MORE ON BETWEENNESS-UNIFORM GRAPHS

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Abstract. We study graphs whose vertices possess the same value of betweenness centrality (which is defined as the sum of relative numbers of shortest paths passing through a given vertex). Extending previously known results of S. Gago, J. Hurajová, T. Madaras (2013), we show that, apart of cycles, such graphs cannot contain 2-valent vertices and, moreover, are 3-connected if their diameter is 2. In addition, we prove that the betweenness uniformity is satisfied in a wide graph family of semi-symmetric graphs, which enables us to construct a variety of nontrivial cubic betweenness-uniform graphs.

Keywords: betweenness centrality; betweenness-uniform graph

MSC 2010: 05C15

1. Introduction

This paper continues the research originated in [9]. We recall here some specialized notation and definitions. For a graph G with vertex set $V(G)$ and edge set $E(G)$, $\Delta(G)$ and $\delta(G)$ denote the maximum and the minimum degree of vertices of G, respectively. The set of all neighbours of a vertex u is denoted by $N(u)$. For two vertices u, v of G, $d(u,v)$ denotes their distance (that is, the length of a shortest (u–v)-path); the diameter $\operatorname{diam}(G)$ of G is the maximum of $d(u,v)$ taken over all pairs u,v of vertices of G, and the average distance $\overline{l}(G)$ of G is the arithmetic mean of $d(u,v)$ for all pairs of distinct vertices u,v. An n-vertex cycle is denoted by C_n. Other terminology not defined here is taken from the book [7].

In both pure and applied graph theory, a great attention is paid to the study of local graph characteristics, particularly, of real-valued functions on the vertex set which are isomorphism-invariant (under an additional assumption linking the higher function values with more centrally perceived positions of vertices within a graph, they are known as *vertex centrality indices*); the standard examples are the vertex degree, the vertex eccentricity (that is, the maximum distance from the given vertex)

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or the total distance (the sum of distances of all vertices from given vertex). One of typical areas in their study is the investigation of properties of graphs whose vertices have the same value of the centrality function considered—in the case of vertex degree, these graphs are precisely regular graphs, for the eccentricity, they are known as self-centered graphs (see [1], [3], [4]) and, for the total distance, they are known under the names self-median or farness-selfcentric graphs (see, for example, [2] or [10]). In [9], we initiated the study of *betweenness-uniform graphs* whose vertices have the same value of the *betweenness centrality* defined as follows (see [8]):

Given a graph G and its distinct vertices u, v, x, let $\sigma_{u,v}$ be the number of all shortest (u–v)-paths in G, and $\sigma_{u,v}(x)$ the number of all shortest (u–v)-paths which pass through x. Then the betweenness centrality of x is defined as

$$B(x) = \sum_{u,v \in V(G)} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}}.$$

Among the results of [9], it was shown that each betweenness-uniform graph is 2-connected; furthermore, if it contains a universal or sub-universal vertex (that is, one which is adjacent to all vertices or to all vertices except a single one), then it is isomorphic to a complete graph or has diameter two, respectively. However, the computational results of Section 2 on large collections of graphs suggest that the following stronger conjectures might be true:

Conjecture 1.1. *If G is a betweenness-uniform graph which is not a cycle, then G is 3-connected.*

Conjecture 1.2. *If G is a betweenness-uniform graph and $\Delta(G) = n - k$, then* $\operatorname{diam}(G) \leqslant k$.

We prove the latter result for $k = 3$ with even better upper bound and the former — with the exception of two short cycles — for graphs of diameter 2, and show that betweenness-uniform graph which is not a cycle, cannot contain a vertex of degree 2.

In addition, we study sparse cubic betweenness-uniform graphs and are interested in non-transitive ones. Note that non-transitive cubic graphs cannot be obtained by constructions used in [9]; nevertheless, we show that there exist infinite families of such graphs.

2. The results

First, in Figure 1, we present an updated overview of all betweenness-uniform connected graphs from 4 up to 10 vertices (their list first appeared in [9]) containing

