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ON THE MAXIMAL RUN-LENGTH FUNCTION IN THE LÜROTH EXPANSION

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Abstract. We obtain a metrical property on the asymptotic behaviour of the maximal run-length function in the Lüroth expansion. We also determine the Hausdorff dimension of a class of exceptional sets of points whose maximal run-length function has sub-linear growth rate.

Keywords: Lüroth expansion; run-length function; Hausdorff dimension

MSC 2010: 11K55, 28A80

1. Introduction

The Lüroth expansion was introduced by Lüroth in [9] in 1883. Consider the Lüroth transformation $T\colon (0,1]\to(0,1]$ defined by

$$T(x) := d_1(x)(d_1(x)-1)\Big(x-\frac{1}{d_1(x)}\Big),$$

where $d_1(x)=[1/x]+1$ and $[\cdot]$ denotes the integer part function. Then every $x\in(0,1]$ has the Lüroth expansion

$$x=\frac{1}{d_1(x)}+\frac{1}{d_1(x)(d_1(x)-1)d_2(x)}+\dots+\frac{1}{d_1(x)(d_1(x)-1)d_2(x)\dots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)}+\dots,\qquad (1.1)$$

where the digits $d_n(x)\geqslant 2$ are positive integers and are defined by $d_n(x)=d_1(T^{n-1}x)$ for all $n\geqslant 1$. Lüroth showed that every irrational number has a unique infinite

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expansion of the form (1.1) and each rational number has either a finite or an infinite periodic expansion. For each $x \in (0,1]$, let $\{d_n(x)\}_{n\geqslant 1}$ be the digits sequence in the Lüroth expansion of x. Here, and in what follows, we denote the Lüroth expansion of x by

$$x = [d_1(x), d_2(x), \ldots].$$

Now we use an example to illustrate the algorithm of the Lüroth transformation. Let $x = \sqrt{2} - 1$, expanding this irrational number as its Lüroth expansion, we find

$$d_1(x) = \left[\frac{1}{x}\right] + 1 = [\sqrt{2}+1] + 1 = 3, \quad Tx = 3\cdot 2\Big(x - \frac{1}{3}\Big) = 6x - 2 = 6\sqrt{2} - 8,$$
$$d_2(x) = \left[\frac{1}{Tx}\right] + 1 = 3, \quad T^2x = 6\cdot Tx - 2 = 36\sqrt{2} - 50,$$
$$d_3(x) = 2, \quad T^3x = 72\sqrt{2} - 101, \quad d_4(x) = 2, \quad \ldots,$$

and then

$$\sqrt{2} - 1 = [3, 3, 2, 2, \ldots].$$

We call

$$l_n(x) = \max_{i\geqslant 2}\{k\colon\ d_{j+1}(x) = \ldots = d_{j+k}(x) = i \text{ for some } j,\ 0 \leqslant j \leqslant n-k\}$$

the nth maximal run-length function of x, which represents the longest run of the same symbol in the first n digits of x.

In this paper, we first study the asymptotic behaviour of the maximal run-length function in the Lüroth expansion from the global measure theoretic point of view. We obtain a large number law for $l_n(x)$.

Theorem 1.1. *For almost all* $x \in (0,1]$,

$$\lim_{n\to\infty} \frac{l_n(x)}{\log_2 n} = 1. \tag{1.2}$$

Our second objective is to determine the Hausdorff dimension of the exceptional set of numbers which violate the above metrical property. We show that the corresponding exceptional set is of Hausdorff dimension 1. This follows from the following more general result. Let $\varphi\colon\ \mathbb{N} \to \mathbb{R}^+$ be an increasing function for each pair of numbers $\alpha, \beta \in [0,\infty]$ with $\alpha \leqslant \beta$, define

$$E^{\varphi}_{\alpha,\beta} = \Big\{x \in (0,1]\colon\ \liminf_{n\to\infty} \frac{l_n(x)}{\varphi(n)} = \alpha,\ \limsup_{n\to\infty} \frac{l_n(x)}{\varphi(n)} = \beta\Big\}.$$

Theorem 1.2. *Let* $\varphi\colon \mathbb{N} \to \mathbb{R}^+$ *be a monotonically increasing function satisfying* $\lim_{n\to\infty} \varphi(n) = \infty$ *and* $\lim_{n\to\infty} (\varphi(n+1) - \varphi(n)) = 0$. *For any* $0 \leqslant \alpha \leqslant \beta \leqslant \infty$, *we have*

$$\dim_{\mathrm{H}} E^{\varphi}_{\alpha,\beta} = 1,$$

where $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension.*

Example 1.3. There are several typical examples of φ which fulfill the assumptions of Theorem 1.2:

▷ $\varphi(n) = \log_2 n$;
▷ $\varphi(n) = n^{\gamma}$, $0 < \gamma < 1$;
▷ $\varphi(n) = \dfrac{n}{(\log n)^{\tau}}$, $\tau > 0$.

Remark 1.4. The condition $\lim_{n\to\infty} (\varphi(n+1) - \varphi(n)) = 0$ on φ in Theorem 1.2 cannot be weakened by requiring $\liminf_{n\to\infty} \varphi(n)n^{-1} = 0$ or even $\lim_{n\to\infty} \varphi(n)n^{-1} = 0$ as in [7], [8] for the case $0 < \alpha \leqslant \beta < \infty$. For example, if we take $\varphi(n) = 2^{2^k}$ when $2^{2^{2^k}} \leqslant n < 2^{2^{2^{k+1}}}$ for $k \geqslant 0$, then $\lim_{n\to\infty} \varphi(n)n^{-1} = 0$, but one can verify directly $E^{\varphi}_{\alpha,\beta} = \emptyset$ whenever $0 < \alpha \leqslant \beta < \infty$ by the fact that $0 \leqslant l_{n+1}(x) - l_n(x) \leqslant 1$ for any $x \in [0,1)$.

