{p,\alpha}}\sum_{I\in\mathcal{D}^\beta}|Q_I|_\alpha^{p-1}|Q_I|_{\sigma,\alpha}^{2-p}B_{\sigma,\alpha}(f,Q_I)B_{\omega,\alpha}(g,Q_I).
\end{aligned}$$

We observe that $|Q_I|_\alpha\backsimeq|T_I|_\alpha$ and as $T_I\subset Q_I$ and $p\geqslant 2$, $|Q_I|_{\sigma,\alpha}^{2-p}\lesssim|T_I|_{\sigma,\alpha}^{2-p}$. On the other hand, it is easy to see that

$$|T_I|_\alpha\leqslant|T_I|_{\sigma,\alpha}^{1/q}|T_I|_{\omega,\alpha}^{1/p}.$$

Thus

$$|Q_I|_\alpha^{p-1}|Q_I|_{\sigma,\alpha}^{2-p}\lesssim|T_I|_\alpha^{p-1}|T_I|_{\sigma,\alpha}^{2-p}\leqslant|T_I|_{\sigma,\alpha}^{1/p}|T_I|_{\omega,\alpha}^{1/q}.$$

It follows that

$$
\begin{aligned}
\Pi &:= \langle Q_\alpha^\beta(\sigma f), g\omega\rangle_\alpha \\
&\lesssim [\omega]_{B_{p,\alpha}} \sum_{I\in\mathcal{D}^\beta} |T_I|_{\sigma,\alpha}^{1/p}|T_I|_{\omega,\alpha}^{1/q} B_{\sigma,\alpha}(f,Q_I)B_{\omega,\alpha}(g,Q_I) \\
&\leqslant [\omega]_{B_{p,\alpha}} \Big(\sum_{I\in\mathcal{D}^\beta} |T_I|_{\sigma,\alpha}(B_{\sigma,\alpha}(f,Q_I))^p\Big)^{1/p} \Big(\sum_{I\in\mathcal{D}^\beta} |T_I|_{\omega,\alpha}(B_{\omega,\alpha}(g,Q_I))^q\Big)^{1/q} \\
&= [\omega]_{B_{p,\alpha}} \Big(\sum_{I\in\mathcal{D}^\beta} \int_{T_I} (B_{\sigma,\alpha}(f,Q_I))^p\sigma \,\mathrm{d}V_\alpha\Big)^{1/p} \Big(\sum_{I\in\mathcal{D}^\beta} \int_{T_I} (B_{\omega,\alpha}(g,Q_I))^q\omega \,\mathrm{d}V_\alpha\Big)^{1/p'} \\
&\leqslant [\omega]_{B_{p,\alpha}} \|M_{\sigma,\alpha}^\beta f\|_{L^p(\sigma\,\mathrm{d}V_\alpha)} \|M_{\omega,\alpha}^\beta g\|_{L^q(\omega\,\mathrm{d}V_\alpha)} \\
&\leqslant C(p)[\omega]_{B_{p,\alpha}} \|f\|_{L^p(\sigma\,\mathrm{d}V_\alpha)}\|g\|_{L^q(\omega\,\mathrm{d}V_\alpha)}.
\end{aligned}
$$

For the case $1 < p < 2$, we use the previous inequalities and duality. We observe that Q_α^β is self-adjoint with respect to the duality pairing $\langle,\rangle_\alpha$. Hence

$$
\|Q_\alpha^\beta\|_{L^p(\omega\,\mathrm{d}V_\alpha)\to L^p(\omega\,\mathrm{d}V_\alpha)} = \|Q_\alpha^\beta\|_{L^q(\sigma\,\mathrm{d}V_\alpha)\to L^q(\sigma\,\mathrm{d}V_\alpha)} \leqslant C(q)[\sigma]_{B_{q,\alpha}} \leqslant C(p)[\omega]_{B_{p,\alpha}}^{1/(p-1)}.
$$