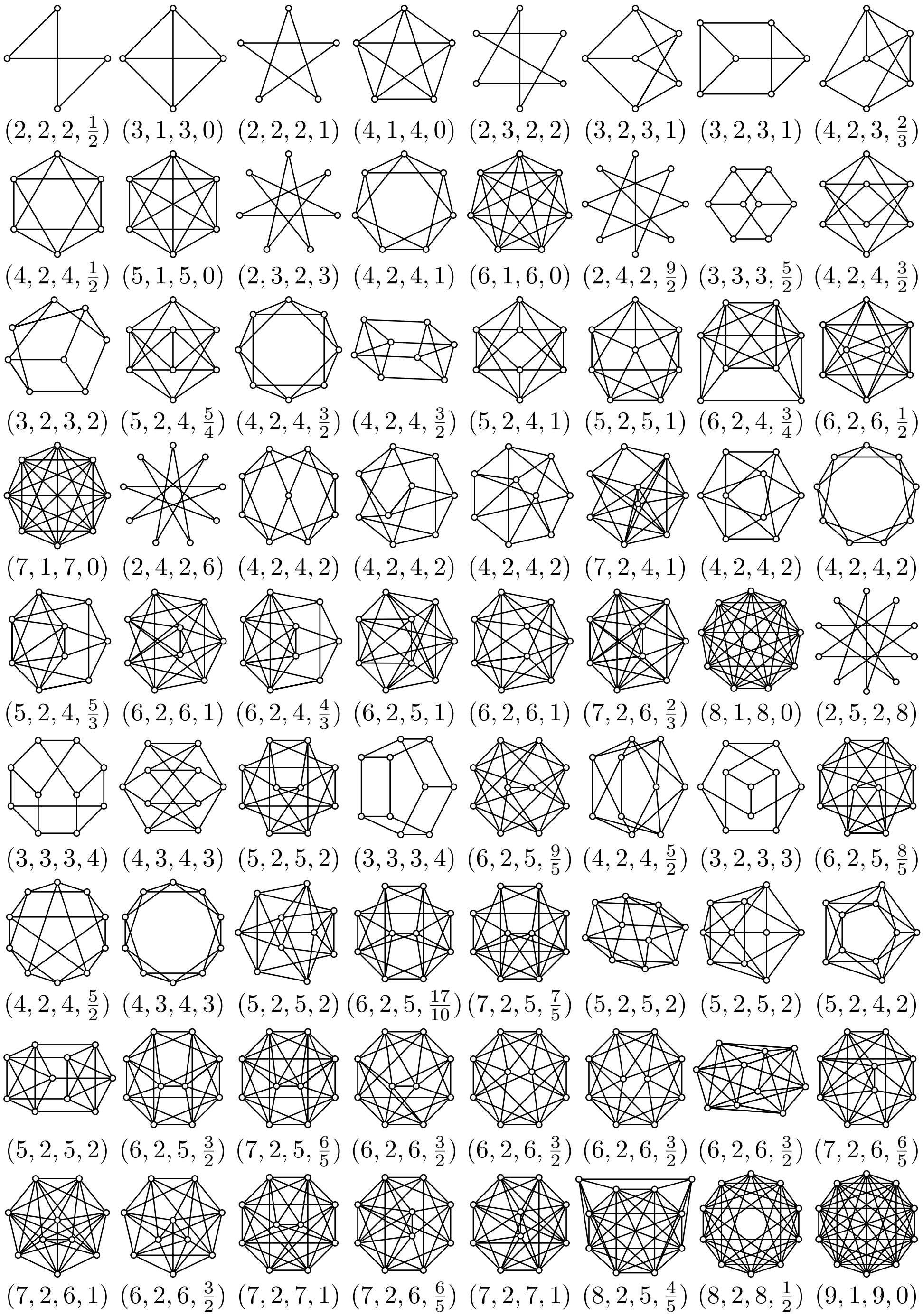

Figure 1. Data for connected betweenness-uniform graphs with 4–10 vertices.

additional information on their maximum degree, diameter, vertex connectivity and betweenness value; all these values support our conjectures stated in Section 1.

Next, we turn our attention to sparse betweenness-uniform graphs. It follows from Theorem 2.4 that such graphs — if distinct from cycles — have minimum degree at least 3, hence, it is natural to consider cubic graphs as candidates for exploring betweenness-uniformity. We have checked all connected cubic graphs up to 20 vertices; despite of large numbers of considered cubic graphs, only 34 of them are betweenness-uniform and, surprisingly, among them, only three are non-transitive (see Figure 2). The lengthy check for 22-vertex cubic graphs revealed that only three of them are betweenness-uniform (see Figure 3).

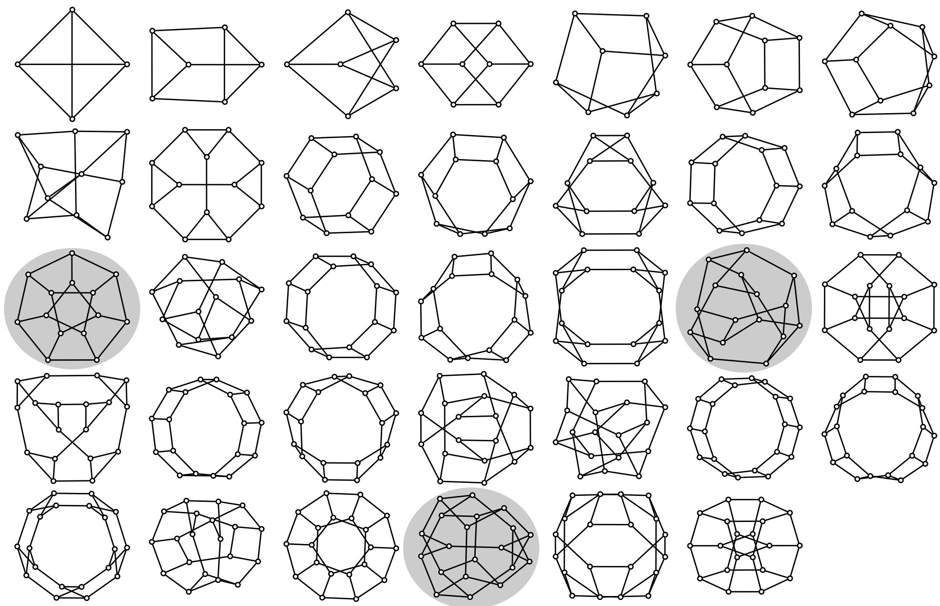

Figure 2. All betweenness-uniform connected cubic graphs up to 20 vertices.

