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CHARACTER CONNES AMENABILITY OF DUAL BANACH ALGEBRAS

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Abstract. We study the notion of character Connes amenability of dual Banach algebras and show that if A is an Arens regular Banach algebra, then A^{**} is character Connes amenable if and only if A is character amenable, which will resolve positively Runde's problem for this concept of amenability. We then characterize character Connes amenability of various dual Banach algebras related to locally compact groups. We also investigate character Connes amenability of Lau product and module extension of Banach algebras. These help us to give examples of dual Banach algebras which are not Connes amenable.

Keywords: dual Banach algebra; Connes amenability; character amenability; locally compact group

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1. INTRODUCTION AND PRELIMINARIES

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then the dual X^* of X is a dual Banach A-bimodule whose actions are given by $(f\cdot a)(x) = f(a\cdot x)$ and $(a\cdot f)(x) = f(x\cdot a)$ $(a \in A,\ x \in X,\ f \in X^*)$. A derivation $D\colon A \to X$ is a linear map such that $D(ab) = D(a)\cdot b + a\cdot D(b)$ for all $a, b \in A$. For $x \in X$, define $d_x\colon A \to X$ by $d_x(a) = a\cdot x - x\cdot a$ for all $a \in A$. Then d_x is a derivation; these derivations are called inner derivations. A Banach algebra A is called amenable if for every Banach A-bimodule X every continuous derivation $D\colon A \to X^*$ is an inner derivation; i.e., $H^1(A, X^*) = \{0\}$. This concept was first introduced by Johnson in [10]. He proved that a locally compact group G is amenable if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra.

In [11], Johnson, Kadison and Ringrose introduced a notion of amenability for von Neumann algebras, which modified Johnson's original definition for Banach algebras in the sense that it takes the dual space structure of a von Neumann algebra into

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account. This notion of amenability was later called Connes amenability by Helemskiĭ in [8]. Runde in [19] extended the notion of Connes amenability to the larger class of dual Banach algebras and studied certain concrete Banach algebras in the subsequent papers [21], [22] and [23]. Dual Banach algebras form a special class of Banach algebras which includes all von Neumann algebras, the Banach algebra $B(E)$ of all bounded operators on a reflexive Banach space E, the measure algebra $M(G)$, the Fourier-Stieltjes algebra $B(G)$, and the second dual A^{**} of an Arens regular Banach algebra A. A Banach algebra A is called a dual Banach algebra if there exists a closed submodule A_* of A^* such that $A = (A_*)^*$. One can see that a Banach algebra that is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous. A dual Banach A-bimodule X is called normal if, for each $x \in X$, the maps $a \to a \cdot x$ and $b \to x \cdot b$ from A into X are w^*-continuous, see [19]. A dual Banach algebra A is called Connes amenable if for every normal dual Banach A-bimodule X, every w^*-continuous derivation $D\colon A \to X$ is inner; i.e., $H^1_{w^*}(A, X) = \{0\}$, see [19]. Several characterizations and modifications of Connes amenability have been described by many authors, see, for example, [2], [7], [21], [23]. In [19], Runde investigated how, for an Arens regular Banach algebra A, the amenability of A and the Connes amenability of A^{**} are related. Indeed, he showed that if A is an Arens regular Banach algebra, the amenability of A implies Connes amenability of A^{**} (see [19], Corollary 4.3). The converse is still an open problem, see [2], Section 6. In the case where A is an ideal of A^{**} or A is a C^*-algebra, the converse holds, see [19], Theorem 4.4, and [3].