The proof is complete. $\square$

P r o o f of Proposition 1.2. Observe that $P_\alpha = P_{\alpha_1}\dots P_{\alpha_n}$ where P_{α_j} is the one parameter Bergman projection in the jth variable. It follows using the one parameter estimate of these projections in Theorem 1.1 that for any $f \in L^p(\mathcal{H}^n, \omega\,\mathrm{d}V_\alpha(z))$,

$$
\begin{aligned}
\|P_\alpha f\|_{L^p(\mathcal{H}^n,\omega\,\mathrm{d}V_\alpha(z))} &= \|P_{\alpha_1}\dots P_{\alpha_n} f\|_{L^p(\mathcal{H}^n,\omega\,\mathrm{d}V_\alpha(z))} \\
&\leqslant C[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}_\alpha^n)}^{\max\{1,q/p\}} \|P_{\alpha_2}\dots P_{\alpha_n} f\|_{L^p(\mathcal{H}^n,\omega\,\mathrm{d}V_\alpha(z))} \\
&\leqslant C[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}_\alpha^n)}^{2\max\{1,q/p\}} \|P_{\alpha_3}\dots P_{\alpha_n} f\|_{L^p(\mathcal{H}^n,\omega\,\mathrm{d}V_\alpha(z))} \cdots \\
&\leqslant C[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}_\alpha^n)}^{n\max\{1,q/p\}} \|f\|_{L^p(\mathcal{H}^n,\omega\,\mathrm{d}V_\alpha(z))}.
\end{aligned}
$$

The proof is complete. $\square$

4. Applications

Before giving our applications, we recall a brief history of the so-called Sarason's conjecture. Let us denote by $\mathrm{d}\nu$ the normalized Lebesgue measure on the unit disc $\mathbb{D}$ of $\mathbb{C}$.

For $\alpha > -1$, we denote by $\mathrm{d}\nu_\alpha$ the normalized Lebesgue measure $\mathrm{d}\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha \, \mathrm{d}\nu(z)$, c_α being the normalizing constant. The weighted Bergman space $A^p_\alpha(\mathbb{D})$ is the space of holomorphic functions f such that

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}\nu_\alpha(z) < \infty.$$

The Bergman space $A^2_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space with kernel $K^\alpha_w(z) = K^\alpha(w,z) = (1-w\overline{z})^{-(2+\alpha)}$. That is, for any $f \in A^2_\alpha(\mathbb{D})$, the following representation holds:

$$f(w) = P_\alpha f(w) = \langle f, K^\alpha_w\rangle_\alpha = \int_{\mathbb{D}} f(z) K^\alpha(w,z) \, \mathrm{d}\nu_\alpha(z), \quad w \in \mathbb{D}. \tag{4.1}$$

For $f \in L^2(\mathbb{D}, \mathrm{d}\nu_\alpha)$, we can densely define the Toeplitz operator T_f with symbol f on $A^2_\alpha(\mathbb{D})$ as

$$T_f(g) = P_\alpha(M_f)(g) = P_\alpha(fg) \tag{4.2}$$

where M_f is the multiplication operator by f. The Berezin transform is the operator defined on $L^1(\mathbb{D}, \mathrm{d}\nu_\alpha)$ by

$$B_\alpha(f)(w) = \int_{\mathbb{D}} f(z) |k^\alpha_w(z)|^2 \, \mathrm{d}v_\alpha(z)$$

where k^α_w is the normalized reproducing kernel of $A^2_\alpha(\mathbb{D})$.