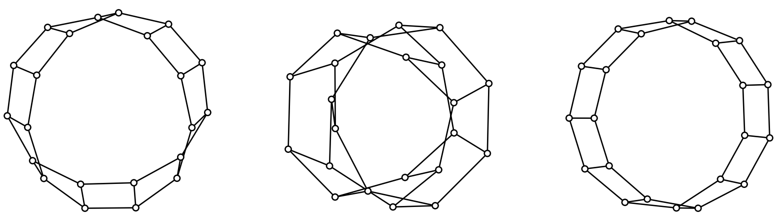

Figure 3. All betweenness-uniform connected cubic graphs on 22 vertices.

Observe that, among these graphs, one can find several generalized Petersen graphs $\mathrm{GP}(n, k)$ (see [13]). This suggests to test for which values n and k the graph $\mathrm{GP}(n, k)$

is betweenness-uniform. Note that the vertex-non-transitivity of $\mathrm{GP}(n,k)$ is easy to detect: by [13], $\mathrm{GP}(n,k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$ or $[n,k] = [10,2]$. However, testing the betweenness-uniformity for all vertex-non-transitive $\mathrm{GP}(n,k)$ with $n \leqslant 500$ revealed that only $\mathrm{GP}(7,2)$ (which is isomorphic to $\mathrm{GP}(7,3)$), $\mathrm{GP}(34,10)$ and $\mathrm{GP}(58,8)$ are betweenness-uniform. All these findings might suggest that cubic non-transitive betweenness-uniform graphs are extremely rare; nevertheless, we show that the betweenness uniformity holds for semi-symmetric graphs (that is, the graphs which are edge-transitive and regular, but not vertex-transitive):

Theorem 2.1. *Every semi-symmetric graph is betweenness-uniform.*

P r o o f. Let G be a semi-symmetric graph of order n. For the purpose of this proof, we use the notion of the edge betweenness centrality defined, for an edge $e = uv$ of G, as the sum $B(e) = \sum\limits_{x,y \in V(G)} \sigma_{x,y}(e)/\sigma_{x,y}$ where $\sigma_{x,y}(e)$ is the number of shortest (x–y)-paths containing the edge e; accordingly, the adjusted betweenness centrality of a vertex u (see [5]) is defined as $c(u) = \sum\limits_{v \in N(u)} B(uv)$. Then the standard and the adjusted betweenness centrality satisfy, by [5], the formula $B(u) = (c(u) - n + 1)/2$. Now, since G is edge-transitive, we get $B(e) = B(f) = b$, for each pair e, f of edges of G. Note that G is also k-regular for some k, so, using the above formula, the betweenness centrality of an arbitrary vertex $u \in V(G)$ is equal to $B(u) = (kb - n + 1)/2$. We can see that $B(u)$ does not depend on the choice of u, thus G is betweenness-uniform. $\square$

By [11], there exist infinitely many cubic semi-symmetric graphs, the smallest one being the Gray graph.

The following auxiliary lemma establishes an upper bound for the arithmetic mean $\overline{B}(G)$ of betweenness centralities of vertices of G:

Lemma 2.2. *Let G be a betweenness-uniform graph on n vertices. Then $\overline{B}(G) \leqslant \overline{B}(C_n)$.*

P r o o f. From [9], we obtain that G is 2-connected. Then, by a result of Plesník [12], the sum of all distances in G does not exceed the sum of all distances in C_n. Hence, for the average distance in G and in C_n, we obtain $\overline{l}(G) \leqslant \overline{l}(C_n)$ which yields $\overline{B}(G) \leqslant \overline{B}(C_n)$ due to the fact that $\overline{B}(G) = (n-1)(\overline{l}(G) - 1)$, see [6]. $\square$

To better understand the structure of the betweenness-uniform graphs, we look at some of their properties. The previous lemma together with the list of all betweenness-uniform graphs up to 10 vertices in [9] indicate that except of the cycles every betweenness-uniform graph should be 3-connected. This conjecture is supported by the following two theorems: we show that if G is a betweenness-uniform graph and G is not a cycle then it has minimum degree at least 3 and is 3-connected if the diameter of G is equal to 2.

Theorem 2.3. *Let G be a betweenness-uniform graph of diameter 2. Then $G \cong C_k$, $k = 4, 5$ or G is 3-connected.*

P r o o f. By contradiction. Let G be a betweenness-uniform graph of vertex-connectivity 2 with $\mathrm{diam}(G) = 2$. Then there exist two vertices $u, v \in V(G)$ such that $G \setminus \{u, v\} = \bigcup_{i=1}^{t} G_i$, $t \geqslant 2$.

Let $U = \{w \in V(G)\colon\ d(w,u) < d(w,v)\}$, $V = \{w \in V(G)\colon\ d(w,v) < d(w,u)\}$ and $S = \{w \in V(G)\colon\ d(w,v) = d(w,u)\}$. Take G_1, G_2 on n_1 and n_2 vertices, and consider three cases based on the cardinality of S.

