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A DISPERSION INEQUALITY IN THE HANKEL SETTING

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Abstract. The aim of this paper is to prove a quantitative version of Shapiro's uncertainty principle for orthonormal sequences in the setting of Gabor-Hankel theory.

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1. Introduction

Time-frequency analysis is an important field in signal processing. The windowed Hankel transform plays a fundamental part in this analysis. To be more precise, let $d \geqslant 1$ be the dimension, and let us denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$. Our starting point is the following Heisenberg's uncertainty inequality:

$$\| |x| f\|^2_{L^2(\mathbb{R}^d)} + \| |\xi| \mathcal{F}(f)\|^2_{L^2(\mathbb{R}^d)} \geqslant d \|f\|^2_{L^2(\mathbb{R}^d)}, \tag{1.1}$$

where the Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}\langle x, \xi \rangle} \, \mathrm{d}x,$$

and it is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way. With this normalization, if $f(x) = \tilde{f}(|x|)$ is a radial function on $\mathbb{R}^d$, then $\mathcal{F}(f)(\xi) = \mathcal{H}_{d/2-1}(\tilde{f})(|\xi|)$, where for $\alpha > -1/2$, $\mathcal{H}_\alpha$ is the Hankel transform (also known as the Fourier-Bessel transform) defined by

$$\mathcal{H}_\alpha(f)(\xi) = \int_0^\infty f(x) j_\alpha(x\xi) \, \mathrm{d}\mu_\alpha(x), \quad \xi \in \mathbb{R}_+ = [0, \infty).$$

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Here $\mathrm{d}\mu_\alpha(x) = (x^{2\alpha+1}/2^\alpha\Gamma(\alpha+1))\,\mathrm{d}x$ and j_α is the spherical Bessel function given by

$$j_\alpha(x) := \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)}\Big(\frac{x}{2}\Big)^{2n}.$$

Therefore Heisenberg's inequality (1.1) for the Hankel transform leads to (see [1], [3])

$$\|xf\|^2_{L^2_\alpha} + \|\xi\,\mathcal{H}_\alpha(f)\|^2_{L^2_\alpha} \geqslant (2\alpha+2)\|f\|^2_{L^2_\alpha}, \tag{1.2}$$

where for $1 \leqslant p < \infty$ we denote by $L^p_\alpha(\mathbb{R}_+)$ the Banach space consisting of measurable functions f on $\mathbb{R}_+$ equipped with the norms

$$\|f\|_{L^p_\alpha} = \Big(\int_0^\infty |f(x)|^p\,\mathrm{d}\mu_\alpha(x)\Big)^{1/p}.$$

Shapiro in [8] observed that Heisenberg's inequality (1.1) can be refined for infinite orthonormal sequences, that is, if $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, then

$$\sup_n \big(\||x|\varphi_n\|^2_{L^2(\mathbb{R}^d)} + \||\xi|\mathcal{F}(\varphi_n)\|^2_{L^2(\mathbb{R}^d)}\big) = \infty. \tag{1.3}$$

A quantitative version of (1.3) can be written in the following form (see [7]): If $s > 0$ and $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, then

$$\sum_{n=1}^{N}\big(\||x|^s\varphi_n\|^2_{L^2(\mathbb{R}^d)} + \||\xi|^s\mathcal{F}(\varphi_n)\|^2_{L^2(\mathbb{R}^d)}\big) \geqslant CN^{1+s/d}. \tag{1.4}$$

Time-frequency analysis has emerged as an important field in signal processing as it can be used to represent time-varying signals in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Usually, the time-frequency resolution is associated with the windowed Fourier transform also known as the (continuous) Gabor transform, or the short-time Fourier transform. To be more precise, fix $g \in L^2(\mathbb{R}^d)$ a nonzero window function, and define for $f \in L^2(\mathbb{R}^d)$ its windowed Fourier transform with respect to the window g as

$$\mathcal{F}_g(f)(x,\xi) = \mathcal{F}[f\overline{g(\cdot - x)}](\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(t)\overline{g(t-x)}\mathrm{e}^{-\mathrm{i}\langle t,\xi\rangle}\,\mathrm{d}t.$$

A considerable attention has been devoted recently to discovering new mathematical formulations of the uncertainty principle for the windowed Fourier transform. In particular, we recall the Heisenberg-type uncertainty inequality

$$\||x|\mathcal{F}_g(f)\|^2_{L^2(\mathbb{R}^{2d})} + \||\xi|\mathcal{F}_g(f)\|^2_{L^2(\mathbb{R}^{2d})} \geqslant C(d)\|g\|^2_{L^2(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{1.5}$$

Further limitations are given by various versions of the uncertainty principle for the windowed Fourier transform. These state in particular that if we concentrate $\mathcal{F}_g(f)$ in the x-variable, then we loose concentration in the ξ-variable. One might then prove an extension of the result of Shapiro for the windowed Fourier transform, which has been recently stated in [5].