For a Banach algebra A, let $\sigma(A)$ be the set of all nonzero multiplicative linear functionals on A. If $\phi \in \sigma(A) \cup \{0\}$, then A is called ϕ-amenable if $H^1(A, X^*) = \{0\}$ for all Banach A-bimodules X for which the left module action is given by $a \cdot x = \phi(a)x$ for all $a \in A$ and $x \in X$; A is called character amenable if it is ϕ-amenable for every $\phi \in \sigma(A) \cup \{0\}$. The concept of character amenability is introduced by Sangani Monfared in [17] under the name of right character amenability, see also [9]. The notion of ϕ-amenability is introduced and studied by Kaniuth, Lau and Pym in [13], see also [12]. This notion is a generalization of left amenability of a class of Banach algebras studied by Lau in [14], known as Lau algebras. Authors in [12], [13] gave several characterizations of ϕ-amenability; for example, they showed that if $\phi \in \sigma(A)$ then ϕ-amenability of A is equivalent to the existence of topological invariant mean on A^*; that is, an $m \in A^{**}$ such that $m(\phi) = 1$ and $a \cdot m = \phi(a)m$ for all $a \in A$.

If A is a dual Banach algebra and $\phi \in \sigma(A) \cup \{0\}$ is w^*-continuous, then A is called ϕ-Connes amenable if $H^1_{w^*}(A, X) = \{0\}$ for all normal dual Banach A-bimodules X for which the right module action of A on X is given by $xa = \phi(a)x$ for each $a \in A$ and $x \in X$. This notion was recently introduced and studied by Mahmoodi in [15].

The Banach algebra A is called ϕ-contractible if $H^1(A,X)=\{0\}$ for all Banach A-bimodules X for which the right module action is given by $x\cdot a=\phi(a)x$ ($a\in A$, $x\in X$). The notion of ϕ-contractibility was recently introduced and studied by Hu et al. in [9] under the name of right ϕ-contractibility. Let us point out that ϕ-contractibility is significantly stronger than ϕ-amenability.

It is of interest to know whether Connes amenability can be replaced by a new variation of Connes amenability of A^{**}, which makes it equivalent to character amenability of A.

In this paper, for a w^*-continuous $\phi\in\sigma(A)\cup\{0\}$, we show that the concepts of ϕ-amenability, ϕ-contractibility and ϕ-Connes amenability for A are equivalent. This shows that the assumption of A being an ideal in A^{**} in [18], Corollary 3.6, for a large class of Banach algebras is not needed. We then introduce the notion of character Connes amenable dual Banach algebras and give some examples to show that the class of character Connes amenable dual Banach algebras is larger than that of character amenable, character contractible and Connes amenable dual Banach algebras. We investigate the hereditary properties of character Connes amenable dual Banach algebras. We then apply these results to characterize character Connes amenability of various dual Banach algebras related to locally compact groups. Among other results, we show that if A is an Arens regular Banach algebra, then A^{**} is character Connes amenable if and only if A is character amenable, which will resolve positively Runde's problem for this concept of amenability. We also investigate character Connes amenability of certain classes of Banach algebras consisting of Lau product $A\times_\varphi B$ and module extension $A\oplus_1 X$. From this we give examples of dual Banach algebras which are not Connes amenable.

2. Characterization of character Connes amenability

Let A be a dual Banach algebra and let $\sigma_{w^*}(A)$ denote the set of all nonzero w^*-continuous multiplicative linear functionals on A. In this paper, for $\phi\in\sigma_{w^*}(A)\cup\{0\}$, we denote by $\mathcal{N}_\phi^A$ the class of all normal dual Banach A-bimodules X for which the right module action of A on X is given by $x\cdot a=\phi(a)x$ for each $a\in A$ and $x\in X$.

Definition 2.1. Let A be a dual Banach algebra. We say that A is character Connes amenable if it is ϕ-Connes amenable for every $\phi\in\sigma_{w^*}(A)\cup\{0\}$.

Clearly every Connes amenable dual Banach algebra is ϕ-Connes amenable. The following example shows that the converse is not true.

Example 2.2. Let X be a Banach space and take $x_0 \in X \setminus \{0\}$ with $\|x_0\| \leqslant 1$. Then X^* with the product given by

$$fg = f(x_0)g, \quad f, g \in X^*,$$

is a dual Banach algebra. It is clear that $\sigma(X^*) = \sigma_{w^*}(X^*) = \{x_0\}$. This dual Banach algebra is x_0-Connes amenable, but if $\dim X^* > 1$, it is not unital and so it is not Connes amenable.