Case 1: There are two vertices u_1, v_1 where $u_1 \in U \cap V(G_1)$ and $v_1 \in V \cap V(G_1)$. As $\mathrm{diam}(G) = 2$, both u and v are adjacent to every vertex in G_2. Let $x \in V(G_2)$. Then

$$B(x) = \sum_{y,z \in V(G)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} = \sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} \leqslant \sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + 1.$$

Now, for every vertex $y \in V(G_2)$, each $(u_1 - y)$-shortest path passes through u, hence

$$\begin{aligned} B(u) = \sum_{y,z \in V(G)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} &\geqslant \sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} + \sum_{y \in V(G_2)} \frac{\sigma_{u_1,y}(u)}{\sigma_{u_1,y}} \\ &\geqslant \sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} + n_2. \end{aligned}$$

The graph G is betweenness-uniform, so $B(x) = B(u)$, which yields

$$\sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + 1 \geqslant \sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} + n_2 \geqslant \sum_{y,z \in V(G_2)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + n_2.$$

Therefore $n_2 = 1$, $\deg(x) = 2$ and $B(x) = \sigma_{u,v}(x)/\sigma_{u,v} \leqslant 1$. One can see that $n_1 = 2$, or else $B(u) + B(v) \geqslant n_1 + n_2 \geqslant 3 > 2B(x)$ and G is not betweenness-uniform. For $n_1 = 2$, there is exactly one betweenness-uniform graph, namely C_5.

Case 2: Assume that, for every vertex y in $V(G_1)$, $d(u,y) \leqslant d(v,y)$ and there exists a vertex $u_1 \in U \cap V(G_1)$. Again, u is adjacent to every other vertex of $V(G) \setminus \{v\}$

and

$$B(u_1) = \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(u_1)}{\sigma_{y,z}} = \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(u_1)}{\sigma_{y,z}},$$

$$\begin{aligned} B(u) = \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} &\geqslant \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} + \sum_{y\in V(G_2)} \frac{\sigma_{u_1,y}(u)}{\sigma_{u_1,y}} \\ &\geqslant \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} + n_2 \geqslant \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(u_1)}{\sigma_{y,z}} + n_2. \end{aligned}$$

Therefore $n_2 = 0$, a contradiction. The situation where, for each vertex $y \in V(G_1)$, $d(v, y) \leqslant d(u, y)$ holds, leads to the same conclusion.

Case 3: Finally, assume that, for every vertex x in $V(G_1) \cup V(G_2)$, $x \in S$. Without loss of generality, let $x \in V(G_1)$. Then

$$B(x) = \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} = \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} < \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + 1$$

and

$$B(u) = \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} \geqslant \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} + \frac{n_1 n_2}{2} \geqslant \sum_{y,z\in V(G_1)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} + \frac{n_1 n_2}{2}.$$

From the above two inequalities it follows that $1 > n_1 n_2/2$, therefore $n_1 = n_2 = 1$ and G is a cycle on 4 vertices. $\square$

Theorem 2.4. *Let G be a betweenness-uniform graph on $n \geqslant 4$ vertices. Then $G \cong C_n$ or $\delta(G) \geqslant 3$.*

P r o o f. By contradiction. The statement clearly holds for graphs up to 10 vertices, see Figure 1. Let G be a betweenness-uniform graph of order n having a vertex x such that $\deg(x) = 2$ and $N(x) = \{u, v\}$ (let us recall that G is 2-connected). It is easy to see that $uv \notin E(G)$, otherwise $B(x) = 0$ and $G \cong K_3$.

Let $U = \bigcup_{i=1}^{p} U_i$, $V = \bigcup_{i=1}^{p} V_i$ and $S = \bigcup_{i=1}^{p} S_i$ where $p = \operatorname{diam}(G)$ and

$$\begin{aligned} U_i &= \{w \in V(G)\colon\ i = d(w, u) < d(w, v)\}, \quad i = 1, 2, \ldots, p, \\ V_i &= \{w \in V(G)\colon\ i = d(w, v) < d(w, u)\}, \quad i = 1, 2, \ldots, p, \\ S_i &= \{w \in V(G),\ w \neq x\colon\ d(w, u) = d(w, v) = i\}, \quad i = 1, 2, \ldots, p. \end{aligned}$$

In the following we use the following three relations:

$$\begin{aligned}
B(x) &= B(u) = B(v),\\
2B(x) &= B(u) + B(v),\\
B(x) + B(y) &= B(u) + B(v) \geqslant \sum_{\substack{z\in V(G)\\ z\neq x,u,v}} \frac{\sigma_{x,z}(u)}{\sigma_{x,z}} + \sum_{\substack{z\in V(G)\\ z\neq x,u,v}} \frac{\sigma_{x,z}(v)}{\sigma_{x,z}}\\
&= \sum_{\substack{z\in V(G)\\ z\neq x,u,v}} \frac{\sigma_{x,z}(u)+\sigma_{x,z}(v)}{\sigma_{x,z}} = \sum_{\substack{z\in V(G)\\ z\neq x,u,v}} \frac{\sigma_{x,z}}{\sigma_{x,z}} = n-3, \quad y \in V(G).
\end{aligned}$$