The so-called Sarason conjecture said that given two functions $f, g \in A^2_\alpha(\mathbb{D})$, the product $T_f T_{\overline{g}}$ is bounded on $A^2_\alpha(\mathbb{D})$ if and only if the following relation holds:

$$\sup_{w\in\mathbb{D}} (B_\alpha(|f|^2)(w))^{1/2} (B_\alpha(|g|^2)(w))^{1/2} = \sup_{w\in\mathbb{D}} \|f k^\alpha_w\|_{2,\alpha} \|g k^\alpha_w\|_{2,\alpha} < \infty. \tag{4.3}$$

We call (4.3) the Sarason condition. For $\alpha = -1$, that is in the case of the Hardy space $H^2(\mathbb{D})$, Nazarov has proved that the conjecture fails (see [12]), although (4.3) is necessary as proved by Treil (see [15]). For the usual Bergman spaces, $\alpha > -1$, there have been many works on the problem. It has been proved in [18] that condition (4.3) is necessary but the authors did not manage to prove whether the same condition is sufficient or not (see also [8], [10], [14], [16], [17] for related discussions and other domains). It is only two years ago that Aleman, Pott and Reguera exhibited in [1] an example of $f, g \in A^2(\mathbb{D}) = A_0^2(\mathbb{D})$ such that (4.3) holds but the product $T_f T_{\overline{g}}$ is not bounded on $A^2(\mathbb{D})$.

For $f \in L^2(\mathcal{H}, \mathrm{d}V_\alpha)$, we can densely define the Toeplitz operator T_f with symbol f on $A_\alpha^2(\mathcal{H})$ as in (4.2). We recall that the normalized reproducing kernel k_w^α, $w = u + \mathrm{i}v$, of $A_\alpha^2(\mathcal{H})$ is given by

$$k_w^\alpha(z) := \frac{K_w^\alpha(z)}{\|K_w^\alpha\|_{2,\alpha}} = \frac{v^{1+\alpha/2}}{(z-\overline{w})^{2+\alpha}}.$$

The following assertion can be observed as in [4]. It says that the boundedness of a Toeplitz product is equivalent to a two-weight problem for the Bergman projection.

Proposition 4.1. *Let $1 < p < \infty$ and $-1 < \alpha < \infty$. Then $T_f T_{\overline{g}}$ is bounded on $A_\alpha^p(\mathcal{H})$ if and only if P_α is bounded from $L^p(\mathcal{H}, |g|^{-p}\,\mathrm{d}V_\alpha)$ to $L^p(\mathcal{H}, |f|^p\,\mathrm{d}V_\alpha)$.*

The next result says that if one of the symbols satisfies an invariant condition, then the Sarason condition is necessary and sufficient for the associated Toeplitz product to be bounded.

Theorem 4.2. *Let $1 < p, q < \infty$, $p = q(p-1)$, and $-1 < \alpha < \infty$. Suppose that f and g are analytic in $\mathcal{H}$ with*

$$[f]_{p,\alpha} := \sup_{w\in\mathcal{H}} \|f k_w^\alpha\|_{p,\alpha} \|f^{-1} k_w^\alpha\|_{q,\alpha} < \infty. \tag{4.4}$$

Then $T_f T_{\overline{g}}$ is bounded on $A_\alpha^p(\mathcal{H})$ if and only if

$$[f,g]_{p,\alpha} := \sup_{w\in\mathcal{H}} \|f k_w^\alpha\|_{p,\alpha} \|g k_w^\alpha\|_{q,\alpha} < \infty. \tag{4.5}$$

Moreover,

$$\|T_f T_{\overline{g}}\| \leqslant C(p)[f,g]_{p,\alpha}[f]_{p,\alpha}^{\max\{p,q\}}$$

and

$$[f,g]_{p,\alpha} \leqslant C(p)\|T_g T_{\overline{f}}\| [f]_{p,\alpha}.$$

Proof. Let us start with the sufficient part. We first observe that the condition on f provides in particular that the weight $\omega = |f|^p$ is in $B_{p,\alpha}(\mathcal{H})$. Clearly, for the interval $I \subset \mathbb{R}$, let Q_I be its associated Carleson box and w its center. Then