An other fundamental tool in time-frequency analysis is the windowed Hankel transform introduced in [2]. Precisely, we define the translation operator by

$$\tau_x^\alpha f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^\pi f(\sqrt{x^2+y^2-2xy\cos\theta})(\sin\theta)^{2\alpha}\,\mathrm{d}\theta,$$

and the modulation operator by

$$\mathcal{M}_\xi^\alpha g := \mathcal{H}_\alpha\Big(\sqrt{\tau_\xi^\alpha |\mathcal{H}_\alpha(g)|^2}\Big).$$

Then for any nonzero window function $g \in L^2_\alpha(\mathbb{R}_+)$, the windowed Hankel transform of any signal $f \in L^2_\alpha(\mathbb{R}_+)$ with respect to the window g is given by

$$\mathcal{V}_g^\alpha(f)(x,\xi) = \int_0^\infty f(s)\overline{\tau_x^\alpha \mathcal{M}_\xi^\alpha g(s)}\,\mathrm{d}\mu_\alpha(s), \quad (x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+,$$

where $\widehat{\mathbb{R}}_+$ denotes the half real line thought of as the frequency axis and the bar denoting complex conjugation.

Notice that the windowed Hankel transform cannot be obtained from the windowed Fourier transform by taking spherical averages (see e.g. [2], Example 1, Example 2), i.e. if $f, g \in L^2_{d/2-1}(\mathbb{R}_+)$ are the radial parts of $F, G \in L^2(\mathbb{R}^d)$, then it is not true in general that

$$\mathcal{V}_g^{d/2-1}(f)(|x|,|\xi|) = \mathcal{F}_G(F)(x,\xi), \quad x,\xi \in \mathbb{R}^d,$$

because generally $\mathcal{F}_G(F)$ is not radial in any of the two variables. So, the windowed Hankel transform is a new object and not just an average of the standard windowed Fourier transform. Our contribution shows which part of the approach of references [7], [5] is still valid in the Gabor-Hankel context. This involves considering new aspects such as the role of Gabor-Toeplitz operators and spectrograms.

Heisenberg-type uncertainty inequality for the windowed Hankel transform can be written in the form (see [4], Theorem 4.5)

$$\|x^s \mathcal{V}_g^\alpha(f)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} \|\xi^s \mathcal{V}_g^\alpha(f)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} \geqslant C(\alpha,s)\|f\|^2_{L^2_\alpha}\|g\|^2_{L^2_\alpha}. \tag{1.6}$$

It states that the windowed Hankel transform of a nonzero function with respect to a nonzero window function cannot be time and frequency concentrated around zero, and a dilation argument (see [4], Lemma 2.2) shows that inequality (1.6) is equivalent to

$$(1.7)\qquad \|x^s\mathcal{V}_g^\alpha(f)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}+\|\xi^s\mathcal{V}_g^\alpha(f)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\geqslant 2C(\alpha,s)\|f\|^2_{L^2_\alpha}\|g\|^2_{L^2_\alpha}.$$

Here we denote by $L^p_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$, $1\leqslant p<\infty$, the Banach space consisting of measurable functions F on $\mathbb{R}_+\times\widehat{\mathbb{R}}_+$ equipped with the norms

$$\|F\|_{L^p_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}=\biggl(\iint_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+}|F(x,\xi)|^p\,\mathrm{d}\nu_\alpha(x,\xi)\biggr)^{1/p},$$

where $\mathrm{d}\nu_\alpha(x,\xi)=\mathrm{d}\mu_\alpha(x)\,\mathrm{d}\mu_\alpha(\xi)$.

In this paper, we will adopt the broader view that the uncertainty principle can be seen not only as a statement about the time-frequency localization of a single function but also as a statement on the degradation of localization when one considers successive elements of an orthonormal sequence. In particular, Heisenberg-type inequality (1.7) states that a unit-norm function in $L^2_\alpha(\mathbb{R}_+)$ cannot occupy an arbitrarily small region in the time-frequency plane. The main aim of this paper is to refine inequality (1.7) for orthonormal sequences, and, motivated by the Malinnikova's process in [7], we show the following analogue of the dispersion inequality (1.4) for the windowed Hankel transform.

Theorem 1.1. *Let $s>0$, $g\in L^2_\alpha(\mathbb{R}_+)$ be a nonzero window function of unit L^2_α-norm and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$. Then for every $N\geqslant 1$,*

$$(1.8)\quad \sum_{n=1}^N\bigl(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}+\|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\bigr)\geqslant c(s,\alpha)N^{1+s/(2\alpha+2)}.$$

This result shows how an orthonormal sequence can cover the time-frequency plane $\mathbb{R}_+\times\widehat{\mathbb{R}}_+$, and this is an important factor in determining which applications the sequence is suited for. In particular, inequality (1.8) implies that the elements of an orthonormal sequence cannot be uniformly concentrated in the time-frequency plane (see Corollary 4.7). This problem was first studied by Shapiro in [8] in order to bound the time and frequency dispersions of an orthonormal sequence by means of the Fourier transform.

Other consequence of inequality (1.8) is the following so-called strong uncertainty principle for infinite orthonormal sequences, that improves its corresponding inequality (1.7) for a single function,

$$(1.9)\qquad \sup_n\bigl(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}+\|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\bigr)=\infty.$$

It is also interesting to note that if one takes $N = 1$ in (1.8), then this yields the usual form of Heisenberg-type uncertainty principle (1.7) for the windowed Hankel transform.