Further,

$$\begin{aligned}
B(x) &= \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(x)}{\sigma_{y,z}} = \sum_{\substack{y,z\in V(G)\\ y,z\neq u,v}} \frac{\sigma_{y,z}(u,x,v)}{\sigma_{y,z}} + \sum_{\substack{y\in V(G)\\ y\neq v}} \frac{\sigma_{y,v}(u,x)}{\sigma_{y,v}}\\
&\quad + \sum_{\substack{y\in V(G)\\ y\neq u}} \frac{\sigma_{u,y}(x,v)}{\sigma_{u,y}} + \frac{\sigma_{u,v}(x)}{\sigma_{u,v}}\\
&= A + B + C + \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} \leqslant A + B + C + 1,\\
B(u) &= \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} = \sum_{\substack{y,z\in V(G)\\ y,z\neq u,v}} \frac{\sigma_{y,z}(u,x,v)}{\sigma_{y,z}} + \sum_{\substack{y\in V(G)\\ y\neq u}} \frac{\sigma_{y,v}(u,x)}{\sigma_{y,v}}\\
&\quad + \sum_{\substack{y\in V(G)\\ y\neq u}} \frac{\sigma_{x,y}(u)}{\sigma_{x,y}} + \sum_{y,z\in V(G)\setminus x} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} = A + B + D + F,\\
B(v) &= \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(v)}{\sigma_{y,z}} = \sum_{\substack{y,z\in V(G)\\ y,z\neq u,v}} \frac{\sigma_{y,z}(u,x,v)}{\sigma_{y,z}} + \sum_{\substack{y\in V(G)\\ y\neq v}} \frac{\sigma_{u,y}(x,v)}{\sigma_{u,y}}\\
&\quad + \sum_{\substack{y\in V(G)\\ y\neq v}} \frac{\sigma_{x,y}(v)}{\sigma_{x,y}} + \sum_{y,z\in V(G)\setminus x} \frac{\sigma_{y,z}(v)}{\sigma_{y,z}} = A + C + H + J,
\end{aligned}$$

where $\sigma_{y,z}(w_1, w_2, \ldots, w_k)$ denotes the number of shortest $(y$–$z)$-paths that pass through all of the vertices $w_1, w_2, \ldots, w_k$ and the shortest paths included in the sums F and J of $B(u)$ and $B(v)$ do not pass through x. Moreover, each geodesic $(y$–$v)$-path that goes through u, x is also a geodesic $(y$–$x)$-path and each geodesic $(y$–$u)$-path that passes through v, x is also a geodesic $(y$–$x)$-path. Therefore $D \geqslant B$ and $H \geqslant C$, and it is easy to see that if $D + H > B + C + 2$ then $B(u) + B(v) = 2A+B+C+D+H+F+J > 2A+2B+2C+F+J+2 \geqslant 2(A+B+C+1) \geqslant 2B(x)$. Similarly, if $F > 0$ or $J > 0$ and $D + H \geqslant B + C + 2$ then $B(u) + B(v) > 2B(x)$. In both these cases, G is not betweenness-uniform.

First we show that $|S| < 3$. The pairs of vertices where at least one vertex is from S do not contribute to $B(x)$ but if one vertex is from S and the other is x, then such a pair adds 1 to $B(u) + B(v)$. So, for $|S| \geqslant 3$, we have $D + H \geqslant B + C + 3$, a contradiction.

In the following, we take $|S| \leqslant 2$ and consider two cases based on the eccentricity $e(x)$.

Case 1: Let $e(x) = 2$.

(a) There exists a vertex $w \in S_1$. We calculate the betweenness centrality of $B(w)$ and compare $B(x) + B(w)$ and $B(u) + B(v)$:

$$\begin{aligned}
B(w) = \sum_{y,z\in V(G)} \frac{\sigma_{y,z}(w)}{\sigma_{y,z}} &= \sum_{\substack{y,z\in V(G)\\ y,z\neq u,v}} \frac{\sigma_{y,z}(u,w,v)}{\sigma_{y,z}} + \sum_{\substack{y\in V(G)\\ y\neq u}} \frac{\sigma_{y,v}(u,w)}{\sigma_{y,v}} \\
&+ \sum_{\substack{y\in V(G)\\ y\neq v}} \frac{\sigma_{u,y}(w,v)}{\sigma_{u,y}} + \sum_{y,z\in V(G)\setminus\{u,v\}} \frac{\sigma_{y,z}(w)}{\sigma_{y,z}} + \frac{\sigma_{u,v}(w)}{\sigma_{u,v}} \\
&= A + B + C + \frac{\sigma_{u,v}(w)}{\sigma_{u,v}}.
\end{aligned}$$

Both the vertices x, w have the same value of betweenness centrality, hence

$$\sum_{y,z\in V(G)\setminus\{u,v\}} \frac{\sigma_{y,z}(w)}{\sigma_{y,z}} = 0, \quad \frac{\sigma_{u,v}(w)}{\sigma_{u,v}} = \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} \leqslant \frac{1}{2}$$

and

$$B(w) = B(x) \leqslant A + B + C + \frac{1}{2}.$$

Now $D + H \geqslant n - 3 \geqslant 1$ and

$$\begin{aligned}
B(u) + B(v) &\geqslant 2A + 2B + D + 2A + 2C + H \geqslant 4A + 2B + 2C + n - 3 \\
&\geqslant 2A + 2B + 2C + 1 \geqslant B(x) + B(w) \geqslant n - 3.
\end{aligned}$$

The equality $B(x) + B(w) = B(u) + B(v)$ is achieved if and only if $A = B = C = 0$ and $n \leqslant 4$, a contradiction.