$$\begin{aligned}
\infty > [f]_{p,\alpha} &> \left(\int_{\mathcal{H}} |f|^p |k_w^\alpha|^p \,\mathrm{d}V_\alpha\right)^{1/p} \left(\int_{\mathcal{H}} |f|^{-q} |k_w^\alpha|^q \,\mathrm{d}V_\alpha\right)^{1/q} \\
&\geqslant \left(\int_{Q_I} |f|^p |k_w^\alpha|^p \,\mathrm{d}V_\alpha\right)^{1/p} \left(\int_{Q_I} |f|^{-q} |k_w^\alpha|^q \,\mathrm{d}V_\alpha\right)^{1/q} \\
&\backsimeq \left(\frac{1}{|Q_I|_\alpha^{p/2}} \int_{Q_I} |f|^p \,\mathrm{d}V_\alpha\right)^{1/p} \left(\frac{1}{|Q_I|_\alpha^{q/2}} \int_{Q_I} |f|^{-q} \,\mathrm{d}V_\alpha\right)^{1/q} \\
&= \left(\frac{1}{|Q_I|_\alpha} \int_{Q_I} \omega \,\mathrm{d}V_\alpha \left(\frac{1}{|Q_I|_\alpha} \int_{Q_I} \omega^{1-q}\right)^{p-1}\right)^{1/p}.
\end{aligned}$$

Hence if $\omega = |f|^p$, then

$$[\omega]_{B_{p,\alpha}} \lesssim [f]_{p,\alpha}^p.$$

Next, using Lemma 2.3, we obtain

$$\begin{aligned}
|T_f T_{\overline{g}} h(z)| &= |f(z)| |P_\alpha(\overline{g} h)(z)| \\
&\leqslant |f(z)| \int_{\mathcal{H}} \frac{|g(w)||h(w)|}{|z - \overline{w}|^{2+\alpha}} \,\mathrm{d}V_\alpha(w) \\
&= |f(z)| \int_{\mathcal{H}} \frac{|g(w)||f(w)||f(w)|^{-1}|h(w)|}{|z - \overline{w}|^{2+\alpha}} \,\mathrm{d}V_\alpha(w) \\
&\leqslant [f,g]_{p,\alpha} |f(z)| \int_{\mathcal{H}} \frac{|f(w)|^{-1}|h(w)|}{|z - \overline{w}|^{2+\alpha}} \,\mathrm{d}V_\alpha(w).
\end{aligned}$$

Hence to prove that $T_f T_{\overline{g}}$ is bounded on $A_\alpha^p(\mathcal{H})$, it is enough to prove that the positive operator

$$h \mapsto |f(z)| \int_{\mathcal{H}} \frac{|f(w)|^{-1} h(w)}{|z - \overline{w}|^{2+\alpha}} \,\mathrm{d}V_\alpha(w), \quad z \in \mathcal{H}$$

is bounded on $L^p(\mathcal{H}, \mathrm{d}V_\alpha)$. The boundedness of the latter is equivalent to the boundedness of P_α^+ on $L^p(\mathcal{H}, |f|^p \,\mathrm{d}V_\alpha)$ which holds by Theorem 1.1 since the weight $\omega = |f|^p$ is in the class $B_{p,\alpha}(\mathcal{H})$, and with the right estimate. Thus

$$\|T_f T_{\overline{g}}\| \leqslant C(p) [\omega]_{B_{p,\alpha}}^{\max\{1, q/p\}} [f,g]_{p,\alpha} \leqslant C(p) [f,g]_{p,\alpha} [f]_{p,\alpha}^{\max\{p,q\}}.$$

Let us now suppose that $T_f T_{\overline{g}}$ is bounded on $A_\alpha^p(\mathcal{H})$. Then in particular we have

$$\|k_w^\alpha\|_{q,\alpha}^{-1} |f(w)| \|g k_w^\alpha\|_{q,\alpha} = \|k_w^\alpha\|_{q,\alpha}^{-1} \|\overline{f(w)} g k_w^\alpha\|_{q,\alpha} = \|k_w^\alpha\|_{q,\alpha}^{-1} \|T_g T_{\overline{f}} k_w^\alpha\|_{q,\alpha} \leqslant \|T_g T_{\overline{f}}\|.$$