The remainder of the paper is organized as follows. Next section is devoted to some preliminaries on the windowed Hankel transform and in Section 3 we introduce the Gabor-Toeplitz and the phase-space restriction operators and we prove a trace formula in terms of the spectrogram. Finally, in Section 4 we prove the dispersion inequality (1.8).

2. Preliminaries

2.1. Notation. Throughout this paper, s and r will be two real numbers such that $s, r > 0$ and $\mathcal{B}_r = \{(x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+ : |(x,\xi)| \leqslant r\}$ is the closed ball in $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ centered at 0 and of radius r. If A is a subset of $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$, then we denote by $A^c = (\mathbb{R}_+ \times \widehat{\mathbb{R}}_+) \setminus A$ the complement of A in $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ and the characteristic function of A will be denoted by χ_A.

We write $c_{s,\alpha}$ for a constant that depends only on the parameters s and α. This constant may change from line to line.

Finally, if a compact operator $\mathcal{A}$ on the Hilbert space $L^2_\alpha(\mathbb{R}_+)$ is Hilbert-Schmidt, then the positive operator $\mathcal{A}^*\mathcal{A}$ is in the space of trace class and

$$\|\mathcal{A}\|^2_{HS} = \operatorname{tr}(\mathcal{A}^*\mathcal{A}) = \sum_{n=1}^{\infty} \|\mathcal{A}\varphi_n\|^2_{L^2_\alpha} \tag{2.1}$$

for any orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ for $L^2_\alpha(\mathbb{R}_+)$.

2.2. Generalities. For $\alpha > -1/2$, let us recall the *Poisson representation formula*

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\Gamma(1/2)} \int_{-1}^{1} (1-s^2)^{\alpha-1/2} \cos(sx)\,\mathrm{d}x.$$

Therefore j_α is bounded with $|j_\alpha(x)| \leqslant j_\alpha(0) = 1$. As a consequence,

$$\|\mathcal{H}_\alpha(f)\|_\infty \leqslant \|f\|_{L^1_\alpha}. \tag{2.2}$$

Here $\|\cdot\|_\infty$ is the usual essential supremum norm and $L^\infty(\mathbb{R}_+)$ will denote the usual space of essentially bounded functions.

It is also well known that the Hankel transform extends to an isometry on $L^2_\alpha(\mathbb{R}_+)$:

$$\|\mathcal{H}_\alpha(f)\|_{L^2_\alpha} = \|f\|_{L^2_\alpha}. \tag{2.3}$$

2.3. Generalized translation. Following Levitan in [6], for any function $f \in C^2(\mathbb{R}_+)$ we define the *generalized Bessel translation operator*

$$\tau_y^\alpha f(x) = u(x,y), \quad x,y \in \mathbb{R}_+,$$

as a solution of the following Cauchy problem:

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x}\Big)u(x,y) = \Big(\frac{\partial^2}{\partial y^2} + \frac{2\alpha+1}{y}\frac{\partial}{\partial y}\Big)u(x,y)$$

with initial conditions $u(x,0) = f(x)$ and $\partial u(x,0)/\partial y = 0$; here

$$\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x}$$

is the differential Bessel operator. The solution of the Cauchy problem can be written out in explicit form:

$$\tau_x^\alpha f(y) = \tau_y^\alpha f(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_0^\pi f\big(\sqrt{x^2+y^2-2xy\cos\theta}\big)(\sin\theta)^{2\alpha}\,\mathrm{d}\theta. \tag{2.4}$$

The operator τ_x^α can be also expressed by the formula

$$\tau_x^\alpha f(y) = \int_0^\infty f(t)W(x,y,t)\,\mathrm{d}\mu_\alpha(t), \tag{2.5}$$

where $W(x,y,t)\,\mathrm{d}\mu_\alpha(t)$ is a probability measure and $W(x,y,t)$ is defined by

$$W(x,y,t) = \begin{cases} \dfrac{2\pi^{\alpha+1/2}\Gamma(\alpha+1)^2}{\Gamma(\alpha+1/2)}\dfrac{\Delta(x,y,t)^{2\alpha-1}}{(xyt)^{2\alpha}}, & \text{if } |x-y| < t < x+y, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Delta(x,y,t) = \sqrt{(x+y)^2-t^2}\sqrt{t^2-(x-y)^2}$$

is the area of the triangle with side length x, y, t. Further, $W(x,y,t)\,\mathrm{d}\mu_\alpha(t)$ is a probability measure, so that, for $p \geqslant 1$, $|\tau_x^\alpha f|^p \leqslant \tau_x^\alpha |f|^p$, thus

$$\|\tau_x^\alpha f\|_{L^p_\alpha} \leqslant \|f\|_{L^p_\alpha}. \tag{2.6}$$

This allows to extend the definition of $\tau_x^\alpha f$ to functions $f \in L^p_\alpha(\mathbb{R}_+)$. It is also well known that for every $r, x, \xi > 0$, $f \in L^1_\alpha(\mathbb{R}_+)$

$$\int_0^\infty \tau_x^\alpha f(s)\,\mathrm{d}\mu_\alpha(s) = \int_0^\infty f(s)\,\mathrm{d}\mu_\alpha(s), \tag{2.7}$$

and

$$\tau_x^\alpha j_\alpha(r\,\cdot)(\xi) = j_\alpha(rx)j_\alpha(r\xi). \tag{2.8}$$