Analogous problems were first considered for the dyadic expansion. For any $x \in [0,1)$, let $r_n(x)$ be the dyadic run-length function of x, namely, the longest run of 1's in the first n digits of the terminating dyadic expansion of x. In a pioneering work, Erdős and Rényi [2] proved that for almost all $x \in (0,1]$,

$$\lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = 1.$$

The Hausdorff dimension of various exceptional sets of points whose dyadic run-length functions obey other asymptotic behaviour instead of $\log_2 n$ were studied in [10], [15], [7], [8], [13]. A result similar to Theorem 1.1 also holds in continued fraction expansion, see [14]. For more details about run-length function and Hausdorff dimension, one can refer to the books [12], [3] and references therein. The generalized Lüroth expansion dynamical system and several other concrete examples of number-theoretic dynamical systems were used to illuminate various aspects of infinite ergodic theory by Kesseböhmer, Munday and Stratmann in their recent book [6]; they also used these dynamical systems to analyze some explicit questions to illustrate not only the powerful methods from the infinite ergodic theory but also the strong connection between the infinite ergodic theory and the renewal theory.

2. Preliminaries

This section is devoted to some elementary properties and dimensional results of Lüroth expansion that will be used later. For a wealth of classical results about Lüroth expansion and Hausdorff dimension, see the books by Galambos [4] and Falconer [3], respectively.

2.1. Elementary properties.

Lemma 2.1 ([4]). *The series expansion in* (1.1) *is the Lüroth expansion of some* $x \in (0,1]$ *if and only if* $d_n(x) \geqslant 2$ *for all* $n \geqslant 1$.

Lemma 2.2 ([1]). *The random variable sequence* $\{d_n(x)\}_{n\geqslant 1}$ *is independent and identically distributed.*

Let $\Sigma = \{2, 3, 4, \ldots\}$. For any $n \in \mathbb{N}$ and $(d_1, \ldots, d_n) \in \Sigma^n$, we call

$$I_n(d_1, \ldots, d_n) = \{x \in (0,1] \colon\ d_i(x) = d_i \text{ for } 1 \leqslant i \leqslant n\}$$

a cylinder set of order n. The $I_n(d_1, \ldots, d_n)$ represents the set of numbers in $(0,1]$ which have the Lüroth expansion beginning with $d_1, \ldots, d_n$.

Lemma 2.3 ([4]). *For any* $(d_1, \ldots, d_n) \in \Sigma^n$, $n \in \mathbb{N}$, $I_n(d_1, \ldots, d_n)$ *is the interval with endpoints*

$$\sum_{i=1}^{n} \frac{1}{d_1(d_1-1)\ldots d_{i-1}(d_{i-1}-1)d_i}$$

and

$$\sum_{i=1}^{n} \frac{1}{d_1(d_1-1)\ldots d_{i-1}(d_{i-1}-1)d_i} + \frac{1}{d_1(d_1-1)d_2(d_2-1)\ldots d_n(d_n-1)}.$$

Thus

$$|I_n(d_1, d_2, \ldots, d_n)| = \frac{1}{d_1(d_1-1)d_2(d_2-1)\ldots d_n(d_n-1)},$$

where $|I|$ *denotes the length of the interval* I.

The next lemma and figure (see Figure 1) describe the positions of cylinders I_{n+1} of order $n+1$ inside the nth order cylinder I_n.

Lemma 2.4 ([4]). *Let* $I_n = I_n(d_1, \ldots, d_n)$ *be a cylinder of order* n, *which is partitioned into sub-cylinders* $\{I_{n+1}(d_1, \ldots, d_n, d_{n+1}) \colon\ d_{n+1} \in \Sigma\}$. *These sub-cylinders are positioned from right to left, as* d_{n+1} *increases from* 2 *to* ∞.

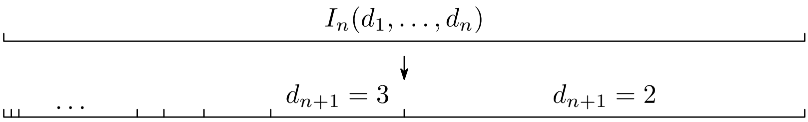

Figure 1. Distribution of cylinders.

2.2. Dimensional results. We first recall the following lemma, which is an important tool to obtain a lower bound of the Hausdorff dimension.