Our first result shows that the concepts of ϕ-contractibility, ϕ-amenability and ϕ-Connes amenability for dual Banach algebras are equivalent; we shall frequently use it without explicit reference. This result shows that the hypothesis of A being an ideal in A^{**} in [18], Corollary 3.6, for a large class of Banach algebras is not needed.

Proposition 2.3. *Let A be a dual Banach algebra and $\phi \in \sigma_{w^*}(A) \cup \{0\}$. Then the following are equivalent.*

(i) *A is ϕ-contractible.*
(ii) *A is ϕ-amenable.*
(iii) *A is ϕ-Connes amenable.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from definition. For (ii) $\Rightarrow$ (iii), suppose that $X \in \mathcal{N}_\phi^A$. Also, let $D\colon A \longrightarrow X$ be a w^*-continuous derivation. It is enough to show that D is bounded. If D is unbounded, then there exists a sequence $\{a_n\}$ in A such that $\lim \|a_n\| = 0$ and $\lim \|D(a_n)\| = \infty$. The uniform boundedness principle, see [6], Theorem 5.13, implies that $D(a_n)$ does not converge to 0 in weak* topology. This is a contradiction, since $a_n \xrightarrow{w^*} 0$.

To show that (iii) $\Rightarrow$ (i), suppose that A is ϕ-Connes amenable. Note that $\ker\phi$ with the right action $x \cdot a = \phi(a)x$ for all $a \in A$ and $x \in \ker\phi$, and the natural left action, is a normal dual Banach A-bimodule. If $\phi \neq 0$, choose $a_0 \in A$ such that $\phi(a_0) = 1$ and define $D(a) = aa_0 - \phi(a)a_0$ for each $a \in A$. Then D is a w^*-continuous derivation from A into $\ker\phi$. By assumption, there exists $a_1 \in \ker\phi$ such that $D = d_{a_1}$. Now consider $m = a_0 - a_1$, and note that $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in A$. This shows that A is ϕ-contractible, see [18], Theorem 2.1. If $\phi = 0$ then $D(a) = a$ is also a w^*-continuous derivation from A into $\ker\phi$. So there is $e \in A$ such that $a = D(a) = ae$ for all $a \in A$. Therefore, A is 0-contractible, see [9], Theorem 6.3. □

Let A be a Banach algebra. Then the second dual A^{**} of A is a Banach algebra with the first Arens product defined by the equations

$$(f \cdot a)(b) = f(ab), \quad (G \cdot f)(a) = G(f \cdot a) \quad \text{and} \quad (F \cdot G)(f) = F(G \cdot f)$$

for $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$. A Banach algebra A is called Arens regular if for each $F \in A^{**}$ the mapping $G \to F \cdot G$ from A^{**} into A^{**} is w^*-continuous. If A is Arens regular then A^{**} is a dual Banach algebra.

In [19], Corollary 4.3, it is shown that if A is an Arens regular Banach algebra, the amenability of A implies Connes amenability of A^{**}. The converse is still an open problem, see [2], Section 6. The next result resolves this problem for the concept of character Connes amenability of dual Banach algebras. This result was also obtained by Mahmoodi in [15], Theorem 2.6, with an extra assumption. For $\phi \in \sigma(A)$, we denote by $\hat{\phi} \in \sigma(A^{**})$ the unique extension of ϕ, defined by $\hat{\phi}(F) = F(\phi)$ for all $F \in A^{**}$.

