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GENERALIZED OUTERPLANAR INDEX OF A GRAPH

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Abstract. We define the generalized outerplanar index of a graph and give a full characterization of graphs with respect to this index.

Keywords: generalized outerplanar graph; iterated line graph; planar index; outerplanar index

MSC 2010: 05C10, 05C76

1. Introduction

A generalized outerplanar graph is a planar graph which can be embedded in the plane in such a way that at least one end-vertex of each edge lies on the external face. In [4], it was shown that G is generalized outerplanar if and only if no subgraph of G is homeomorphic to any of the graphs in Figure 1.

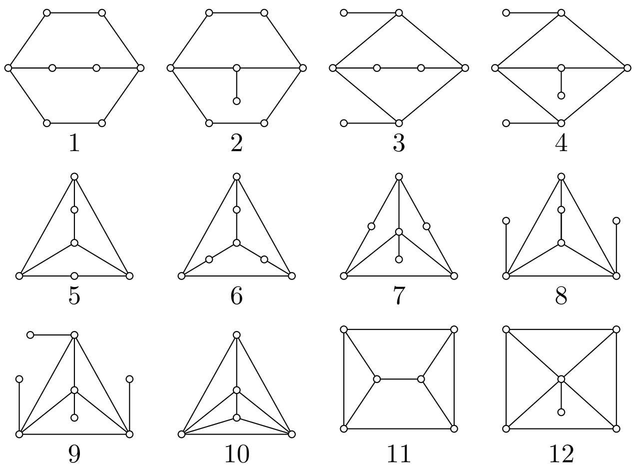

Figure 1.

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Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the class of all outerplanar graphs and the class of all generalized outerplanar graphs, respectively. Also, let $L(G)$ stand for the line graph of G. In [5], Sedláček characterized all graphs whose line graph is generalized outerplanar.

Theorem 1.1 ([5], Theorem 3)**.** *Let G be a graph. Then the following three statements are equivalent:*

(1) $L(G) \in \mathcal{A}_2$.

(2) *G has no subgraph homeomorphic to any of the seven graphs shown in Figure* 2.

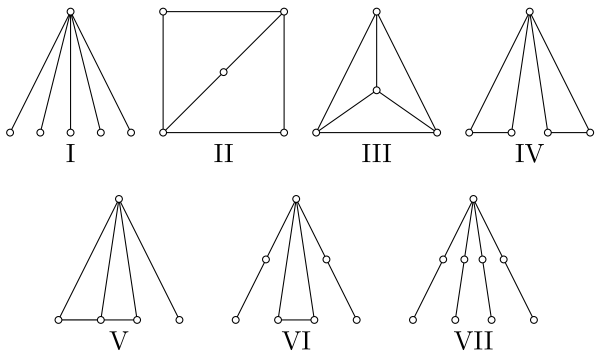

Figure 2.

(3) *The following two conditions hold:*

  (i) $G \in \mathcal{A}_1$.

  (ii) *The degree of each vertex is at most four, each vertex c of degree four is a cut-vertex, for every such c there are at least two bridges incident with c, and at least one of them is an end-bridge.*

We denote the kth iterated line graph of G by $L^k(G)$, where $L^k(G) = L(L^{k-1}(G))$, $L^0(G) = G$ and $L^1(G) = L(G)$. In [2], Ghebleh et al. defined the planar index for a graph G as follows: the planar index of a graph G is the smallest k such that $L^k(G)$ is non-planar and this index equals to infinity if $L^k(G)$ is planar for all $k \geqslant 0$. We denote the planar index of G by $\zeta(G)$. The authors gave a full characterization of all graphs with respect to their planar index; it is stated in the following theorem.

Theorem 1.2 ([2], Theorem 10)**.** *Let G be a connected graph. Then:*

(i) $\xi(G) = 0$ *if and only if G is non-planar.*

(ii) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or $K_{1,3}$.*

(iii) $\xi(G) = 1$ *if and only if G is planar and either $\Delta(G) \geqslant 5$ or G has a vertex of degree 4 and it is not a cut-vertex.*

(iv) $\xi(G) = 2$ *if and only if $L(G)$ is planar and G contains one of the graphs E_i in Figure 3 as a subgraph.*

(v) $\xi(G) = 4$ *if and only if* G *is one of the graphs* X_k *or* Y_k *(Figure 3) for some* $k \geqslant 2$.