Therefore for $f \in L^p_\alpha(\mathbb{R}_+)$, $p = 1$ or 2,

$$\mathcal{H}_\alpha\big(\tau_x^\alpha f\big)(\xi) = j_\alpha(x\xi)\mathcal{H}_\alpha(f)(\xi).$$

The *Bessel convolution* $f *_\alpha g$ of two functions f and g in $L^1_\alpha(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is defined by

$$f *_\alpha g(x) = \int_0^\infty f(t)\tau_x^\alpha g(t)\,\mathrm{d}\mu_\alpha(t) = \int_0^\infty \tau_x^\alpha f(t)g(t)\,\mathrm{d}\mu_\alpha(t), \quad x \geqslant 0.$$

Then if $1 \leqslant p, q, r \leqslant \infty$ are such that $1/p + 1/q - 1 = 1/r$, then $f *_\alpha g \in L^r_\alpha(\mathbb{R}_+)$ and

$$\|f *_\alpha g\|_{L^r_\alpha} \leqslant \|f\|_{L^p_\alpha}\|g\|_{L^q_\alpha}.$$

This then allows to define $f *_\alpha g$ for $f \in L^p_\alpha(\mathbb{R}_+)$ and $g \in L^q_\alpha(\mathbb{R}_+)$. In particular, if $f \in L^1_\alpha(\mathbb{R}_+)$ and $g \in L^q_\alpha(\mathbb{R}_+)$, $q = 1$ or 2, then

$$\mathcal{H}_\alpha(f *_\alpha g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g). \tag{2.9}$$

Moreover, for $f, g \in L^2_\alpha(\mathbb{R}_+)$ the function $f *_\alpha g$ belongs to $L^2_\alpha(\mathbb{R}_+)$ if and only if the function $\mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g)$ belongs to $L^2_\alpha(\mathbb{R}_+)$ and then (2.9) holds.

2.4. The windowed Hankel transform. Following [2] for every $g \in L^2_\alpha(\mathbb{R}_+)$ the modulation of g by $\xi \in \widehat{\mathbb{R}}_+$ is defined by

$$\mathcal{M}_\xi^\alpha g := g_\xi^\alpha := \mathcal{H}_\alpha\Big(\sqrt{\tau_\xi^\alpha|\mathcal{H}_\alpha(g)|^2}\Big). \tag{2.10}$$

Then for every $g \in L^2_\alpha(\mathbb{R}_+)$ and $\xi \in \widehat{\mathbb{R}}_+$ we have:

$$\|g_\xi^\alpha\|_{L^2_\alpha} = \|g\|_{L^2_\alpha} \quad \text{and} \quad \|\mathcal{H}_\alpha(g_\xi^\alpha)\|_\infty \leqslant \|\mathcal{H}_\alpha(g)\|_\infty. \tag{2.11}$$

For a nonzero window function $g \in L^2_\alpha(\mathbb{R}_+)$ and $(x, \xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ we consider the function $g^\alpha_{x,\xi}$ defined by

$$g^\alpha_{x,\xi} = \tau_x^\alpha g_\xi^\alpha. \tag{2.12}$$

Therefore, for any function $f \in L^2_\alpha(\mathbb{R}_+)$ we define its windowed Hankel transform with respect to the window g by

$$\mathcal{V}_g^\alpha(f)(x,\xi) = \int_0^\infty f(s)\overline{g^\alpha_{x,\xi}(s)}\,\mathrm{d}\mu_\alpha(s), \quad (x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+, \tag{2.13}$$

which can also be written in the form

$$\mathcal{V}_g^\alpha(f)(x,\xi) = f *_\alpha g^\alpha_\xi(x). \tag{2.14}$$

Thus, from the Cauchy-Schwartz inequality and from (2.6), (2.11) we have

$$\|\mathcal{V}_g^\alpha(f)\|_\infty \leqslant \|f\|_{L^2_\alpha}\|g\|_{L^2_\alpha}. \tag{2.15}$$

Moreover, the windowed Hankel transform satisfies the following properties (see [2]).

Proposition 2.1. *Let $g \in L^2_\alpha(\mathbb{R}_+)$ be a nonzero window function. Then we have:*

(1) *A Plancherel's theorem: for every $f \in L^2_\alpha(\mathbb{R}_+)$*

$$\|\mathcal{V}_g^\alpha(f)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} = \|f\|_{L^2_\alpha}\|g\|_{L^2_\alpha}. \tag{2.16}$$

(2) *An orthogonality relation: for every $f, h \in L^2_\alpha(\mathbb{R}_+)$ we have*

$$\begin{aligned}\langle \mathcal{V}_g^\alpha(f), \mathcal{V}_g^\alpha(h)\rangle_{\nu_\alpha} &= \int_0^\infty\int_0^\infty \mathcal{V}_g^\alpha(f)(x,\xi)\overline{\mathcal{V}_g^\alpha(h)(x,\xi)}\,\mathrm{d}\nu_\alpha(x,\xi)\\ &= \|g\|^2_{L^2_\alpha}\int_0^\infty f(t)\overline{h(t)}\,\mathrm{d}\mu_\alpha(t),\end{aligned} \tag{2.17}$$

where $\langle\cdot,\cdot\rangle_{\nu_\alpha}$ is the usual inner product in the Hilbert space $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$.