Theorem 2.4. *Let A be an Arens regular Banach algebra, then the following statements hold.*

(i) *If $\phi \in \sigma(A) \cup \{0\}$, then A^{**} is $\hat{\phi}$-Connes amenable if and only if A is ϕ-amenable.*
(ii) *A^{**} is character Connes amenable if and only if A is character amenable.*

Proof. With a direct verification, we can show that $\sigma_{w^*}(A^{**}) = \{\hat{\phi};\ \phi \in \sigma(A)\}$. So, it is enough to prove (i). For this, first note that A has a bounded right approximate identity if and only if A^{**} has a right identity, see for example [1], Proposition III.28.7. Thus, A^{**} is 0-Connes amenable if and only if A is 0-amenable. Now, suppose that $\phi \in \sigma(A)$. Then $\hat{\phi} \circ \kappa_A = \phi$, where $\kappa_A \colon A \to A^{**}$ is the canonical embedding. If A is ϕ-amenable, from [15], Theorem 2.5, we get that A^{**} is also $\hat{\phi}$-Connes amenable.

Conversely, if A^{**} is $\hat{\phi}$-Connes amenable, then by Proposition 2.3, there exists an element $m \in A^{**}$ with $m(\phi) = 1$ and

$$F \cdot m = F(\phi)m, \quad F \in A^{**}.$$

From this we have $a \cdot m = \phi(a)m$ for all $a \in A$. Thus A is ϕ-amenable. □

Let A be an Arens regular dual Banach algebra. If A^{**} is Connes amenable, then so is A, see [2], Section 3. We have the following similar result for character Connes amenable dual Banach algebras, which follows from Theorem 2.3 and [13], Proposition 3.4. We do not know if the converse of part (ii) in this result is true.

Corollary 2.5. *Let A be an Arens regular dual Banach algebra, then the following statements hold.*

(i) *If $\phi \in \sigma_{w^*}(A) \cup \{0\}$, then A^{**} is $\hat{\phi}$-Connes amenable if and only if A is ϕ-Connes amenable.*
(ii) *If A^{**} is character Connes amenable, then A is also character Connes amenable.*

It is well-known that, if A is a C^*-algebra, then A is amenable if and only if A^{**} is Connes amenable, see for example [20], Corollary 6.5.12. Since A is always character amenable, see [17], Corollary 2.7, from Theorem 2.4, we have the following result. This result shows that there are character Connes amenable dual Banach algebras which are not Connes amenable.

Corollary 2.6. *If A is a C^*-algebra, then A^{**} is character Connes amenable.*

Our next result describes an interaction between the character Connes amenability of a dual Banach algebra and of its w^*-closed ideals.

Proposition 2.7. *Let A be a character Connes amenable dual Banach algebra and J be a w^*-closed ideal of A. Then J is character Connes amenable if and only if J has a right identity.*

Proof. Let b_0 be a right identity for J, and $\phi \in \sigma_{w^*}(J)$. Define $\varphi\colon A \to \mathbb{C}$ by $\varphi(a) = \phi(b_0 a)$. Then $\varphi|_J = \phi$ and $\varphi \in \sigma_{w^*}(A)$. Since A is φ-Connes amenable, Theorem 2.3 and [18], Proposition 3.8, imply that J is ϕ-Connes amenable. Thus J is character Connes amenable. The converse is clear. $\square$

Before we give our next result, note that if J is a w^*-closed ideal of a dual Banach algebra A, then the quotient algebra A/J is a dual Banach algebra with predual

$$J_\perp = \{a_* \in A_*;\ b(a_*) = 0\ \forall\, b \in J\}.$$

If $\phi \in \sigma_{w^*}(A)$, then there is a unique $\phi_q \in \sigma_{w^*}(A/J)$ with $\phi_q \circ q = \phi$ if and only if $J \subseteq \ker\phi$, where $q\colon A \to A/J$ is the quotient map.