Lemma 2.5 ([3]). *Let $E \in \mathbb{R}$ and let $f\colon E \to \mathbb{R}$ be an η-Hölder function, i.e., for any $x, y \in E$,*

$$|f(x) - f(y)| \ll |x-y|^{\eta},$$

where the constant in $\ll$ is an absolute constant. Then

$$\dim_{\mathrm{H}} f(E) \leqslant \frac{1}{\eta} \dim_{\mathrm{H}} E.$$

Lemma 2.6. *For any $M \in \Sigma$, let*

$$E_M = \{x \in (0,1]\colon\ 2 \leqslant d_i(x) \leqslant M \text{ for all } i \geqslant 1\}.$$

Then $\dim_{\mathrm{H}} E_M = s_M$, where s_M is given by the Moran formula

$$\sum_{k=2}^{M} \Big(\frac{1}{k(k-1)}\Big)^{s_M} = 1.$$

P r o o f. For any $M \geqslant 2$, we notice that the Cantor set E_M is the attractor of the similitudes $\{f_k(x) = x(k(k-1))^{-1} + k^{-1}\}_{k=2}^{M}$ with contraction ratios $\{(k(k-1))^{-1}\}_{k=2}^{M}$. Then Lemma 2.6 follows from the classical dimensional result of Moran [11] on self-similar sets (see also [3], [5]). □

By simple calculation, we have

Lemma 2.7. $\lim\limits_{M\to\infty} s_M = 1$.

Now we prove an elementary result which is similar to Lemma 4 in [10].

Lemma 2.8. *Given a set of positive integers $\mathcal{J} = \{j_1 < j_2 < \ldots\}$ and an infinite bounded sequence $\{b_i\}_{i\geqslant 1}$ with $2 \leqslant b_i \leqslant B$ for some $B \in \mathbb{N}$, let*

$$E(\mathcal{J}, \{b_i\}) = \{x = [d_1(x), d_2(x), \ldots] \in (0,1]\colon\ d_i(x) = b_i,\ \forall\, i \in \mathcal{J}\}.$$

If the density of $\mathcal{J}$ is zero, that is,

$$\lim_{n\to\infty} \frac{\#\{i \leqslant n\colon\ i \in \mathcal{J}\}}{n} = 0,$$

then $\dim_{\mathrm{H}} E(\mathcal{J}, \{b_i\}) = 1$, where $\#$ denotes the number of elements in a set.

P r o o f. The main idea of the proof is showing that $E(\mathcal{J},\{b_i\})$ contains subsets with Hausdorff dimensions converging to one. Let

$$E_M(\mathcal{J},\{b_i\}) = \{x = [d_1(x), d_2(x), \ldots] \in (0,1]\colon\ d_i(x) = b_i,\ \forall\, i \in \mathcal{J} \\ \text{and } 2 \leqslant d_i(x) \leqslant M \text{ for } i \notin \mathcal{J}\}.$$

Clearly $E_M(\mathcal{J},\{b_i\}) \subset E(\mathcal{J},\{b_i\})$. We will construct an $(1-\varepsilon)^{-1}$-Hölder function f from $E_M(\mathcal{J},\{b_i\})$ to E_M. By Lemma 2.5, this means that $\dim_{\rm H} E_M(\mathcal{J},\{b_i\}) \geqslant (1-\varepsilon)\dim_{\rm H} E_M$ and thus by Lemmas 2.6 and 2.7, $\dim_{\rm H} E(\mathcal{J},\{b_i\}) \geqslant 1-\varepsilon$ as we desired.

Define $f\colon E_M(\mathcal{J},\{b_i\}) \to E_M$ as follows. For any $x = [d_1(x), d_2(x), \ldots] \in E_M(\mathcal{J},\{b_i\})$, let

$$f(x) = \bar{x} = \lim_{n\to\infty} \overline{[d_1(x), \ldots, d_n(x)]},$$

where the Lüroth expansion $\overline{[d_1(x), \ldots, d_n(x)]}$ is obtained by eliminating the terms $d_i(x)$ with $i \in \mathcal{J}$ in the first n-digits of the Lüroth expansion $[d_1(x), \ldots, d_n(x)]$ of x.

Let $M > B$ and $\varepsilon > 0$. It is clear from the construction $f(E_M(\mathcal{J},\{b_i\})) = E_M$ for f. Now we show that f is an $(1-\varepsilon)^{-1}$-Hölder function. Let $t(n) = \#\{i \leqslant n\colon i \in \mathcal{J}\}$, choose N such that $2^{n\varepsilon} > (M(M-1))^{t(n)}$ for all $n \geqslant N$, this is possible since $\lim\limits_{n\to\infty} t(n)n^{-1} = 0$. For any $x, y \in E_M(\mathcal{J},\{b_i\})$ with $|x-y| < (M(M-1))^{-N}$ and $x \neq y$, let n be the greatest integer such that x, y are contained in the same cylinder of order n. By Lemma 2.3, we see that $n \geqslant N$ and $n+1 \notin \mathcal{J}$. Without loss of generality, we assume $x > y$. Then there exist $2 \leqslant d_1, \ldots, d_n \leqslant M$ and $2 \leqslant \tau_{n+1} < \sigma_{n+1} \leqslant M$ such that $x \in I_{n+1}(d_1, \ldots, d_n, \tau_{n+1})$, $y \in I_{n+1}(d_1, \ldots, d_n, \sigma_{n+1})$. From the construction of $E_M(\mathcal{J},\{b_i\})$ and the distribution of cylinders (Figure 1), we notice that $|x-y|$ is greater than the length of the cylinder $I_{n+2}(d_1, \ldots, d_n, \tau_{n+1}, M+1)$, that is

$$\begin{aligned} |x-y| &\geqslant |I_{n+2}(d_1, \ldots, d_n, \tau_{n+1}, M+1)| \\ &\geqslant \frac{1}{(M+1)M}\,\frac{1}{M(M-1)}|I_n(d_1, \ldots, d_n)| \geqslant \frac{1}{M^4}|I_n(d_1, \ldots, d_n)|. \end{aligned} \tag{2.1}$$