3. The Gabor-Toeplitz and the phase-space restriction operators

In the remainder of this section, $g \in L^2_\alpha(\mathbb{R}_+)$ will be a nonzero window function such that $\|g\|_{L^2_\alpha} = 1$ and $\Sigma \subset \mathbb{R}_+ \times \widehat{\mathbb{R}}_+$ will be a subset of finite measure $0 < \nu_\alpha(\Sigma) < \infty$.

Definition 3.1. The spectrogram of a signal $f \in L^2_\alpha(\mathbb{R}_+)$ with respect to the window g is defined to be

$$\mathrm{SPEC}_g f(x,\xi) = |\mathcal{V}_g^\alpha(f)(x,\xi)|^2, \quad (x,\xi) \in \mathbb{R}_+ \times \widehat{\mathbb{R}}_+. \tag{3.1}$$

The spectrogram measures the distribution of the time-frequency content of f, and it is often interpreted as an energy density in time-frequency space. Its size depends on the window g and from the definition, it is non-negative. Moreover, by Plancherel's theorem (2.16), it is energy-preserving, i.e.

$$\|\mathrm{SPEC}_g f\|_{L^1_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} = \|f\|^2_{L^2_\alpha}. \tag{3.2}$$

Note that the uncertainty principle in Fourier analysis states that if a unit L^2_α-norm signal f is concentrated inside a region Σ in the time-frequency plane $\mathbb{R}_+\times\widehat{\mathbb{R}}_+$, then the area of Σ must be at least 1. More precisely (see [2], Proposition 3.7), if the spectrogram of f with respect to g satisfies

$$\int_\Sigma \mathrm{SPEC}_g f(x,\xi)\,\mathrm{d}\nu_\alpha(x,\xi) \geqslant 1-\varepsilon,$$

then $\nu_\alpha(\Sigma)\geqslant 1-\varepsilon^2$.

Let $\mathbb{H}=\mathcal{V}^\alpha_g[L^2_\alpha(\mathbb{R}_+)]\subset L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$ be the (closed) range of the $\mathcal{V}^\alpha_g$, and let P_g (or $P_\mathbb{H}$) be the orthogonal projection from $L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$ onto $\mathbb{H}$. The orthogonal projector P_g is an integral operator explicitly given by

$$P_gF(z) = \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} F(z')\mathcal{K}_g(z;z')\,\mathrm{d}\nu_\alpha(z'),\quad z=(x,\xi)\in\mathbb{R}_+\times\widehat{\mathbb{R}}_+. \tag{3.3}$$

Using this description, it follows that (see [4], Proposition 4.1) $\mathbb{H}$ is a reproducing kernel Hilbert space in $L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)$ with kernel function $\mathcal{K}_g$ defined by

$$\mathcal{K}_g((x',\xi');(x,\xi)) = g^\alpha_{x,\xi} *_\alpha g^\alpha_{\xi'}(x') = \mathcal{V}^\alpha_g(g^\alpha_{x,\xi})(x',\xi'). \tag{3.4}$$

This means that each function $F\in\mathbb{H}$ is continuous and satisfies:

$$F(x,\xi) = \langle F,\mathcal{K}_g(\cdot\,;(x,\xi))\rangle_{\nu_\alpha},\quad (x,\xi)\in\mathbb{R}_+\times\widehat{\mathbb{R}}_+. \tag{3.5}$$

Since $\mathcal{K}_g$ is the integral kernel of an orthogonal projection, it satisfies

$$\overline{\mathcal{K}_g(z';z)} = \mathcal{K}_g(z;z'),\quad z=(x,\xi),\ z'=(x',\xi')\in\mathbb{R}_+\times\widehat{\mathbb{R}}_+, \tag{3.6}$$

and

$$\mathcal{K}_g(z;z') = \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+}\mathcal{K}_g(z;z'')\mathcal{K}_g(z'';z')\,\mathrm{d}\nu_\alpha(z''). \tag{3.7}$$

Note also that for all $z=(x,\xi)$, $z'=(x',\xi')\in\mathbb{R}_+\times\widehat{\mathbb{R}}_+$,

$$|\mathcal{K}_g(z;z')|^2 = \mathrm{SPEC}_g g^\alpha_{x,\xi}(z'). \tag{3.8}$$

We introduce the orthogonal projections P_Σ on $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ known as the time-frequency limiting operator defined by:

$$P_\Sigma F = F\chi_\Sigma, \quad F \in L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+).$$