(b) The set S is empty. In this case we get $N(u) \cup N(v) \cup \{u, v\} = V(G)$ and $N(u) \cap N(v) = \{x\}$. Since $e(x) = 2$ and G is 2-connected on greater than or equal to 11 vertices, there exist, without loss of generality, vertices w_1, w_2, w_3 such that $w_1 \in N(u)$, $w_2, w_3 \in N(v)$ and w_1w_2, w_2w_3 are the edges in G (if $w_3 \in N(u)$ then, instead of w_2w_3, we consider the edge w_1w_3). Such vertex w_3 surely exists, otherwise $\sigma_{w_2,w_3}(v)/\sigma_{w_2,w_3} > 0$, i.e., $J > 0$ and $D+H \geqslant B+C+2$. Now $d(w_1, x) = d(w_2, x) = d(w_1, u) + 1 = d(w_2, v) + 1$. If $d(w_3, u) = 2$, then one can see that the pairs (w_3, u),

(w_3, v) do not contribute to $B(x)$ but the pair (w_3, x) gives 1 to $D+H$ which results in $D + H \geqslant B + C + 3$. For $d(w_3, u) = 3$, there are at least two shortest $(w_3–u)$-paths, namely w_3vxu and $w_3w_2w_1u$; thus the contribution of (w_3, u) to C is less than 1 while (w_3, x) contributes 1 to $D+H$. Taking into account that $(w_1, x), (w_2, x)$ contribute 2 to $D + H$, we obtain $D + H > B + C + 2$, a contradiction (which yields $G \cong C_5$).

Case 2: Let $e(x) \geqslant 3$.

(a) There are exactly two vertices $w_1, w_2 \in S$ where pairs (w_1, x), (w_2, x) contribute to $D + H$ together by 2 and $D + H \geqslant B + C + 2$. Now it is enough to show that another pair of vertices (y, z) contributes to $B(u) + B(v)$ more than to $2B(x)$. Note that neither of these two vertices w_1, w_2 is in S_1 otherwise it is a similar situation as in Case 1 (a).

Without loss of generality, choose $w_1 \in S_i$ such that the index i is the smallest possible. Then there are vertices $u_1 \in U_{i-1}$, $v_1 \in V_{i-1}$ such that w_1 is adjacent to both of them. Consider two vertices $u_j \in U_j$, $v_k \in V_k$. If there is an edge u_jv_k, $j = k$ in G then $D+H > B+C+2$ because neither (u_j, y) nor (v_j, y), $y \in U \cup V$ contribute to $B(x)$ but, for $y = x$, both pairs contribute 1 to $D + H$. If $u_jv_k \in E(G)$, $k = j - 1$ or $j = k - 1$ then $u_j \in S \cap U_j$ and $v_k \in S \cap V_k$, respectively, but this cannot occur. Also, there is no edge $u_jv_k \in E(G)$, $k \leqslant j - 2$ or $j \leqslant k - 2$ because of the definition of U_j, V_k. Thus, we obtain that w_1 is adjacent to u_1, v_1 and there exist at least two shortest $(u_1–v)$-paths where at least one of them does not pass through x; hence

$$\frac{\sigma_{u_1,x}(u)}{\sigma_{u_1,x}} = 1 > \frac{\sigma_{u_1,v}(x)}{\sigma_{u_1,v}} = \frac{\sigma_{u_1,v}(u)}{\sigma_{u_1,v}}$$

which implies that $D + H > B + C + 2$.

(b) There is exactly one vertex $w \in S_i$ and $u_1 \in U_{i-1}$, $v_1 \in V_{i-1}$ such that $(u_1, w), (v_1, w) \in E(G)$. It is easy to see that $i \geqslant 2$ otherwise we obtain Case 1 (a).

Moreover, there is no edge $e = (u_j, v_k)$, where $u_j \in U_j$ and $v_k \in V_j$. The reason is the same as in the former Case 2 (a), because the endvertices of such an edge contribute 2 to $B(u) + B(v)$ but $B(x)$ gets 0 from this pair.