On the other hand, we have $\bar{x}, \overline{y} \in I_{n-t(n)}(\overline{[d_1(x), \ldots, d_n(x)]})$, i.e.,

$$d_j(\bar{x}) = d_j(\overline{y}), \quad 1 \leqslant j \leqslant n - t(n).$$

Thus

$$\begin{aligned} |f(x) - f(y)| = |\bar{x} - \overline{y}| &\leqslant |I_{n-t(n)}(\overline{[d_1(x), \ldots, d_n(x)]})| \\ &\leqslant (M(M-1))^{t(n)}|I_n(d_1, \ldots, d_n)| \\ &\leqslant 2^{n\varepsilon}|I_n(d_1, \ldots, d_n)| \leqslant |I_n(d_1, \ldots, d_n)|^{1-\varepsilon}. \end{aligned} \tag{2.2}$$

Combined with (2.1), this yields the Hölder continuity exponent of f. The proof of Lemma 2.8 is complete. □

An application of the above lemma leads to the following technical result.

Lemma 2.9. *Let $\{m_k\}_{k\geqslant 1}$, $\{n_k\}_{k\geqslant 1}$ be two increasing sequence of natural numbers satisfying the following conditions:*

(1) $\exists\, K \geqslant 1$, $n_k < m_k < n_{k+1}$ *for any* $k \geqslant K$;

(2) $\lim\limits_{k\to\infty}(m_k - n_k) = \infty$;

(3) $\lim\limits_{k\to\infty}\dfrac{m_k - n_k}{m_k} = 0$;

(4) $\lim\limits_{k\to\infty}\dfrac{m_k - n_k}{n_{k+1} - n_k} = 0.$

For $k \geqslant K$, let t_k be the largest integer such that $m_k + t_k(m_k - n_k) < n_{k+1}$. Define a set of positive integers $\mathcal{D}$ and an infinite sequence $\{a_i\}_{i\geqslant 1}$ as follows:

$$\begin{aligned}\mathcal{D} := \mathcal{D}(\{m_k\},\{n_k\}) = \{1,2,\ldots,n_K - 1\} \cup \bigcup_{k\geqslant K}\{&n_k, n_k+1,\ldots,m_k-1,m_k,\\ &m_k+(m_k-n_k)-1,\ldots,m_k+(t_k-1)(m_k-n_k)-1, m_k+t_k(m_k-n_k)-1,\\ &m_k+(m_k-n_k),\ldots,m_k+(t_k-1)(m_k-n_k), m_k+t_k(m_k-n_k)\}.\end{aligned}$$

For $1 \leqslant i < n_K$, set

$$a_i = 3.$$

For $k \geqslant K$, set

$$a_{n_k} = 3, \quad a_{n_k+1} = \ldots = a_{m_k-1} = 2, \quad a_{m_k} = 3;$$

$$a_{m_k+(m_k-n_k)-1} = a_{m_k+2(m_k-n_k)-1} = \ldots = a_{m_k+t_k(m_k-n_k)-1} = 2;$$

and

$$a_{m_k+(m_k-n_k)} = a_{m_k+2(m_k-n_k)} = \ldots = a_{m_k+t_k(m_k-n_k)} = 3.$$

Then the set

$$E(\mathcal{D},\{a_i\}) = \{x = [d_1(x), d_2(x),\ldots] \in (0,1]\colon\ d_i(x) = a_i,\ \forall\, i \in \mathcal{D}\}.$$

is of Hausdorff dimension 1.

P r o o f. We need only to show that the density of $\mathcal{D}$ is zero. For any $n \geqslant n_{K+1}$, there exists $k \geqslant K+1$ such that $n_k \leqslant n < n_{k+1}$, and

▷ if $n_k \leqslant n \leqslant m_k$, then

$$\#\{i \leqslant n,\ i \in \mathcal{D}\} = n_K + \sum_{j=K}^{k-1}[(m_j - n_j + 1) + 2t_j] + n - n_k;$$

▷ if $m_k + t(m_k - n_k) \leqslant n < m_k + (t+1)(m_k - n_k)$ for some $0 \leqslant t \leqslant t_k - 1$, then

$$\#\{i \leqslant n,\ i \in \mathcal{D}\} \leqslant n_K + \sum_{j=K}^{k-1} [(m_j - n_j + 1) + 2t_j] + m_k - n_k + 2t + 1;$$

▷ if $m_k + t_k(m_k - n_k) \leqslant n < n_{k+1}$, then

$$\#\{i \leqslant n,\ i \in \mathcal{D}\} = n_K + \sum_{j=K}^{k-1} [(m_j - n_j + 1) + 2t_j] + m_k - n_k + 2t_k.$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \#\{i \leqslant n,\ i \in \mathcal{D}\} \\
&\quad \leqslant \limsup_{k\to\infty} \max_{0 \leqslant t \leqslant t_k} \frac{n_K + \sum_{j=K}^{k-1} [(m_j - n_j + 1) + 2t_j] + m_k - n_k + 2t + 1}{m_k + t(m_k - n_k)} \\
&\quad \leqslant \limsup_{k\to\infty} \left\{ \frac{n_K + \sum_{j=K}^{k-1} [(m_j - n_j + 1) + 2t_j] + m_k - n_k + 1}{m_k} + \frac{2}{m_k - n_k} \right\} \\
&\quad = \limsup_{k\to\infty} \frac{\sum_{j=K}^{k-1} [(m_j - n_j + 1) + 2t_j]}{m_k} \leqslant \limsup_{k\to\infty} \frac{m_k - n_k + 1 + 2t_k}{m_{k+1} - m_k} = 0.
\end{aligned}$$

Therefore, by Lemma 2.8, we have $\dim_{\mathrm{H}} E(\mathcal{D}, \{a_i\}) = 1$. □

3. Proof of Theorem 1.1

In this section, we will give the details of the proof of Theorem 1.1. Our argument utilizes ideas presented in [12], [14].