Since $\mathbb{H}$ is a reproducing kernel Hilbert space in $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$, then $P_\Sigma P_g$ is a Hilbert-Schmidt operator with (see [4], inequality (4.8))

$$\|P_\Sigma P_g\|^2_{HS} \leqslant \nu_\alpha(\Sigma). \tag{3.9}$$

Definition 3.2. We define the *Gabor-Toeplitz operator* $T_{g,\Sigma}\colon \mathbb{H} \to \mathbb{H}$ by

$$T_{g,\Sigma}\, F = P_g P_\Sigma F, \tag{3.10}$$

and the phase space restriction operator by

$$L_{g,\Sigma} = T_{g,\Sigma} T^*_{g,\Sigma} = P_g P_\Sigma P_g. \tag{3.11}$$

Since $T_{g,\Sigma}$ is Hilbert-Schmidt, then $L_{g,\Sigma}$ is trace-class and from (2.1), (3.9),

$$\operatorname{tr}(L_{g,\Sigma}) = \|T_{g,\Sigma}\|^2_{HS} \leqslant \nu_\alpha(\Sigma). \tag{3.12}$$

Moreover, for $F \in \mathbb{H}$,

$$\langle T_{g,\Sigma}F, F\rangle_{\nu_\alpha} = \langle P_g(P_\Sigma F), F\rangle_{\nu_\alpha} = \langle P_\Sigma F, F\rangle_{\nu_\alpha}. \tag{3.13}$$

Then

$$0 \leqslant T_{g,\Sigma} \leqslant P_\Sigma. \tag{3.14}$$

In particular, $T_{g,\Sigma}$ is bounded and positive. Furthermore, it is explicitly given by

$$T_{g,\Sigma}F(z) = \int_\Sigma F(z')\overline{\mathcal{K}_g(z';z)}\,\mathrm{d}\nu_\alpha(z') = \int_\Sigma F(z')\mathcal{K}_g(z;z')\,\mathrm{d}\nu_\alpha(z'). \tag{3.15}$$

The advantage of working with $L_{g,\Sigma}$ instead of $T_{g,\Sigma}$ is that it is defined on $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+)$ and consequently its spectral properties can be easily related to its integral kernel. Hence, $L_{g,\Sigma}$ is positive and with respect to the decomposition $L^2_\alpha(\mathbb{R}_+ \times \widehat{\mathbb{R}}_+) = \mathbb{H} \oplus \mathbb{H}^\perp$ we deduce that

$$\operatorname{tr}(L_{g,\Sigma}) = \operatorname{tr}(T_{g,\Sigma}). \tag{3.16}$$

In addition, we have the following result.

Proposition 3.3. *The trace of $T^2_{g,\Sigma}$ is given by*

$$\operatorname{tr}(T^2_{g,\Sigma}) = \int_\Sigma\int_\Sigma \mathrm{SPEC}_g g^\alpha_{x,\xi}(x',\xi')\,\mathrm{d}\nu_\alpha(x,\xi)\,\mathrm{d}\nu_\alpha(x',\xi'). \tag{3.17}$$

P r o o f. Since $L_{g,\Sigma}$ is positive, then

$$\operatorname{tr}(T^2_{g,\Sigma}) = \operatorname{tr}(L^2_{g,\Sigma}). \tag{3.18}$$

Now from formulas (3.3) and (3.15) we obtain

$$\begin{aligned} L_{g,\Sigma}F(z) &= T_{g,\Sigma}(P_gF)(z) \\ &= \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} F(z') \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \chi_\Sigma(z'')\mathcal{K}_g(z;z'')\mathcal{K}_g(z'';z')\,\mathrm{d}\nu_\alpha(z'')\,\mathrm{d}\nu_\alpha(z'). \end{aligned}$$

That is, $L_{g,\Sigma}$ has integral kernel

$$\mathcal{N}_{g,\Sigma}(z,z') = \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \chi_\Sigma(z'')\mathcal{K}_g(z;z'')\mathcal{K}_g(z'';z')\,\mathrm{d}\nu_\alpha(z''). \tag{3.19}$$

Then

$$L^2_{g,\Sigma}F(z) = \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} F(z'')\mathbf{N}_{g,\Sigma}(z,z'')\,\mathrm{d}\nu_\alpha(z'') \tag{3.20}$$

with

$$\mathbf{N}_{g,\Sigma}(z,z'') = \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \mathcal{N}_{g,\Sigma}(z,z')\mathcal{N}_{g,\Sigma}(z',z'')\,\mathrm{d}\nu_\alpha(z'). \tag{3.21}$$

Therefore

$$\begin{aligned} \operatorname{tr}(L^2_{g,\Sigma}) &= \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \mathbf{N}_{g,\Sigma}(z,z)\,\mathrm{d}\nu_\alpha(z) \\ &= \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+}\int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \mathcal{N}_{g,\Sigma}(z,z')\mathcal{N}_{g,\Sigma}(z',z)\,\mathrm{d}\nu_\alpha(z)\,\mathrm{d}\nu_\alpha(z') \\ &= \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+}\int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} \chi_\Sigma(z_1)\chi_\Sigma(z_2)K(z_1,z_2)\,\mathrm{d}\nu_\alpha(z_1)\,\mathrm{d}\nu_\alpha(z_2), \end{aligned}$$

where

$$\begin{aligned} K(z_1,z_2) = \int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+}\int_{\mathbb{R}_+\times\widehat{\mathbb{R}}_+} &\mathcal{K}_g(z_2;z)\mathcal{K}_g(z;z_1) \\ &\times \mathcal{K}_g(z_1;z')\mathcal{K}_g(z';z_2)\,\mathrm{d}\nu_\alpha(z)\,\mathrm{d}\nu_\alpha(z'). \end{aligned} \tag{3.22}$$