(vi) $\xi(G) = 3$ *otherwise.*

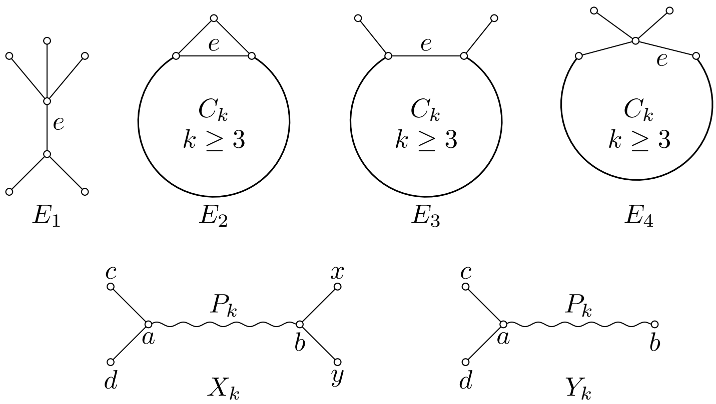

Figure 3.

Also, in [3], the authors present a definition for the outerplanar index of a graph. For a given graph G, the outerplanar index of G is the smallest k such that $L^k(G)$ is non-outerplanar and this index equals to infinity if $L^k(G)$ is outerplanar for all $k \geqslant 0$. In this paper, we denote the outerplanar index of G by $\xi(G)$. In [3], the authors gave a full characterization of all graphs with respect to their outerplanarity index.

Theorem 1.3 ([3], Theorem 3.4)**.** *Let* G *be a connected graph. Then:*

(a) $\zeta(G) = 0$ *if and only if* G *is non-outerplanar.*

(b) $\zeta(G) = \infty$ *if and only if* G *is a path, a cycle, or* $K_{1,3}$.

(c) $\zeta(G) = 1$ *if and only if* G *is planar and* G *has a subgraph homeomorphic to* $K_{1,4}$ *or* $K_1 + P_3$ *in Figure 4.*

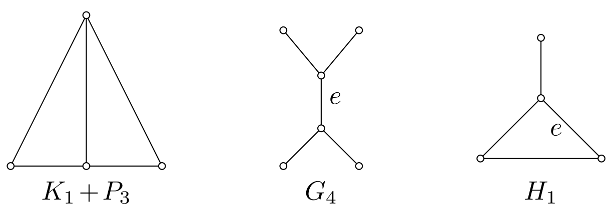

Figure 4.

(d) $\zeta(G) = 2$ *if and only if* $L(G)$ *is planar and* G *has a subgraph isomorphic to one of the graphs* G_4 *and* H_1 *in Figure 4.*

(e) $\zeta(G) = 3$ *if and only if* G *is the graph which is drawn in Figure* 5.

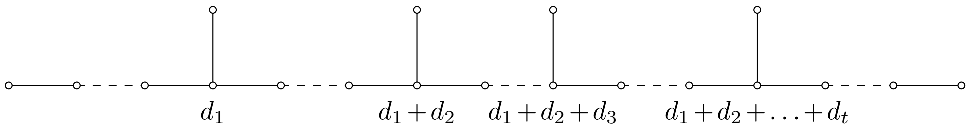

Figure 5. $I(d_1, d_2, \ldots, d_t)$, where $d_i \geqslant 2$ for $i = 2, \ldots, t-1$, and $d_1 \geqslant 1$.

In the following definition, we define the generalized outerplanar index of a graph.

Definition 1.4. The generalized outerplanar index of a graph G is the smallest k such that $L^k(G)$ is not generalized outerplanar. We denote the generalized outerplanar index of G by $\gamma(G)$. If $L^k(G)$ is generalized outerplanar for all $k \geqslant 0$, we define $\gamma(G) = \infty$.