Proof of Theorem 1.1. We prove (1.2) in two stages. First, for any $\varepsilon > 0$, we establish that

$$\limsup_{n\to\infty} \frac{l_n(x)}{\log_2 n} \leqslant 1 + \varepsilon, \quad \text{a.e.} \tag{3.1}$$

From this inequality we instantly get

$$\limsup_{n\to\infty} \frac{l_n(x)}{\log_2 n} \leqslant 1, \quad \text{a.e.},$$

which means that we need to obtain for any $\varepsilon > 0$

$$\liminf_{n\to\infty} \frac{l_n(x)}{\log_2 n} \geqslant 1 - \varepsilon, \quad \text{a.e.} \tag{3.2}$$

to complete the proof.

For any $a > 0$ and $n > m \geqslant 0$, we put

$$A(a,n) = \{x \in (0,1]\colon\ l_n(x) > a\log_2 n\},$$
$$B(a,n) = \{x \in (0,1]\colon\ l_n(x) < a\log_2 n\},$$

and

$$l_{m,n}(x) = l_{n-m}(d_{m+1}(x),\dots,d_n(x)),$$

which represents the longest run of the same symbol in the first $n-m$ digits in the Lüroth expansion of $T^m x$. Before proceeding with the proof, we show the following lemma.

Lemma 3.1. *For any $a > 0$ and all large $n > \max\{2^{1/a}, 4a^2\}$, we have*

$$\mu(A(a,n)) \leqslant \frac{8}{n^{a-1}}, \tag{3.3}$$

and

$$\mu(B(a,n)) \leqslant \mathrm{e}^{-n^{1-a}/(2a\log_2 n)}, \tag{3.4}$$

where μ denotes the 1-dimensional Lebesgue measure.

P r o o f. For $n > 2^{1/a}$, by Lemmas 2.2 and 2.3 we have

$$\begin{aligned}
\mu(A(a,n)) &= \sum_{k>a\log_2 n} \mu(\{x\in(0,1]\colon\ l_n(x)=k\})\\
&\leqslant \sum_{k>a\log_2 n}\sum_{i=2}^{\infty}\sum_{j=0}^{n-k}\mu\{x\in(0,1]\colon\ d_{j+1}(x)=\ldots=d_{j+k}(x)=i\}\\
&= \sum_{k>a\log_2 n}\sum_{i=2}^{\infty}\sum_{j=0}^{n-k}\mu\{x\in(0,1]\colon\ d_1(x)=\ldots=d_k(x)=i\}\\
&\leqslant \sum_{k>a\log_2 n}\sum_{i=2}^{\infty} n\Big(\frac{1}{i(i-1)}\Big)^k = n\sum_{i=2}^{\infty}\sum_{k>a\log_2 n}\Big(\frac{1}{i(i-1)}\Big)^k\\
&\leqslant n\sum_{i=2}^{\infty}\frac{\big(\frac{1}{i(i-1)}\big)^{a\log_2 n}}{1-\frac{1}{i(i-1)}} \leqslant 2n\sum_{k=1}^{\infty}\sum_{i=2^k}^{2^{k+1}-1}\Big(\frac{1}{i(i-1)}\Big)^{a\log_2 n}\\
&\leqslant 2n\sum_{k=1}^{\infty}2^k\Big(\frac{1}{2^k(2^k-1)}\Big)^{a\log_2 n} \leqslant 4n\sum_{k=1}^{\infty}\Big(\frac{1}{2^k}\Big)^{a\log_2 n} \leqslant \frac{8}{n^{a-1}},
\end{aligned}$$

which yields (3.3).

Denote $u_n = [a \log_2 n]$ and $k_n = [n/u_n]$. For $n > 4a^2$, we note that $k_n \geqslant 2$ and

$$\{x \in (0,1]\colon\ l_n(x) \leqslant u_n\} \subset \{x \in (0,1]\colon\ l_{iu_n+1,(i+1)u_n}(x) \leqslant u_n,\ \forall\, 0 \leqslant i < k_n\}. \tag{3.5}$$

Hence, by (3.5) and Lemmas 2.2 and 2.3,

$$\begin{aligned}
\mu(B(a,n)) &\leqslant \mu(\{x \in (0,1]\colon\ l_n(x) \leqslant u_n\}) \\
&= (\mu\{x \in (0,1]\colon\ l_{u_n}(x) \leqslant u_n\})^{k_n} \\
&= \Big(1 - \sum_{i=2}^{\infty} \mu(I_{u_n}(i,i,\ldots,i))\Big)^{k_n} \\
&= \Big(1 - \sum_{i=2}^{\infty} \Big(\frac{1}{i(i-1)}\Big)^{u_n}\Big)^{k_n} \\
&\leqslant \Big(1 - \Big(\frac{1}{2}\Big)^{u_n}\Big)^{k_n} \leqslant \mathrm{e}^{-k_n t 2^{u_n}} \leqslant \mathrm{e}^{-n^{1-a}/(2a\log_2 n)},
\end{aligned}$$

so that the required (3.4) follows. □

We now consider $\mu(A(a,n))$ for

$$a = 1 + \varepsilon, \quad n = 2^k, \quad k \geqslant 1.$$

We have

$$\mu(\{x \in (0,1]\colon\ l_{2^k}(x) > (1+\varepsilon)k\}) \leqslant \frac{8}{2^{k\varepsilon}}.$$