Proposition 2.8. *Let A be a dual Banach algebra and J be a w^*-closed ideal of A. Then the following statements hold.*

(i) *If J has a right identity, $\phi \in \sigma_{w^*}(A) \cup \{0\}$ and $J \subseteq \ker\phi$, then A/J is ϕ_q-Connes amenable if and only if A is ϕ-Connes amenable.*
(ii) *J and A/J are character Connes amenable if and only if J has a right identity and A is character Connes amenable.*

Proof. (i): If ϕ is nonzero, then (i) follows from Theorem 2.3 and [18], Propositions 3.10, 3.12. So let $\phi = 0$. If e is a right identity for A, then $q(e)$ is a right identity for A/J. Conversely, let $a_0 \in A$ be such that $q(a_0)$ is a right identity for A/J and b_0 is a right identity of J. Then for each $a \in A$, from $q(a)q(a_0) = q(a)$ we get $aa_0 - a \in J$. Therefore,

$$(aa_0 - a)b_0 = aa_0 - a.$$

This shows that $a(a_0 + b_0 - a_0 b_0) = a$. So $a_0 + b_0 - a_0 b_0$ is a right identity for A.

(ii): It follows from (i), Proposition 2.7 and the fact that: if $\chi \in \sigma_{w^*}(A/J)$, then $\phi = \chi \circ q \in \sigma_{w^*}(A)$ and $\phi_q = \chi$. □

As a consequence of Proposition 2.7, [17], Theorem 2.6, and [9], Lemma 6.8, we have the next result.

Corollary 2.9. *Let A be a character contractible dual Banach algebra. For each w^*-closed ideal J of A, the following are equivalent.*

(i) *J is character contractible.*

(ii) *J is character amenable.*

(iii) *J is character Connes amenable.*

3. Character Connes amenability of group algebras

Let G be a locally compact group with a fixed left Haar measure λ. Let $L^1(G)$ denote the group algebra of G. Then $L^1(G)$ is a Banach algebra with convolution as its multiplication and can be identified with the predual of the von Neumann algebra $L^\infty(G)$; the usual Lebesgue space with the essential supremum norm. Let $M(G)$ be the measure algebra of G as defined in [5]. Then $M(G)$ is a dual Banach algebra with natural predual $C_0(G)$, the collection of all continuous functions on G that vanish at infinity.

Let $A(G)$ be the Fourier algebra of G as introduced by Eymard in [4]. Then $A(G)$ with the pointwise multiplication is a commutative Banach algebra and can be identified with the predual of the group von Neumann algebra $VN(G)$ generated by left translations on $L^2(G)$ by the pairing $u(\lambda(t)) = u(t)$.

We know from [21] that $M(G)$ is Connes amenable if and only if G is amenable. It is (character) amenable if and only if G is discrete and amenable, see [17], Corollary 2.5. Also it is character contractible if and only if G is finite, see [18], Corollary 6.2. In the next result, we show that it is always character Connes amenable. This result shows that there is a character Connes amenable dual Banach algebra which is neither (character) amenable nor character contractible.

Proposition 3.1. *If G is a locally compact group, then the following statements hold.*

(i) *$M(G)$ is character Connes amenable.*

(ii) *$L^\infty(G)$ is character Connes amenable.*

Proof. (i): If $\varphi \in \sigma_{w^*}(M(G))$, then $\varphi \in C_0(G)$ and $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in G$. This implies that G is compact. From [21], Proposition 3.4, we get that

$M(G)$ is Connes amenable and so it is φ-Connes amenable. Therefore, $M(G)$ is character Connes amenable.

(ii): Assume that $\varphi \in \sigma_{w^*}(L^\infty(G))$. From the w^*-density of $A(G)$ in $L^\infty(G)$ it follows that $\varphi|_{A(G)} \in \sigma(A(G))$. Then, by [13], Example 2.6, $A(G)$ is $\varphi|_{A(G)}$-amenable. It follows from [15], Theorem 2.5, that $L^\infty(G)$ is φ-Connes amenable. Therefore, $L^\infty(G)$ is character Connes amenable. □

For a locally compact group G, the Fourier-Stieltjes algebra $B(G)$ of G is the collection of all coefficient functions of continuous unitary representations of G. As is well known, see [4], $B(G)$ can be identified with the dual of the (full) group C^*-algebra $C^*(G)$ of G. With the norm defined by this duality and the pointwise multiplication, $B(G)$ is a dual Banach algebra containing $A(G)$ as a closed ideal.