By Theorem 1.1, it is easy to see that if G is not generalized outerplanar, then $L(G)$ is not generalized outerplanar. This implies that if G is generalized outerplanar, then

$$\gamma(G) = 1 + \max\{k;\ L^k(G) \text{ is generalized outerplanar}\}.$$

In the next section, we consider the problem of the generalized outerplanarity of iterated line graphs of a graph G. We show that the generalized outerplanar index of a graph G is either infinite or at most 4. Moreover, all graphs are completely characterized with respect to this index.

In this paper, all graphs are finite and simple and we use the standard terminology of graphs as in [6]. In order to make this paper easier to follow, we recall here the various notions from graph theory which will be used in the sequel. Let G be a graph. Then the *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges of G incident to v. The graph G is said to be *connected* if there exists a path between any two distinct vertices and otherwise it is *disconnected.* A *connected component* is a maximal connected subgraph of G. For a positive integer r, an r*-partite* graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in the same subset. A *complete* r*-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A *cut vertex* v is any vertex whose removal increases the number of connected components of G. A *bridge* is an edge e of G whose removal increases the number of connected components of G. Also, an *end-bridge* is a bridge uv of G where one of the vertices u and v has degree 1.

2. Generalized outerplanar index of iterated line graphs

By the definition, the following lemma is clear; however, since it is used several times throughout this paper, we should state it.

Lemma 2.1. *If H is a subgraph of G, then $\gamma(G) \leqslant \gamma(H)$.*

By Lemma 2.1, the generalized outerplanar index of a graph is the minimum of the generalized outerplanar indexes of its connected components. Therefore, from now on we may assume that all graphs are connected.

By the definitions of planar and generalized outerplanar index, it is clear that $\gamma(G) \leqslant \zeta(G)$. Using this fact, we can state the following lemma.

Lemma 2.2. *If G is a graph with $\Delta(G) \geqslant 4$, then $\gamma(G) \leqslant 3$.*

Proof. It was proved that if G is a graph with $\Delta(G) \geqslant 4$, then $\zeta(G) \leqslant 3$ (see [2], Lemma 5). Now, since $\gamma(G) \leqslant \zeta(G)$, we are done. $\square$

Proposition 2.3. *Let G be a connected graph. Then $\gamma(G) = \infty$ if and only if G is a path, cycle or $K_{1,3}$.*

Proof. Let P_n and C_n denote, respectively, the path and cycle on n vertices and let P_0 be the empty graph. It is easy to see that $L(P_n) = P_{n-1}$ for $n \geqslant 1$, $L(P_0) = P_0$, $L(C_n) = C_n$ and $L(K_{1,3}) = C_3$. Therefore if G is a path, a cycle or $K_{1,3}$, then $\gamma(G) = \infty$.

Conversely, assume that G is neither a path, a cycle, nor $K_{1,3}$. Since G is connected and it is not a path or a cycle, we have that $\Delta(G) \geqslant 3$. If $\Delta(G) \geqslant 4$, by Lemma 2.2, we can conclude that $\gamma(G) \leqslant 3$. If $\Delta(G) = 3$, then it has H_1 or H_2 (Figure 6) as a subgraph, since G is not $K_{1,3}$.

Figure 6.

It may be easily verified that $\gamma(H_1) = 3$ and $\gamma(H_2) = 4$. So, by Lemma 2.1, $\gamma(G) \leqslant 4$. $\square$

Chartrand et al. characterized all graphs whose line graph is outerplanar. We need to use this characterization in the proof of the following theorem, so we just state it here.

Theorem 2.4 ([1]). *The line graph $L(G)$ of a graph G is outerplanar if and only if $\Delta(G) \leqslant 3$, and if $\deg(v) = 3$ for a vertex v of G, then v is a cut vertex.*

Theorem 2.5. *Let G be a connected graph. Then $\gamma(G) = 2$ if and only if $L(G)$ is generalized outerplanar and G has a homeomorphic subgraph to one of the five graphs shown in Figure 7.*

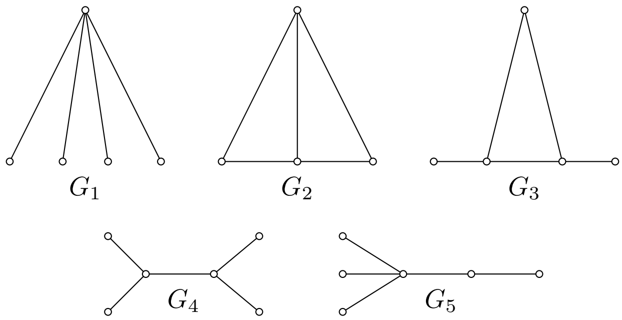

Figure 7.