Now, there are $k + l$ geodesic $(u_1–v)$-paths, k of them pass through x and the remaining l paths go through w. Let a be a real number (not necessarily positive). We have

$$\frac{\sigma_{u_1,v}(x)}{\sigma_{u_1,v}} = \frac{k}{k+l} = \frac{1}{2} + a \quad \text{and} \quad \frac{\sigma_{u_1,v}(w)}{\sigma_{u_1,v}} = \frac{l}{k+l} = \frac{1}{2} - a.$$

For (v_1, u), we have $t + s$ shortest $(v_1–u)$-paths, t of them go through w and the other s paths pass through x. If Pxv is the shortest $(u_1–v)$-path going through x then v_1wP is also a geodesic $(v_1–u)$-path passing through w, and so $t \geqslant k$; conversely, if u_1wQ is the shortest $(u_1–v)$-path going through v_1 then Qxu is also geodesic $(v_1–u)$-path passing through x which gives $s \leqslant l$ and

$$\frac{\sigma_{v_1,u}(x)}{\sigma_{v_1,u}} = \frac{s}{s+t} \quad \text{and} \quad \frac{\sigma_{v_1,u}(w)}{\sigma_{v_1,u}} = \frac{t}{t+s}.$$

Now, we discuss the following possibilities:

▷ For $\dfrac{1}{k+l} \leqslant \dfrac{1}{s+t}$,

$$\frac{1}{2} + a = \frac{k}{k+l} \leqslant \frac{k}{s+t} \leqslant \frac{t}{s+t} \quad \text{and} \quad \frac{s}{s+t} = 1 - \frac{t}{s+t} \leqslant \frac{1}{2} - a,$$

▷ for $\dfrac{1}{k+l} \geqslant \dfrac{1}{s+t}$,

$$\frac{1}{2} - a = \frac{l}{k+l} \geqslant \frac{l}{s+t} \geqslant \frac{s}{s+t}.$$

As we can see the contribution of the pairs $(u_1, v), (v_1, u)$ to $B + C$ is $k/(k+l) + s/(s+t) \leqslant 1/2 + a + 1/2 - a = 1$ but each of the pairs (u_1, x), (v_1, x) and (w, x) contribute 1 to $D + H$. Consequently, $D + H \geqslant B + C + 2$. This implies that, for $F + J > 0$, G is not betweenness uniform, hence the induced subgraphs $G[U_1]$ and $G[V_1]$ are complete graphs. Further, there is no other vertex y, $y \in U \setminus \{u_1\}$ or $y \in V \setminus \{v_1\}$ such that the pair (y, v) or (u, y) contributes to $B(x)$ or to $B(w)$, respectively, at the same time (any pair with y contributes $h < 1$ to $B + C$ but 1 to $D + H$; it means $2B(x)$ increases by $2h$ but $B(u) + B(v)$ by $1 + h$). Hence $\{x, w\}$ is a cutset. If we consider that $|U| = |V|$ to satisfy $B(u) = B(v)$, we get

$$B(x) + B(w) = \frac{n-2}{2}\frac{n-2}{2} = \frac{n^2 - 4n + 4}{4} = 2B(C_n).$$

This shows that if there is a unique vertex at the same distance from u and v then G is isomorphic to a cycle on n vertices.

(c) S is empty. G is 2-connected, therefore there exists an edge $u_1v_1 \in E(G)$, where $u_1 \in U_i$ and $v_1 \in V_j$. It is easy to check that $i = j$. The pairs (u_1, x), (v_1, x) contribute 2 to $B(u) + B(v)$ and zero to $B(x)$. Further, one can see that there is no other edge u_tv_s, $u_t \in U_t$, $v_s \in V_t$, $(u_t, v_s) \neq (u_1, v_1)$, otherwise $D + H \geqslant B + C + 3$. If there is a vertex $y \in U \cup V$ such that $d(u, y) = d(v, y) + 1$ or $d(v, y) = d(u, y) + 1$, then the contribution of any pair containing y to $B(x)$ is zero and again $D + H$ gets 1 yielding $D + H \geqslant B + C + 3$. Therefore the sets U_j, V_j are empty for all $j > i$. Further, all neighbours of u_1 are in U_{i-1} and all neighbours of v_1 are in V_{i-1}. (If not, then there exists a vertex $y \in U_i \setminus \{u_1\}$ such that (y, v) contributes 1 to D but less than 1 to B due to the fact that there are at least two shortest $(y–v)$-paths and at least one of them passes through w and not through x. The same holds if there is another vertex $y \in V_i \setminus \{v_1\}$. Hence the inequality $D + H \geqslant B + C + 2$ is not preserved.) Moreover, $F = J = 0$, so $G[U_1]$ and $G[V_1]$ are again complete graphs.

Now all pairs (s, x), (s, v) with $s \in U \setminus \{u_1\}$ contribute 2 to $B + D$, and all pairs (t, x), (t, v) $t \in V \setminus \{v_1\}$ contribute 2 to $C + H$. If $|U| \neq |V|$ then $B + D \neq C + H$, hence $B(u) \neq B(v)$ and G is not betweenness-uniform.

If we consider the above findings, we see that $\{x, u_1\}$ as well as $\{x, v_1\}$ are cutsets and $B(x) + B(y) \geqslant (n-3)(n-1)/4 = 2B(C_n)$ for $y \in \{u, v\}$ and n odd. So $B(x) = B(C_n)$ for each vertex x; this gives, according to Lemma 2.2, that G is an n-vertex cycle. □

In [9] it is shown that every n-vertex betweenness-uniform graph G with $\Delta(G) = n-2$ has $\operatorname{diam}(G) = 2$. We prove a similar theorem for $\Delta(G) = n-3$.