Recently, Runde and Uygul in [24] showed that $B(G)$ is Connes amenable if and only if G is almost abelian, i.e., it has an abelian subgroup of finite index. Character amenability of $B(G)$ is also studied in [9], Section 4. It is well-known that if G is amenable, then $VN(G)$ is Connes amenable; the converse is also true for inner amenable groups, see [19], Theorem 5.3. For character Connes amenability of $B(G)$ and $VN(G)$ we have the next result.

Proposition 3.2. *If G is a locally compact group, then*

(i) *$B(G)$ is character Connes amenable;*

(ii) *$VN(G)$ is character Connes amenable.*

P r o o f. (i): Assume that $\varphi \in \sigma_{w^*}(B(G))$. From w^*-density of $A(G)$ in $B(G)$ it follows that $\varphi|_{A(G)} \in \sigma(A(G))$. Then, by [13], Example 2.6, $A(G)$ is $\varphi|_{A(G)}$-amenable. It follows from [15], Theorem 2.4, 2.5, that $B(G)$ is φ-Connes amenable. Therefore, $B(G)$ is character Connes amenable.

(ii): If $\varphi \in \sigma_{w^*}(VN(G))$, then $\varphi \in A(G) \subseteq C_0(G)$ and $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in G$. This shows that G is compact. From [19], Theorem 5.3, we get that $VN(G)$ is Connes amenable and so it is φ-Connes amenable. Therefore, $VN(G)$ is character Connes amenable. □

4. Character Connes amenability of Lau product and module extension

Let A and B be Banach algebras, and $\varphi \in \sigma(B)$. Then the vector space $A \times B$ equipped with the algebra multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + \varphi(b_2)a_1 + \varphi(b_1)a_2, b_1b_2), \quad a_1, a_2 \in A,\ b_1, b_2 \in B,$$

and the norm $\|(a,b)\| = \|a\| + \|b\|$ is a Banach algebra which is called the φ-Lau product of A and B and denoted by $A \times_\varphi B$. This type of product was introduced by Lau in [14] for certain class of Banach algebras and was extended by Sangani Monfared in [16] for the general case. We note that in the special case where B is the set of complex numbers $\mathbb{C}$ and φ is the identity map on $\mathbb{C}$, then $A \times_\varphi B$ is the unitization A_e of A.

Now, if A and B are dual Banach algebras and $\varphi \in \sigma_{w^*}(B)$, then the φ-Lau product $A \times_\varphi B$ is also a dual Banach algebra with $A_* \times_\infty B_*$ as predual. Moreover, following [16], Proposition 2.4, we can show that

$$\sigma_{w^*}(A \times_\varphi B) = (\sigma_{w^*}(A) \times \{\varphi\}) \cup (\{0\} \times \sigma_{w^*}(B)).$$

Theorem 4.1. *Let A and B be dual Banach algebras and let $\varphi \in \sigma_{w^*}(B)$. Then the following statements hold.*

(i) *If $\phi \in \sigma_{w^*}(A)$, then $A \times_\varphi B$ is (ϕ, φ)-Connes amenable if and only if A is ϕ-Connes amenable.*

(ii) *$A \times_\varphi B$ is $(0, \varphi)$-Connes amenable if and only if A has a right identity and B is φ-Connes amenable.*

(iii) *Let $\psi \in \sigma_{w^*}(B)$ and $\psi \neq \varphi$. Then $A \times_\varphi B$ is $(0, \psi)$-Connes amenable if and only if B is ψ-Connes amenable.*

P r o o f. (i): Let $(m,n) \in A \times_\varphi B$ be such that $(\phi, \varphi)(m,n) = 1$ and $(a,b)(m,n) = (\phi, \varphi)(a,b)(m,n)$ for all $a \in A$ and $b \in B$. Choosing $b = 0$ and $a_0 \in A$ such that $\phi(a_0) \neq 0$, we conclude that $n = 0$, $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in A$. So A is ϕ-Connes amenable.

