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L^p HARMONIC 1-FORM ON SUBMANIFOLD WITH WEIGHTED POINCARÉ INEQUALITY

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Abstract. We deal with complete submanifolds with weighted Poincaré inequality. By assuming the submanifold is δ-stable or has sufficiently small total curvature, we establish two vanishing theorems for L^p harmonic 1-forms, which are extensions of the results of Dung-Seo and Cavalcante-Mirandola-Vitório.

Keywords: weighted Poincaré inequality; δ-stability; L^p harmonic 1-form; property $(\mathcal{P}_\varrho)$

MSC 2010: 53C42, 53C50

1. Introduction

The topological property and vanishing theorems of submanifolds in various ambient spaces have been studied during a few past years. Specially, the nonexistence of nontrivial L^2 harmonic 1-forms on a complete noncompact submanifold has been studied by many geometricians. Palmer in [17] proved that a complete minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ has no nontrivial L^2 harmonic 1-forms. Thereafter, using Bochner's vanishing technique, Miyaoka in [16] showed that a complete orientable noncompact stable minimal hypersurface in a Riemannnian manifold with nonnegative sectional curvature has no nontrivial L^2 harmonic 1-forms. Later, this result was extended to more general ambient spaces, see [12], [15], [24]. Seo in [19] proved the vanishing theorem holds for a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with the first eigenvalue of the Laplacian satisfying $\lambda_1 > (2n-1)(n-1)$. Moreover, Dung and Seo in [5] obtained the vanishing result holds for a complete noncompact stable non-totally geodesic minimal hypersurface in a Riemannian manifold N with $K \leqslant K_N$, $K \leqslant 0$ and $\lambda_1(M) > -K(2n-1)(n-1)$. Moreover, it turned

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out that these vanishing theorems hold for more general Riemannian manifolds with property $(\mathcal{P}_\varrho)$. We say that an n-dimensional complete Riemannian manifold M has *property* $(\mathcal{P}_\varrho)$, if a weighted Poincaré inequality is valid on M with some nonnegative weight function $\varrho(x)$, namely

$$\int_M \varrho(x)\eta^2 \leqslant \int_M |\nabla\eta|^2, \quad \forall\, \eta \in C_0^\infty(M). \tag{1.1}$$

Moreover, the ϱ-metric, defined by $\mathrm{d}s_\varrho^2 = \varrho\,\mathrm{d}s_M^2$, is complete. In particular, if $\lambda_1(M)$ is assumed to be positive, then obviously M possesses property $(\mathcal{P}_\varrho)$ with $\varrho = \lambda_1(M)$. So, the notion of property $(\mathcal{P}_\varrho)$ may be viewed as a generalization of the assumption $\lambda_1(M) > 0$. Recently, Sang and Thanh in [18] proved that a complete noncompact stable minimal hypersurface with property $(\mathcal{P}_\varrho)$ in a Riemannian manifold N has no nontrivial L^2 harmonic 1-form if the sectional curvature of N satisfies $K_N(x) \geqslant -(1-\tau)\varrho(x)/((2n-1)(n-1))$, $0 < \tau \leqslant 1$, and $\varrho(x)$ satisfies a certain growth condition.

A natural question is how about the nonexistence results of nontrivial L^p, $p \neq 2$, harmonic 1-forms of submanifolds? Yau in [25] proved that there are no nonconstant L^p, $1 < p < \infty$, harmonic functions on a complete Riemannian manifold. Li and Schoen in [14] proved that Yau's result is valid for L^p, $0 < p < \infty$, harmonic functions on a complete manifold with nonnegative Ricci curvature. For L^p harmonic forms, Greene and Wu in [8] and [9] showed that there are no nontrivial L^p, $1 \leqslant p < \infty$, ones on a complete Riemannian manifold or a Kähler manifold of nonnegative curvature. Recently, Seo in [21] considered this problem, and proved that there are no nontrivial L^{2p} harmonic 1-forms on a stable minimal hypersurface M^n of a Riemannian manifold N with $K_N \geqslant K$, $K \leqslant 0$, under the assumption

$$\lambda_1(M) > \frac{-2n(n-1)^2p^2K}{2n-[(n-1)p-n]^2}$$

for $0 < p < n/(n-1) + \sqrt{2n}$. Besides, Dung and Seo in [6] considered the problem on a complete δ-stability hypersurface in a Riemannian manifold with nonnegative sectional curvature. First, we recall the definition of δ-stability which is a generalization of the usual stability.

Definition 1.1. Let M^n be an n-dimensional orientable hypersurface in a Riemannian manifold N. We say M is δ-stable for $0 < \delta \leqslant 1$ if the inequality

$$\int_M |\nabla\eta|^2 \geqslant \delta \int_M (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\eta^2 \tag{1.2}$$

holds for any $\eta \in C_0^\infty(M)$, where ν is a unit normal vector field on M, $\overline{\mathrm{Ric}}$ is the Ricci curvature of N and A is the second fundamental form of M.