Using Lemma 2.3 again and the equivalence $\|k_w^\alpha\|_{p,\alpha}\|k_w^\alpha\|_{q,\alpha} \backsimeq 1$, we obtain

$$\begin{aligned}\|fk_w^\alpha\|_{p,\alpha}\|gk_w^\alpha\|_{q,\alpha} &\backsimeq \|k_w^\alpha\|_{p,\alpha}^{-1}\|k_w^\alpha\|_{q,\alpha}^{-1}|f(w)||f(w)|^{-1}\|fk_w^\alpha\|_{p,\alpha}\|gk_w^\alpha\|_{q,\alpha}\\ &\leqslant C\|T_gT_{\overline{f}}\|\|k_w^\alpha\|_{p,\alpha}^{-1}|f(w)|^{-1}\|fk_w^\alpha\|_{p,\alpha}\\ &\leqslant C\|T_gT_{\overline{f}}\|\|k_w^\alpha\|_{p,\alpha}^{-1}\|k_w^\alpha\|_{q,\alpha}^{-1}\|fk_w^\alpha\|_{p,\alpha}\|f^{-1}k_w^\alpha\|_{q,\alpha}\\ &\leqslant C\|T_gT_{\overline{f}}\|[f]_{p,\alpha} < \infty.\end{aligned}$$

The proof is complete. □

Note that in the above theorem, we do not ask f and g to belong to $A_\alpha^p(\mathcal{H})$ and $A_\alpha^q(\mathcal{H})$, respectively, giving ourselves the flexibility to recover the Bergman projection by taking $f = g = 1$. This assumption is also motivated by the fact that constants (except zero) are not in $A_\alpha^p(\mathcal{H})$. Thus in the case of the unit disc or the unit ball, we can suppose that $f \in A_\alpha^p$ and $g \in A_\alpha^q$.

Taking $g = 1/f$ in the above theorem, we obtain that the invariant condition $[f]_{p,\alpha} < \infty$ is actually necessary and sufficient for the boundedness of $T_fT_{1/\overline{f}}$ on $A_\alpha^p(\mathcal{H})$, and we can even deduce more.

Proposition 4.3. *Let $1 < p, q < \infty$, $pq = p + q$ and let f be analytic on $\mathcal{H}$. Then the following assertions are equivalent.*

(i) *$T_fT_{1/\overline{f}}$ is bounded on $A_\alpha^p(\mathcal{H})$.*

(ii)

$$\sup_{w\in\mathcal{H}} \|fk_w^\alpha\|_{p,\alpha}\|f^{-1}k_w^\alpha\|_{q,\alpha} < \infty.$$

(iii) *If $\omega = |f|^p$ and $\sigma = \omega^{1-q} = |f|^{-q}$, then*

$$\sup_I \frac{|Q_I|_{\omega,\alpha}|Q_I|_{\sigma,\alpha}^{p-1}}{|Q_{I,\alpha}|^p} < \infty.$$

(iv) *P_α^+ is bounded on $L^p(\mathcal{H}, |f|^p\,\mathrm{d}\nu_\alpha)$.*

P r o o f. That (ii) ⇒ (iii) is the beginning of the proof of the above theorem while (iii) ⇒ (iv) is Theorem 1.1. The implication (iv) ⇒ (i) is Proposition 4.1. To finish, we only have to prove that (i) ⇒ (ii).