Theorem 2.5. *Let G be a betweenness-uniform graph of order $n \geqslant 4$. If $\Delta(G) = n-3$ then* $\operatorname{diam}(G) = 2$.

P r o o f. Let G be an n-vertex betweenness-uniform graph with three vertices $u, x, y \in V(G)$ such that $\deg(u) = n-3$, $N(u) = \{v_1, v_2, \ldots, v_{n-3}\}$ and $x, y \notin N(u)$. For the distances of vertices in G, we have

▷ $d(u, x) = 2$, $d(u, y) = 2$, else $\delta(G) = 1$,

▷ $d(x, v_i) \leqslant 3$, $d(y, v_i) \leqslant 3$, $d(x, y) \leqslant 4$ and $d(v_i, v_j) \leqslant 2$, for each $i, j = 1, 2, \ldots, n-3$.

We discuss several cases:

Case 1: Let $\operatorname{diam}(G) = 4$. It means that $d(x, y) = 4$ and at least one shortest $(x–y)$-path contains u. We show that, in this case, $B(u) > B(x)$:

$$\begin{aligned} B(u) = \sum_{z,w \in V(G)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} &= \sum_{z,w \in N(u)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} + \sum_{z \in N(u)} \frac{\sigma_{x,z}(u)}{\sigma_{x,z}} + \sum_{z \in N(u)} \frac{\sigma_{y,z}(u)}{\sigma_{y,z}} \\ &+ \frac{\sigma_{x,y}(u)}{\sigma_{x,y}} \geqslant \sum_{z,w \in N(u)} \frac{\sigma_{z,w}(x)}{\sigma_{z,w}} + \frac{\sigma_{x,y}(u)}{\sigma_{x,y}} > \sum_{z,w \in V(G)} \frac{\sigma_{z,w}(x)}{\sigma_{z,w}} = B(x). \end{aligned}$$

Case 2: Let $d(x, y) = 3$. In this case x, y do not have a common neighbour, so $N(x) \cap N(y) = \emptyset$ and $N(x) \cup N(y) \subseteq N(u)$. If $B(u) = 0$ then G is a complete graph, which contradicts the assumption $\Delta(G) = n-3$. So $B(u) > 0$ and neither the pair (y,u) contributes to $B(x)$ nor does (x,u) contribute to $B(y)$; therefore

$$\begin{aligned} B(u) = \sum_{z,w\in V(G)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} &\geqslant \sum_{z,w\in N(u)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} \\ &\geqslant \sum_{z,w\in N(x)} \frac{\sigma_{z,w}(x)}{\sigma_{z,w}} + \sum_{z,w\in N(y)} \frac{\sigma_{z,w}(y)}{\sigma_{z,w}} = B(x) + B(y), \end{aligned}$$

so

$$B(u) > B(x).$$

Case 3: There exists a vertex $v \in N(u)$ such that $d(x,v) = 3$, $d(x,y) = 2$, $d(x,z) \leqslant 3$ and $d(y,z) \leqslant 3$ where $z \in N(u) \setminus \{v\}$. Since $N(x) \subset N(u)$ we get $\sigma_{wz}(u) \geqslant \sigma_{wz}(x)$ for all $z,w \in N(x)$. Further, when summing the contributions of pairs of vertices to $B(x)$, we can omit the pair (u,v) because its contribution to $B(x)$ is 0. Hence

$$\begin{aligned} B(x) = \sum_{z,w\in V(G)} \frac{\sigma_{z,w}(x)}{\sigma_{z,w}} &= \sum_{z,w\in N(x)} \frac{\sigma_{z,w}(x)}{\sigma_{z,w}} + \sum_{z\in N(x)} \frac{\sigma_{z,y}(x)}{\sigma_{z,y}} \\ &< \sum_{z,w\in N(u)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} + \sum_{z\in N(u)} \frac{\sigma_{z,y}(u)}{\sigma_{z,y}} + \frac{\sigma_{x,v}(u)}{\sigma_{x,v}} \leqslant B(u), \end{aligned}$$

which yields $B(x) < B(u)$.

Case 4: Similarly, let $\operatorname{diam}(G) = d(x,v) = 3$ and $d(x,y) = 1$ for some $v \in N(u)$. Then $d(y,v) \geqslant 2$ and at least one shortest $(v$–$x)$-path passes through u. According to this, we obtain

$$\begin{aligned} B(v) = \sum_{z,w\in V(G)} \frac{\sigma_{z,w}(v)}{\sigma_{z,w}} &= \sum_{z,w\in N(v)} \frac{\sigma_{z,w}(v)}{\sigma_{z,w}} + \sum_{z\in N(v)} \frac{\sigma_{z,y}(v)}{\sigma_{z,y}} < \sum_{z,w\in N(u)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} \\ &+ \sum_{z\in N(u)} \frac{\sigma_{z,y}(u)}{\sigma_{z,y}} + \frac{\sigma_{x,v}(u)}{\sigma_{x,v}} \leqslant \sum_{z,w\in V(G)} \frac{\sigma_{z,w}(u)}{\sigma_{z,w}} = B(u) \end{aligned}$$

and

$$B(v) < B(u).$$

$\square$

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