Hence the set on which $l_{2^k}(x) > (1+\varepsilon)k$ has infinitely many solutions has Lebesgue measure zero by the Borel-Cantelli lemma, that is for a.e. $x \in (0,1]$, $l_{2^k}(x) \leqslant (1+\varepsilon)k$ ultimately. Thus, for a.e. $x \in (0,1]$,

$$\limsup \frac{l_n(x)}{\log_2 n} \leqslant \limsup \frac{l_{2^{k+1}}(x)}{\log_2 2^k} = \limsup \frac{l_{2^{k+1}}(x)}{k+1} \leqslant 1 + \varepsilon.$$

This establishes (4.5).

For any $0 < \varepsilon < 1$, we consider $\mu(B(a,n))$ for $a = 1 - \varepsilon$. Then for all large n,

$$\mu(\{x \in (0,1]\colon\ l_n(x) < (1-\varepsilon)\log_2 n\}) \leqslant \mathrm{e}^{-n^{\varepsilon}/(2a\log_2 n)} \leqslant \mathrm{e}^{-n^{\varepsilon/2}}.$$

Again by the Borel-Cantelli lemma, we have for a.e. $x \in (0,1]$, $l_n(x) \geqslant (1-\varepsilon)\log_2 n$ ultimately. It follows that

$$\liminf_{n\to\infty} \frac{l_n(x)}{\log_2 n} \geqslant 1 - \varepsilon \quad \text{a.e.},$$

which completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

P r o o f of Theorem 1.2. Assume without loss of generality that $\varphi(n) \geqslant 1$ and $\varphi(n+1)-\varphi(n) \leqslant 1$ for any $n \geqslant 1$. The proof relies on the application of Lemma 2.9 by constructing proper sequences $\{m_k\}_{k\geqslant 1}$, $\{n_k\}_{k\geqslant 1}$, and verifying that the corresponding $E(\mathcal{D}, \{a_i\})$ is a subset of $E^{\varphi}_{\alpha,\beta}$. We divide the whole proof into two parts: a detailed proof for the case $0<\alpha\leqslant\beta<\infty$ and a sketch of proof for the remaining cases.

Case 1: $0<\alpha<\beta<\infty$, let the sequence $\{n_k\}_{k\geqslant 1}$ be defined by

$$n_k = \max\Big\{n\colon\ \varphi(n) < \Big(\frac{\beta}{\alpha}\Big)^k\Big\},$$

then

$$\Big(\frac{\beta}{\alpha}\Big)^k - 1 \leqslant \varphi(n_k) < \Big(\frac{\beta}{\alpha}\Big)^k. \tag{4.1}$$

Take $m_k = n_k + [\beta\varphi(n_k)]$. Clearly, $\{n_k\}_{k\geqslant 1}$ increases super-exponentially, since $\lim\limits_{n\to\infty}(\varphi(n+1)-\varphi(n)) = 0$. One can verify that $\{m_k\}_{k\geqslant 1}$ and $\{n_k\}_{k\geqslant 1}$ satisfy the conditions of Lemma 2.9. Let $K \geqslant 1$ be such that $n_k < m_k < n_{k+1}$ for any $k \geqslant K$ and $\varphi(n+1)-\varphi(n) < 1/\beta$ for any $n \geqslant n_K$. We first prove the following two facts.

Fact 1:

$$\lim_{k\to\infty}\frac{\varphi(m_k)}{\varphi(n_k)} = 1. \tag{4.2}$$

To obtain this, we note that

$$\begin{aligned}
\varphi(m_k) &= \varphi(n_k + [\beta\varphi(n_k)])\\
&= \varphi(n_k) + \varphi(n_k+1) - \varphi(n_k) + \ldots + \varphi(n_k + [\beta\varphi(n_k)]) - \varphi(n_k + [\beta\varphi(n_k)] - 1)\\
&\leqslant \varphi(n_k) + [\beta\varphi(n_k)] \max_{n_k\leqslant n<m_k}\{\varphi(n+1)-\varphi(n)\}.
\end{aligned}$$

Thus, by the monotonicity of $\varphi(n)$,

$$1 \leqslant \frac{\varphi(m_k)}{\varphi(n_k)} \leqslant 1 + \beta \max_{n_k\leqslant n<m_k}\{\varphi(n+1)-\varphi(n)\}.$$

This establishes (4.2) by the squeeze theorem.

Fact 2: The function $(n-n_k)/\varphi(n)$ is monotonically increasing from $n_k + m_{k-1} - n_{k-1}$ to $m_k - 1$ for $k \geqslant K+1$.