P r o o f. Suppose that $\gamma(G) = 2$. Then, by the definition, G and $L(G)$ are generalized outerplanar. So G and $L(G)$ are planar. Also, $L^2(G)$ is not generalized outerplanar. Now, by Theorem 1.1, we have the following cases:

Case 1. $L(G) \notin \mathcal{A}_1$. Since $L(G) \in \mathcal{A}_2$, then we have that $G \in \mathcal{A}_1$. Thus $\xi(G) = 1$. So, by part (c) of Theorem 1.3, G is planar and G has a subgraph homeomorphic to G_1 or G_2 in Figure 7.

Case 2. $L(G) \in \mathcal{A}_1$. With Theorem 2.4, we have that $\Delta(G) \leqslant 3$, and if $\deg(v) = 3$ for a vertex v, then v is a cut vertex. Since $\Delta(G) \leqslant 3$, then $\Delta(L(G)) \leqslant 4$. Now, since $L^2(G) \notin \mathcal{A}_2$, we have the following subcases:

2-1. There is a vertex e in $L(G)$ which has degree 4 and is not a cut vertex. Therefore e is an edge of G and it is not a bridge. Hence, G has a cycle containing e. Since e has degree 4 in $L(G)$, G can contain a subgraph which is homeomorphic to either G_2, G_3 or H_3 in Figure 8. But $\Delta(G) \leqslant 3$, so G just can have G_2 or G_3.

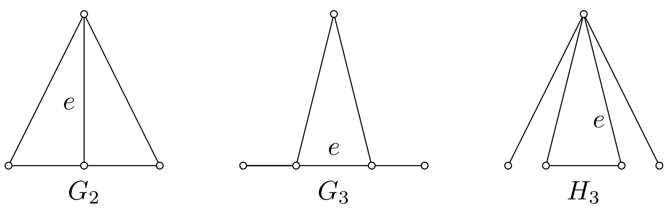

Figure 8.

2-2. There is a vertex e in $L(G)$ which has degree 4 and is a cut vertex but there is at most one bridge incident with e, or none of the bridges is an end-bridge. In this case, $L(G)$ can have a subgragh homeomorphic to IV, V, VI or VII in Figure 2.

It is easy to see that if $L(G)$ has asubgraph homeomorphic to IV and V, then G has a subgraph homeomorphic to G_4 and G_5, respectively (Figure 9). Also, it is not hard to see that $L(G)$ can not have a subgraph homeomorphic to VI or VII.

Figure 9.

To prove the converse statement, at first we calculate the generalized outerplanar index of G_1, G_2, G_3, G_4 and G_5. It is easy to see that the line graph of G_1 has II as a subgraph and the line graph of the graphs G_2, G_3, G_4 and G_5 has IV as a subgraph. So, by Theorem 1.1, $L(L(G_i))$ is not generalized outerplanar for all $1 \leqslant i \leqslant 5$. Therefore, by Lemma 2.1 and our hypothesis, we can conclude that $\gamma(G) = 2$. □

In the following proposition, we study the case when $\gamma(G) = 3$.