(i) ⇒ (ii): Assume that $T_fT_{1/\overline{f}}$ is bounded on $A_\alpha^p(\mathcal{H})$ so that $T_{1/f}T_{\overline{f}}$ is also bounded on $A_\alpha^q(\mathcal{H})$. Put $g = 1/f$. Then we obtain

$$|f(w)|\|gk_w^\alpha\|_{q,\alpha} = \|\overline{f(w)}gk_w^\alpha\|_{q,\alpha} = \|T_gT_{\overline{f}}k_w^\alpha\|_{q,\alpha} \leqslant \|T_gT_{\overline{f}}\|\|k_w^\alpha\|_{q,\alpha}.$$

That is, for all $w \in \mathcal{H}$,

$$|f(w)|\|gk_w^\alpha\|_{q,\alpha} \leqslant \|T_gT_{\overline{f}}\|\|k_w^\alpha\|_{q,\alpha}. \tag{4.6}$$

We also obtain

$$|g(w)|\|fk_w^\alpha\|_{p,\alpha} = \|\overline{g(w)}fk_w^\alpha\|_{p,\alpha} = \|T_fT_{\overline{g}}k_w^\alpha\|_{p,\alpha} \leqslant \|T_fT_{\overline{g}}\|\|k_w^\alpha\|_{p,\alpha}.$$

That is,

$$|g(w)|\|fk_w^\alpha\|_{p,\alpha} \leqslant \|T_fT_{\overline{g}}\|\|k_w^\alpha\|_{p,\alpha}. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$|f(w)||g(w)|\|fk_w^\alpha\|_{p,\alpha}\|gk_w^\alpha\|_{q,\alpha} \leqslant \|T_gT_{\overline{f}}\|\|T_fT_{\overline{g}}\|\|k_w^\alpha\|_{p,\alpha}\|k_w^\alpha\|_{q,\alpha}.$$

Recalling that $g = 1/f$ and using that $\|k_w^\alpha\|_{p,\alpha}\|k_w^\alpha\|_{q,\alpha} \simeq 1$ for all $w \in \mathcal{H}$, we obtain that

$$\sup_{w\in\mathcal{H}} \|fk_w^\alpha\|_{p,\alpha}\|f^{-1}k_w^\alpha\|_{q,\alpha} < \infty.$$

The proof is complete. □

Let us also observe the following result.

Lemma 4.4. *Let* $-\mathbf{1} < \alpha < \underline{\infty}$. *Assume that* $1 < p < \infty$ *and put* $p = q(p-1)$. *If* f *is analytic on* $\mathcal{H}^n$ *with*

$$[f]_{p,\alpha} := \sup_{w\in\mathcal{H}^n} \|fk_w^\alpha\|_{p,\alpha}\|f^{-1}k_w^\alpha\|_{q,\alpha} < \infty,$$

then there is a constant $C > 0$ *such that for any* $1 \leqslant k \leqslant n$ *and any* $w = (w_1,\dots,w_{k-1},w_{k+1},\dots,w_{n-1}) \in \mathcal{H}^{n-1}$,

$$\sup_{z\in\mathcal{H}} \|f_wk_z^{\alpha_k}\|_{p,\alpha_k}\|f_w^{-1}k_z^{\alpha_k}\|_{q,\alpha_k} \leqslant C \sup_{\xi\in\mathcal{H}^n} \|fk_\xi^\alpha\|_{p,\alpha}\|f^{-1}k_\xi^\alpha\|_{q,\alpha},$$

where $f_w(z) = f(\zeta_1,\dots,\zeta_n)$, $\zeta_j = w_j$ *for* $j \neq k$ *and* $\zeta_k = z$.