For any $k \geqslant K+1$ and $n_k + m_{k-1} - n_{k-1} \leqslant n \leqslant m_k - 1$, we have

$$(n - n_k)(\varphi(n+1) - \varphi(n)) < \frac{1}{\beta}\, \beta\varphi(n_k) = \varphi(n_k) < \varphi(n),$$

which is equivalent to

$$\frac{n - n_k}{\varphi(n)} < \frac{n+1-n_k}{\varphi(n+1)}. \tag{4.3}$$

For $k \geqslant K$, let t_k be the largest integer such that $m_k + t_k(m_k - n_k) < n_{k+1}$. Let us recall the definition of

$$E(\mathcal{D}, \{a_i\}) = \{x = [d_1(x), d_2(x), \ldots] \in (0,1]\colon\ d_i(x) = a_i,\ \forall\, i \in \mathcal{D}\},$$

where

$$\begin{aligned}\mathcal{D} := \mathcal{D}(\{m_k\},\{n_k\}) = \{1,2,\ldots,n_K - 1\} \cup \bigcup_{k\geqslant K} \{&n_k, n_k+1, \ldots, m_k - 1, m_k,\\ &m_k + (m_k - n_k) - 1, \ldots, m_k + (t_k - 1)(m_k - n_k) - 1, m_k + t_k(m_k - n_k) - 1,\\ &m_k + (m_k - n_k), \ldots, m_k + (t_k - 1)(m_k - n_k), m_k + t_k(m_k - n_k)\}\end{aligned}$$

and

$$a_i = \begin{cases} 3 & \text{if } 1 \leqslant i < n_K;\\ 3 & \text{if } i = n_k, m_k, m_k + t(m_k - n_k) \text{ for some } t,\ 1 \leqslant t \leqslant t_k,\ k \geqslant K;\\ 2 & \text{if } n_k + 1 \leqslant i \leqslant m_k - 1 \text{ for some }\ k \geqslant K;\\ 2 & \text{if } i = m_k + t(m_k - n_k) - 1 \text{ for some } t,\ 1 \leqslant t \leqslant t_k,\ k \geqslant K.\end{cases}$$

Now we prove that $E(\mathcal{D}, \{a_i\}) \subset E^{\varphi}_{\alpha,\beta}$. Fix $x \in E(\mathcal{D}, \{a_i\})$ for any $n \geqslant n_{K+1}$, let k be the integer such that $n_k \leqslant n < n_{k+1}$. From the construction of the set $E(\mathcal{D}, \{a_i\})$, we see that

$$l_n(x) = \begin{cases} m_{k-1} - n_{k-1} - 1 = [\beta\varphi(n_{k-1})] - 1 & \text{if } n_k \leqslant n \leqslant n_k + m_{k-1} - n_{k-1} - 1,\\ n - n_k & \text{if } n_k + m_{k-1} - n_{k-1} \leqslant n \leqslant m_k - 1,\\ m_k - n_k - 1 = [\beta\varphi(n_k)] - 1 & \text{if } m_k \leqslant n < n_{k+1}.\end{cases}$$

Thus, by (4.1), (4.2) and (4.3),

$$\begin{aligned}\liminf_{n\to\infty} \frac{l_n(x)}{\varphi(n)} &= \liminf_{k\to\infty} \min\Big\{\frac{l_{n_k + m_{k-1} - n_{k-1} - 1}(x)}{\varphi(n_k + m_{k-1} - n_{k-1} - 1)}, \frac{l_{n_{k+1}-1}(x)}{\varphi(n_{k+1} - 1)}\Big\}\\ &= \liminf_{k\to\infty} \min\Big\{\frac{[\beta\varphi(n_{k-1})] - 1}{\varphi(n_k + [\beta\varphi(n_{k-1})] - 1)}, \frac{[\beta\varphi(n_k)] - 1}{\varphi(n_{k+1} - 1)}\Big\}\\ &= \liminf_{k\to\infty} \min\Big\{\frac{[\beta\varphi(n_{k-1})] - 1}{\varphi(n_k)}, \frac{[\beta\varphi(n_k)] - 1}{\varphi(n_{k+1})}\Big\}\\ &= \alpha,\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\frac{l_n(x)}{\varphi(n)} &= \limsup_{k\to\infty}\max\Big\{\frac{l_{n_k}(x)}{\varphi(n_k)},\frac{l_{m_k-1}(x)}{\varphi(m_k-1)}\Big\}\\ &= \limsup_{k\to\infty}\max\Big\{\frac{[\beta\varphi(n_{k-1})]-1}{\varphi(n_k)},\frac{[\beta\varphi(n_k)]-1}{\varphi(m_k)}\Big\}\\ &= \beta.\end{aligned}$$

Hence $x\in E^{\varphi}_{\alpha,\beta}$. Therefore, by Lemma 2.9, we have $\dim_{\mathrm{H}} E^{\varphi}_{\alpha,\beta}=1$.

Case 2: $0<\alpha=\beta<\infty$, let

$$\varepsilon(n)=\inf_{k\geqslant n}\Big(\max\Big\{\varphi(k+1)-\varphi(k),\frac{\varphi(k)}{k}\Big\}\Big)^{-1/2}. \tag{4.4}$$

Since $\varphi(n)$ is monotonically increasing and $\varphi(n)\to\infty$, $\varphi(n+1)-\varphi(n)\to 0$ as $n\to\infty$, it follows that $\varepsilon(n)\to\infty$ as $n\to\infty$ and the infimum in the definition of $\varepsilon(n)$ is achieved.