Conversely, let $m \in A$ be such that $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in A$. Then $(\phi, \varphi)(m, 0) = 1$ and $(a,b)(m,0) = (am + \varphi(b)m, 0) = (\phi, \varphi)(a,b)(m,0)$, for each $(a,b) \in A \times_\varphi B$. This shows that $A \times_\varphi B$ is (ϕ, φ)-Connes amenable.

(ii): It follows from the fact that: if $(m,n) \in A \times_\varphi B$, then $(0, \varphi)(m,n) = 1$ and $(a,b)(m,n) = (0, \varphi)(a,b)(m,n)$ for all $a \in A$ and $b \in B$ if and only if $\varphi(n) = 1$, $bn = \varphi(b)n$ for all $b \in B$, and $-m$ is a right identity for A.

(iii): Let $(m,n) \in A \times_\varphi B$ be such that $(0, \psi)(m,n) = 1$ and $(a,b)(m,n) = (0, \psi)(a,b)(m,n)$ for all $a \in A$ and $b \in B$. From this, we conclude that $\psi(n) = 1$ and $bn = \psi(b)n$ for all $b \in B$. Hence, B is ψ-Connes amenable.

Conversely, let $n_0 \in B$ be such that $\psi(n_0) = 1$ and $bn_0 = \psi(b)n_0$ for all $b \in B$. Since $\varphi \neq \psi$, by Hahn-Banach theorem, there is $b_0 \in B$ such that $\psi(b_0) = 1$ and $\varphi(b_0) = 0$. Put $n = n_0 b_0$. Then $\varphi(n) = 0$ and $bn = \psi(b)n$ for all $b \in B$. Since

$(0, \psi)(0, n) = 1$ and

$$(a, b)(0, n) = (\varphi(n)a, bn) = (0, \psi(b)n) = (0, \psi)(a, b)(0, n)$$

for all $a \in A$, $b \in B$, it follows that $A \times_\varphi B$ is $(0, \psi)$-Connes amenable. $\square$

Corollary 4.2. *Let A and B be dual Banach algebras and let $\varphi \in \sigma_{w^*}(B)$. Then $A \times_\varphi B$ is character Connes amenable if and only if both A and B are character Connes amenable.*

Let A be a dual Banach algebra and let $A_e = A \oplus_1 \mathbb{C}$ be the unitization of A. Let $\phi \in \sigma(A) \cup \{0\}$ and define $\tilde{\phi}(a, \lambda) = \phi(a) + \lambda$. As a corollary of the above theorem we have the next result.

Proposition 4.3. *Let A be a dual Banach algebra, then the following statements hold.*

(i) *If $\phi \in \sigma_{w^*}(A) \cup \{0\}$, then A is ϕ-Connes amenable if and only if A_e is $\tilde{\phi}$-Connes amenable.*

(ii) *A is character Connes amenable if and only if A_e is character Connes amenable.*

For a Banach algebra A and a Banach A-bimodule X, let $A \oplus_1 X$ be the vector space $A \times X$ which is equipped with the norm $\|(a, x)\| = \|a\| + \|x\|$ and the algebra product

$$(a_1, x)(a_2, y) = (a_1 a_2, a_1 \cdot y + x \cdot a_2), \quad a_1, a_2 \in A, \ x, y \in X.$$

Then $A \oplus_1 X$ is a Banach algebra that is called the module extension of A and X. Some aspects of module extension Banach algebras have been discussed in [25].

Now if A is a dual Banach algebra and X is a normal dual Banach A-bimodule, then $A \oplus_1 X$ is also a dual Banach algebra. A direct verification shows that $\sigma_{w^*}(A \oplus_1 X) = \{(\phi, 0); \ \phi \in \sigma_{w^*}(A)\}$.

Proposition 4.4. *Let A be a dual Banach algebra and X be a normal dual Banach A-bimodule. If $A \oplus_1 X$ is character Connes amenable, then so is A.*

Proof. If $(a_0, x_0) \in A \oplus_1 X$ is a right identity of $A \oplus_1 X$, then $aa_0 = a$ for all $a \in A$. Thus a_0 is a right identity of A.