P r o o f. We may suppose that $k = 1$. Let $w = (w_1,\dots,w_{n-1}) \in \mathcal{H}^{n-1}$ be given. For any given $z \in \mathcal{H}$, we put $\zeta = (z,w_1,\dots,w_{n-1})$ and $\tilde{\alpha} = (\alpha_2,\dots,\alpha_n)$. Then using Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}\|f_wk_z^{\alpha_1}\|_{p,\alpha_1}^p &= \int_{\mathcal{H}} |f(u+\mathrm{i}v,w)|^p|k_z^{\alpha_1}(u+\mathrm{i}v)|^pv^{\alpha_1}\,\mathrm{d}V(u+\mathrm{i}v)\\ &= \int_{\mathcal{H}} |f_{u+\mathrm{i}v}(w)|^p|k_z^{\alpha_1}(u+\mathrm{i}v)|^pv^{\alpha_1}\,\mathrm{d}V(u+\mathrm{i}v)\\ &\lesssim \|k_w^{\tilde{\alpha}}\|_{p,\tilde{\alpha}}^{-p}\int_{\mathcal{H}} \|f_{u+\mathrm{i}v}k_w^{\tilde{\alpha}}\|_{p,\tilde{\alpha}}^p|k_z^{\alpha_1}(u+\mathrm{i}v)|^pv^{\alpha_1}\,\mathrm{d}V(u+\mathrm{i}v)\\ &= \|k_w^{\tilde{\alpha}}\|_{p,\tilde{\alpha}}^{-p}\|fk_\zeta^\alpha\|_{p,\alpha}^p.\end{aligned}$$

In the same way, we obtain

$$\|f_w^{-1} k_z^{\alpha_1}\|_{q,\alpha_1}^q \lesssim \|k_w^{\tilde{\alpha}}\|_{q,\tilde{\alpha}}^{-q} \|f^{-1} k_\zeta^{\alpha}\|_{q,\alpha}^q.$$

Thus as $\|k_w^{\tilde{\alpha}}\|_{p,\tilde{\alpha}} \|k_w^{\tilde{\alpha}}\|_{q,\tilde{\alpha}} \backsimeq 1$, we obtain for any $k \in \{1, 2, \ldots, n\}$,

$$\|f_w k_z^{\alpha_k}\|_{p,\alpha_k} \|f_w^{-1} k_z^{\alpha_k}\|_{q,\alpha_k} \lesssim \sup_{\xi \in \mathcal{H}^n} \|f k_\xi^{\alpha}\|_{p,\alpha} \|f^{-1} k_\xi^{\alpha}\|_{q,\alpha}.$$

The proof is complete. $\square$

We have the following extension of Theorem 4.2 to the multi-parameter case.

Theorem 4.5. *Let* $-\mathbf{1} < \alpha < \underline{\infty}$. *Assume that* $1 < p < \infty$ *and put* $p = q(p-1)$. *Suppose that* f *and* g *are analytic in* $\mathcal{H}^n$ *with*

$$[f]_{p,\alpha} := \sup_{w \in \mathcal{H}^n} \|f k_w^{\alpha}\|_{p,\alpha} \|f^{-1} k_w^{\alpha}\|_{q,\alpha} < \infty. \tag{4.8}$$

Then $T_f T_{\overline{g}}$ *is bounded on* $A_\alpha^p(\mathcal{H}^n)$ *if and only if*

$$[f, g]_{p,\alpha} := \sup_{w \in \mathcal{H}^n} \|f k_w^{\alpha}\|_{p,\alpha} \|g k_w^{\alpha}\|_{q,\alpha} < \infty.$$

Moreover,

$$\|T_f T_{\overline{g}}\| \leqslant C(p)[f, g]_{p,\alpha} [f]_{p,\alpha}^{n \times \max\{p,q\}}$$

and

$$[f, g]_{p,\alpha} \leqslant C(p) \|T_g T_{\overline{f}}\| [f]_{p,\alpha}.$$

Proof. From Lemma 4.4 and the beginning of the proof of Theorem 4.2, we obtain that the condition (4.8) implies that the weight $\omega = |f|^p$ belongs to $\mathcal{B}_{p,\alpha}(\mathcal{H}^n)$ with

$$[\omega]_{\mathcal{B}_{p,\alpha}(\mathcal{H}^n)} \lesssim [f]_{p,\alpha}^p.$$

Following the steps of the proof of Theorem 4.2 and using Proposition 1.2 we obtain the result. $\square$

To conclude, we remark that Proposition 4.3 also holds for the tube over the first octant.

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