Take $n_1=2$, $n_{k+1}=n_k+[\varepsilon(n_k)\varphi(n_k)]$ and $m_k=n_k+[\beta\varphi(n_k)]$ for $k\geqslant 1$. There is no difficulty in proving that the sequences $\{m_k\}_{k\geqslant1}$ and $\{n_k\}_{k\geqslant1}$ as defined above satisfy the conditions of Lemma 2.9. The proof of the monotonicity of $(n-n_k)/\varphi(n)$ from $n_k+m_{k-1}-n_{k-1}$ to m_k-1 is the same as that of (4.3) and the next fact should be compared with (4.2).

Fact:

$$\lim_{k\to\infty}\frac{\varphi(n_{k+1})}{\varphi(n_k)}=1. \tag{4.5}$$

Clearly, the formula

$$\begin{aligned}\varphi(n_{k+1}) &= \varphi(n_k+[\varepsilon(n_k)\varphi(n_k)])\\ &\leqslant \varphi(n_k)+[\varepsilon(n_k)\varphi(n_k)]\max_{n_k\leqslant n<n_{k+1}}\{\varphi(n+1)-\varphi(n)\}\\ &\leqslant \varphi(n_k)+\varepsilon(n_k)\varphi(n_k)\max_{n\geqslant n_k}\{\varphi(n+1)-\varphi(n)\}\\ &\leqslant \varphi(n_k)(1+\varepsilon(n_k)^{-1})\end{aligned}$$

implies (4.5).

Recall the notation of $E(\mathcal{D},\{a_i\})$ in Lemma 2.9. For any $x\in E(\mathcal{D},\{a_i\})$, we have

$$l_n(x)=\begin{cases} m_{k-1}-n_{k-1}-1=[\beta\varphi(n_{k-1})]-1 & \text{if } n_k\leqslant n\leqslant n_k+m_{k-1}-n_{k-1}-1,\\ n-n_k & \text{if } n_k+m_{k-1}-n_{k-1}\leqslant n\leqslant m_k-1,\\ m_k-n_k-1=[\beta\varphi(n_k)]-1 & \text{if } m_k\leqslant n<n_{k+1}.\end{cases}$$

Thus, by (4.5),

$$\begin{aligned}\liminf_{n\to\infty}\frac{l_n(x)}{\varphi(n)} &= \liminf_{k\to\infty}\min\Big\{\frac{l_{n_k+m_{k-1}-n_{k-1}-1}(x)}{\varphi(n_k+m_{k-1}-n_{k-1}-1)},\frac{l_{n_{k+1}-1}(x)}{\varphi(n_{k+1}-1)}\Big\}\\ &= \liminf_{k\to\infty}\min\Big\{\frac{[\beta\varphi(n_{k-1})]-1}{\varphi(n_k+[\beta\varphi(n_{k-1})]-1)},\frac{[\beta\varphi(n_k)]-1}{\varphi(n_{k+1}-1)}\Big\}\\ &= \liminf_{k\to\infty}\min\Big\{\frac{[\beta\varphi(n_{k-1})]-1}{\varphi(n_k)},\frac{[\beta\varphi(n_k)]-1}{\varphi(n_{k+1})}\Big\}\\ &= \beta,\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\frac{l_n(x)}{\varphi(n)} &= \limsup_{k\to\infty}\max\Big\{\frac{l_{n_k}(x)}{\varphi(n_k)},\frac{l_{m_k-1}(x)}{\varphi(m_k-1)}\Big\}\\ &= \limsup_{k\to\infty}\max\Big\{\frac{[\beta\varphi(n_{k-1})]-1}{\varphi(n_k)},\frac{[\beta\varphi(n_k)]-1}{\varphi(m_k)}\Big\}\\ &= \beta.\end{aligned}$$

We have thus proved that $x\in E^{\varphi}_{\alpha,\beta}$ for the case $0<\alpha=\beta<\infty$. Hence $E(\mathcal{D},\{a_i\})\subset E^{\varphi}_{\alpha,\beta}$ and $\dim_{\rm H} E^{\varphi}_{\alpha,\beta}=1$.

Since the proof for the remaining cases is similar to that of the cases $0<\alpha<\beta<\infty$ and $0<\alpha=\beta<\infty$, we will only give the constructions of the proper sequences $\{m_k\}_{k\geqslant 1}$ and $\{n_k\}_{k\geqslant 1}$. One can verify that the corresponding $\mathcal{D}(\{m_k\},\{n_k\})$ is of density zero and $E(\mathcal{D},\{a_i\})$ is a subset of $E^{\varphi}_{\alpha,\beta}$ for different cases.

Case 3: $0=\alpha<\beta<\infty$, take $n_k=\max\{n\colon \varphi(n)<2^{2^k}\}$ and $m_k=n_k+[\beta\varphi(n_k)]$ for $k\geqslant 1$.

Case 4: $0<\alpha<\beta=\infty$, take $n_1=2$, $n_{k+1}=\max\{n\colon \varphi(n)<\varepsilon(n_k)\varphi(n_k)\}$ and $m_k=n_k+[\alpha\varepsilon(n_k)\varphi(n_k)]$ for $k\geqslant 1$.

Case 5: $\alpha=\beta=0$, take $n_1=2$, $n_{k+1}=n_k+[\varepsilon(n_k)\varphi(n_k)]$ and $m_k=n_k+[\log\varphi(n_k)]$ for $k\geqslant 1$.

Case 6: $\alpha=\beta=\infty$, take $n_1=2$, $n_{k+1}=n_k+[\varepsilon(n_k)\varphi(n_k)]$ and $m_k=n_k+\big[\sqrt{\varepsilon(n_k)}\varphi(n_k)\big]$ for $k\geqslant 1$. □

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