Proposition 2.6. *For a connected graph G, $\gamma(G) = 3$ if and only if one of the following conditions hold:*

(i) *$L^2(G)$ is generalized outerplanar and G has a subgraph homeomorphic to H_1 (Figure 4).*

(ii) *G is the graph which is drawn in Figure 5.*

Proof. Firstly, let us assume that $\gamma(G) = 3$. So, by definition, G, $L(G)$ and $L^2(G)$ are generalized outerplanar and $L^3(G)$ is not generalized outerplanar. Since $L^2(G)$ is generalized outerplanar, we have that $L(G)$ is outerplanar. Thus $\xi(G) \geqslant 2$. Thus we have the following cases:

Case 1. $\xi(G) = 2$. In this case, by part (d) of Theorem 1.3, we have that $L(G)$ is outerplanar and G has a subgraph homeomorphic to one of the graphs G_4 and H_1 in Figure 4. As we mentioned before, it is easy to see that $\gamma(G_4) = 2$ and $\gamma(H_1) = 3$. Thus, if G has a subgraph homeomorphic to G_4, then $\gamma(G) \leqslant 2$, which is a contradiction. Hence, G can only have a subgraph homeomorphic to H_1.

Case 2. $\xi(G) = 3$. So, by part (e) of Theorem 1.3, we can conclude that G is the graph which is drawn in Figure 5.

Converse statement follows easily. □

Let G be the complete bipartite $K_{1,3}$ with the partite sets $\{a\}$ and $\{b, c, d\}$. Let k be a positive integer. We define Y_k to be the graph obtained from G by subdividing the edge ab to a path of length k. We define X_k to be the graph obtained by adding two new vertices x and y to Y_k and joining them to b. These two families of graphs are drawn in Figure 10.

In the following proposition, we investigate the case when $\gamma(G)=4$.

Proposition 2.7. *For a connected graph G, $\gamma(G)=4$ if and only if G is one of the graphs X_k or Y_k with $k\geqslant 3$ (Figure 10).*

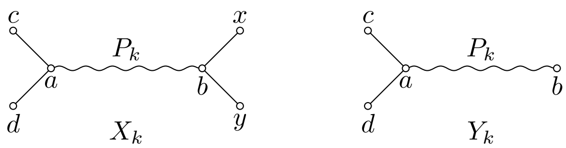

Figure 10.

P r o o f. Suppose that $\gamma(G)=4$. So, $\zeta(G)=4$. Now, by part (v) of Theorem 1.2, G is one of the graphs X_k and Y_k, where $k\geqslant 2$. By Figure 11, one can easily check that $\gamma(X_2)=\gamma(Y_2)=3$. Since $\gamma(G)=4$, we have that G is one of the graphs X_k or Y_k with $k\geqslant 3$.

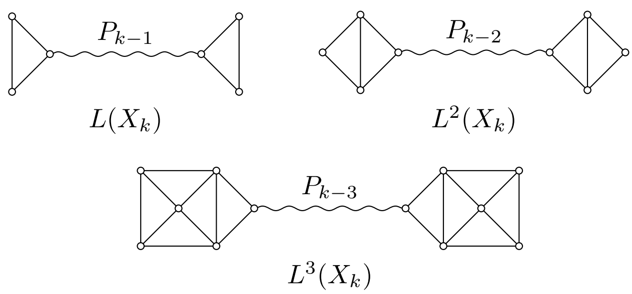

Figure 11.

By Figure 11, the converse statement follows easily. □

The following theorem presents the summary of results of this section, giving a full characterization of graphs with respect to their generalized outerplanar index.

Theorem 2.8. *Let G be a connected graph. Then:*

(1) *$\gamma(G)=0$ if and only if G has a subgraph homeomorphic to one of the twelve graphs shown in Figure 1.*
(2) *$\gamma(G)=\infty$ if and only if G is a path, cycle or $K_{1,3}$.*
(3) *$\gamma(G)=1$ if and only if G is generalized outerplanar and it has a subgraph homeomorphic to one of the seven graphs shown in Figure 2.*
(4) *$\gamma(G)=2$ if and only if $L(G)$ is generalized outerplanar and G has a homeomorphic subgraph from one of the five graphs shown in Figure 7.*
(5) *$\gamma(G)=3$ if and only if one of the following conditions hold:*

(i) $L^2(G)$ *is generalized outerplanar and* G *has a subgraph homeomorphic to* H_1 *(Figure* 4*).*

(ii) G *is the graph which is drawn in Figure* 5*.*

(6) $\gamma(G) = 4$ *if and only if* G *is one of the graphs* X_k *or* Y_k *with* $k \geqslant 3$ *(Figure* 10*).*

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