Now, let $\phi \in \sigma_{w^*}(A)$. Since $A \oplus_1 X$ is $(\phi, 0)$-Connes amenable, there is an element $(m, x_0) \in A \oplus_1 X$ such that $(\phi, 0)(m, x_0) = 1$ and $(a, x)(m, x_0) = (\phi, 0)(a, x)(m, x_0)$ for all $(a, x) \in A \oplus_1 X$. It follows that $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in A$. Hence, A is ϕ-Connes amenable. $\square$

In the case where $X \in \mathcal{N}_\varphi^A$ for some $\varphi \in \sigma_{w^*}(A) \cup \{0\}$, we have the next result.

Theorem 4.5. *Let A be a dual Banach algebra, $\varphi \in \sigma_{w^*}(A) \cup \{0\}$ and let $X \in \mathcal{N}_\varphi^A$. Then the following statements hold.*

(i) *$A \oplus_1 X$ is $(\varphi, 0)$-Connes amenable if and only if $X = 0$ and A is φ-Connes amenable.*

(ii) *If $\varphi \neq \phi \in \sigma_{w^*}(A)$, then $A \oplus_1 X$ is $(\phi, 0)$-Connes amenable if and only if A is ϕ-Connes amenable.*

Proof. (i): First let $\varphi = 0$. Then $(a_0, x_0) \in A \oplus_1 X$ is a right identity of $A \oplus_p X$ if and only if

$$(a, x) = (aa_0, a \cdot x_0), \quad a \in A,\ x \in X.$$

The above equality holds if and only if $X = 0$ and a_0 is a right identity of A. Now let $\varphi \in \sigma_{w^*}(A)$. Then $A \oplus_1 X$ is $(\varphi, 0)$-Connes amenable if and only if there is an element $(m, x_0) \in A \oplus_1 X$ such that $\varphi(m) = 1$ and

$$(am, a \cdot x_0 + x) = (\varphi(a)m, \varphi(a)x_0), \quad a \in A,\ x \in X.$$

These also hold if and only if $X = 0$, $\varphi(m) = 1$ and $am = \varphi(a)m$ for all $a \in A$.

(ii): By Proposition 4.4, the "only if" part is clear. Now suppose that A is ϕ-Connes amenable. Choose $m_1 \in A$ such that $\phi(m_1) = 1$ and $am_1 = \phi(a)m_1$ for each $a \in A$. Since $\phi \neq \varphi$, there is $m_2 \in A$ such that $\varphi(m_2) = 0$ and $\phi(m_2) = 1$. Put $m = m_1 m_2$. Then $\varphi(m) = 0, \phi(m) = 1$ and $am = \phi(a)m$ for each $a \in A$. So $(\phi, 0)(m, 0) = 1$ and

$$(a, x)(m, 0) = (am, x \cdot m) = (\phi(a)m, \varphi(m)x) = (\phi(a)m, 0) = (\phi, 0)(a, x)(m, 0).$$

Thus, $A \oplus_1 X$ is $(\phi, 0)$-Connes amenable. □

Corollary 4.6. *Let A be a dual Banach algebra, $\varphi \in \sigma_{w^*}(A) \cup \{0\}$ and let $X \in \mathcal{N}_\varphi^A$. Then $A \oplus_1 X$ is character Connes amenable if and only if $X = 0$ and A is character Connes amenable.*

Remark 4.7. Since every Connes amenable dual Banach algebra is ϕ-Connes amenable for all $\phi \in \sigma_{w^*}(A) \cup \{0\}$, Theorem 4.5 shows that if A is a dual Banach algebra and X is a normal dual Banach A-bimodule, then in each of the following cases the dual Banach algebra $A \oplus_1 X$ is not Connes amenable.

(i) A is not ϕ-Connes amenable for some $\phi \in \sigma_{w^*}(A) \cup \{0\}$.

(ii) X is nonzero and $X \in \mathcal{N}_\phi^A$ for some $\phi \in \sigma_{w^*}(A) \cup \{0\}$.

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