Using (3.6), (3.7) and (3.8), we get

$$K(z_1,z_2) = \mathrm{SPEC}_g g^\alpha_{x,\xi}(z_2), \quad z_1 = (x,\xi),\ z_2 = (x',\xi'). \tag{3.23}$$

This completes the proof. □

4. Quantitative dispersion inequality for orthonormal sequences

In this section we will prove the main result of this paper. Our proof is inspired by related results established in [7].

Definition 4.1. Let $0 < \varepsilon < 1$ and let f, $g \in L^2_\alpha(\mathbb{R}_+)$ be two nonzero functions and Σ be a measurable subset of $\mathbb{R}_+ \times \widehat{\mathbb{R}}_+$. Then we say that $\mathcal{V}_g^\alpha(f)$ is *ε-time-frequency concentrated on* Σ if

$$\|P_{\Sigma^c}\mathcal{V}_g^\alpha(f)\|_{L^2(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} \leqslant \varepsilon\|f\|_{L^2_\alpha}\|g\|_{L^2_\alpha}. \tag{4.1}$$

If we take $\varepsilon = 0$ in inequality (4.1), then Σ will be the exact support of $\mathcal{V}_g^\alpha(f)$, so when $0 < \varepsilon < 1$, inequality (4.1) means that $\mathcal{V}_g^\alpha(f)$ is "practically zero" outside Σ. Indeed, Σ may be considered as the "essential" support of $\mathcal{V}_g^\alpha(f)$.

Theorem 4.2. *Let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system in $L^2_\alpha(\mathbb{R}_+)$. If $\mathcal{V}_g^\alpha(\varphi_n)$ is ε_n-time-frequency concentrated on Σ, then*

$$\sum_{n=1}^N(1-\varepsilon_n) \leqslant \|g\|^{-2}_{L^2_\alpha}\nu_\alpha(\Sigma). \tag{4.2}$$

Proof. Since

$$\begin{aligned}\sum_{n=1}^N \langle P_\Sigma\mathcal{V}_g^\alpha(\varphi_n), \mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha} &= \sum_{n=1}^N \langle P_g P_\Sigma\mathcal{V}_g^\alpha(\varphi_n), P_g\mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha}\\ &= \sum_{n=1}^N \langle L_{g,\Sigma}\mathcal{V}_g^\alpha(\varphi_n), \mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha}\\ &\leqslant \operatorname{tr}(L_{g,\Sigma}) = \|P_\Sigma P_g\|^2_{HS},\end{aligned}$$

by (3.9) we obtain

$$\sum_{n=1}^N \langle P_\Sigma\mathcal{V}_g^\alpha(\varphi_n), \mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha} \leqslant \nu_\alpha(\Sigma). \tag{4.3}$$

On the other hand, as

$$\langle P_\Sigma\mathcal{V}_g^\alpha(\varphi_n), \mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha} = \|g\|^2_{L^2_\alpha} - \langle P_{\Sigma^c}\mathcal{V}_g^\alpha(\varphi_n), \mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha}, \tag{4.4}$$

by Cauchy-Schwartz inequality,

$$\langle P_\Sigma\mathcal{V}_g^\alpha(\varphi_n), \mathcal{V}_g^\alpha(\varphi_n)\rangle_{\nu_\alpha} \geqslant \|g\|^2_{L^2_\alpha}(1-\varepsilon_n). \tag{4.5}$$

Therefore from (4.3), we deduce the desired result. □

Now we will fix g to be a nonzero window function in $L^2_\alpha(\mathbb{R}_+)$ with $\|g\|_{L^2_\alpha} = 1$. Thus, from Theorem 4.2 we can obtain immediately the following corollary.

Corollary 4.3. *Let $0 < \varepsilon < 1$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system in $L^2_\alpha(\mathbb{R}_+)$. If $\mathcal{V}_g^\alpha(\varphi_n)$ is ε-time-frequency concentrated on $\mathcal{B}_r$, then*

$$N \leqslant \frac{r^{4\alpha+4}}{2^{2\alpha+2}\Gamma(2\alpha+3)}\frac{1}{1-\varepsilon}.$$

Therefore if the generalized dispersion of the elements of an orthonormal sequence is uniformly bounded, then this sequence is finite and we can give a bound on the number of elements in that sequence. More precisely:

Corollary 4.4. *Fix $A > 0$. Let $\{\varphi_n\}_{n=1}^N$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$ that satisfies $\||(x,\xi)|^s\mathcal{V}_g^\alpha(\varphi_n)\|^{1/s}_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} \leqslant A$. Then there exists a positive constant $c_{s,\alpha}$ such that*

$$N \leqslant c_{s,\alpha}A^{4\alpha+4}.$$

Proof. Since

$$\|P_{\mathcal{B}_r^c}\mathcal{V}_g^\alpha(\varphi_n)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} \leqslant r^{-s}\||(x,\xi)|^s\mathcal{V}_g^\alpha(\varphi_n)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}, \tag{4.6}$$

if we choose $r = 4^{1/s}A$, we deduce that φ_n is $1/4$-time-frequency concentrated on $\mathcal{B}_r$. Therefore from Corollary 4.3 we obtain the desired result. □

Our proof of inequality (1.8) formulated in the Introduction is based on the following lemma.

Lemma 4.5. *Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$. Then there exists $j_0 \in \mathbb{Z}$ such that*

$$\forall n \geqslant 1, \quad \max\big(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}, \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\big) \geqslant 2^{s(j_0-1)}. \tag{4.7}$$

Proof. Direct consequence of Heisenberg-type inequality (1.7). □

Theorem 4.6. *Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$. Then for every $N \geqslant 1$,*

$$\sum_{n=1}^N \big(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} + \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\big) \geqslant c_{s,\alpha}N^{1+s/(2\alpha+2)}. \tag{4.8}$$

P r o o f. For each $j \in \mathbb{Z}$ we define

$$P_j = \bigl\{n\colon \max\bigl(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^{1/s}_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}, \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^{1/s}_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\bigr) \in [2^{j-1}, 2^j)\bigr\}.$$

First, by inequality (4.7), we see that P_j is empty for all $j < j_0$. Moreover, using the fact that

$$|a+b|^s \leqslant 2^s(|a|^s + |b|^s), \tag{4.9}$$

we obtain that for each $n \in P_j$, $j \geqslant j_0$,

$$\||(x,\xi)|^s\mathcal{V}_g^\alpha(\varphi_n)\|^{1/s}_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} \leqslant 2^{(s+1)/2s+j}. \tag{4.10}$$

Therefore, by Corollary 4.4, P_j is finite for all $j \geqslant j_0$, and if we denote by N_j the number of elements in P_j, then

$$N_j \leqslant c_{s,\alpha}\, 4^{j(2\alpha+2)}.$$

Therefore, for every $m \geqslant j_0$ the number of elements in $\bigcup\limits_{j=j_0}^{m} P_j$ is less than $c_{s,\alpha}\, 4^{m(2\alpha+2)}$, where $c_{s,\alpha}$ is a constant that does not depend on m.

Now if $N > 2c_{s,\alpha}4^{j_0(2\alpha+2)}$, then we can choose an integer $m > j_0$ such that

$$2c_{s,\alpha}4^{(m-1)(2\alpha+2)} < N \leqslant 2c_{s,\alpha}\, 4^{m(2\alpha+2)}.$$

Therefore at least half of $\{1,\dots,N\}$ does not belong to $\bigcup\limits_{j=j_0}^{m-1} P_j$ and we obtain

$$\begin{aligned}\sum_{n=1}^{N}\bigl(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} + \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\bigr)&\\ \geqslant \frac{N}{2}4^{s(m-1)} \geqslant \frac{1}{2}\frac{N}{4^s}\Bigl(\frac{N}{2c_{s,\alpha}}\Bigr)^{s/(2\alpha+2)} = \frac{(2c_{s,\alpha})^{-s/(2\alpha+2)}}{2^{1+2s}}N^{1+s/(2\alpha+2)}&.\end{aligned}$$

Finally, if $N \leqslant 2c_{s,\alpha}4^{j_0(2\alpha+2)}$, then from Lemma 4.5 we have

$$\begin{aligned}\sum_{n=1}^{N}\bigl(\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} + \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\bigr)&\\ \geqslant N4^{s(j_0-1)} \geqslant \frac{N}{4^s}\Bigl(\frac{N}{2c_{s,\alpha}}\Bigr)^{s/(2\alpha+2)} = \frac{(2c_{s,\alpha})^{-s/(2\alpha+2)}}{4^s}N^{1+s/(2\alpha+2)}&.\end{aligned}$$

This completes the proof. $\square$

The last dispersion inequality implies in particular that there does not exist an infinite sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $L^2_\alpha(\mathbb{R}_+)$ such that the two sequences

$$\{\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\}_{n=1}^{\infty} \quad \text{and} \quad \{\|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\}_{n=1}^{\infty}$$

are bounded. More precisely:

Corollary 4.7. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L^2_\alpha(\mathbb{R}_+)$. Then for every $N \geqslant 1$,*

$$\sup_{1\leqslant n\leqslant N} \{\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}, \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}\} \geqslant c_{s,\alpha}N^{s/(2\alpha+2)}. \tag{4.11}$$

In particular,

$$\sup_n (\|x^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)} + \|\xi^s\mathcal{V}_g^\alpha(\varphi_n)\|^2_{L^2_\alpha(\mathbb{R}_+\times\widehat{\mathbb{R}}_+)}) = \infty. \tag{4.12}$$

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