It is obvious that δ_1-stability implies δ_2-stability for $0 < \delta_2 < \delta_1 \leqslant 1$. In particular, if M is stable, then M is δ-stable for $0 < \delta \leqslant 1$.

For δ-stable complete hypersurfaces in a Riemannian manifold, there have been some vanishing theorems. For $\delta > 1/8$, Kawai in [11] proved that a δ-stable complete minimal surface in $\mathbb{R}^3$ must be a plane. Tam and Zhou in [23] showed that a complete $(n-2)/n$-stable minimal hypersurface in the Euclidean space is either a hyperplane or a catenoid if its second fundamental form satisfies some decay conditions. Dung and Seo in [6] proved the following vanishing theorem.

Theorem 1.2 ([6]). *Let M^n, $2 \leqslant n \leqslant 6$, be a complete orientable noncompact hypersurface in a complete manifold N with nonnegative sectional curvature. If the δ-stability inequality* (1.2) *holds on M for some $(n-2)/(2\sqrt{n-1}) < \delta \leqslant 1$, then there is no nontrivial L^{2p} harmonic 1-form on M for any constant p satisfying*

$$\frac{2\delta}{\sqrt{n-1}}\left(1-\sqrt{1-\frac{n-2}{2\delta\sqrt{n-1}}}\right) < p < \frac{2\delta}{\sqrt{n-1}}\left(1+\sqrt{1-\frac{n-2}{2\delta\sqrt{n-1}}}\right).$$

In the first part of this paper, motivated by all the above results, we will consider the nonexistence of a nontrivial L^p harmonic 1-form of a complete δ-stable hypersurface with property $(\mathcal{P}_\varrho)$ in a Riemannian manifold with sectional curvature bounded below by a nonpositive function. More precisely, we have the following theorem.

Theorem 1.3. *Let M^n, $2 \leqslant n \leqslant 6$, be a complete noncompact hypersurface with property $(\mathcal{P}_\varrho)$ in an $(n+1)$-dimensional Riemannian manifold N. Assume that*

$$K_N(x) \geqslant -\frac{(1-\tau)\varrho(x)}{(2n-1)(n-1)} \quad \forall\, x \in M$$

for some τ: $(122-51\sqrt{5})(12+4\sqrt{5})^{-1} < \tau \leqslant 1$. If the δ-stability inequality (1.2) *holds on M for some δ: $(n-2)(2\sqrt{n-1}-(n-2)C_0)^{-1} < \delta \leqslant 1$, then there is no nontrivial L^{2p} harmonic 1-form on M for any constant p satisfying $C_1(n,\delta,\tau) < p < C_2(n,\delta,\tau)$, where*

$$C_0 = \frac{(2\sqrt{n-1}+n)(1-\tau)}{(2n-1)(n-1)},$$

$$C_1(n,\delta,\tau) = \frac{2\delta\left(1-\sqrt{1-\frac{n-2}{2\delta\sqrt{n-1}}(1+C_0\delta)}\right)}{\sqrt{n-1}(1+C_0\delta)},$$

$$C_2(n,\delta,\tau) = \frac{2\delta\left(1+\sqrt{1-\frac{n-2}{2\delta\sqrt{n-1}}(1+C_0\delta)}\right)}{\sqrt{n-1}(1+C_0\delta)}.$$

Remark 1.4. (i) When $\tau \equiv 1$, i.e. $K_N \geqslant 0$ and $C_0 \equiv 0$ on M, we obtain Theorem 1.2. In this case, $|\omega|$ is a constant from the proof of Theorem 1.3, and using Lemma 2.5 we conclude that $\omega \equiv 0$. As a result, the assumption of property $(\mathcal{P}_\varrho)$ of M is not needed.

(ii) If we relax τ in Theorem 1.3 to $0 < \tau \leqslant 1$, then Theorem 1.3 holds only when the dimension of M is $2, 3, 4, 5$, because from (3.12) we know that $(n-2)/(2\sqrt{n-1} - (n-2)C_0) < \delta \leqslant 1$ holds only when $2 \leqslant n \leqslant 5$.

(iii) When $\delta = 1$ and $p = 1$, Theorem 1.3 is just Theorem 1.2 in [4].

If we choose $\varrho(x) = \lambda_1(M)$ in Theorem 1.3, we have the following corollary.

Corollary 1.5. *Let N^{n+1} be an $(n+1)$-dimensional Riemannian manifold with sectional curvature $K_N \geqslant K$, where K is a nonpositive constant. Let M^n, $2 \leqslant n \leqslant 6$, be a complete noncompact hypersurface in N. Assume further that*

$$\lambda_1(M) \geqslant -\frac{(2n-1)(n-1)K}{1-\tau}$$

for some constant τ: $(122 - 51\sqrt{5})(12 + 4\sqrt{5})^{-1} < \tau < 1$. If the δ-stability inequality (1.2) *holds on M for some $(n-2)(2\sqrt{n-1} - (n-2)C_0)^{-1} < \delta \leqslant 1$, then there is no nontrivial L^{2p} harmonic* 1*-form on M for any constant p satisfying $C_1(n, \delta, \tau) < p < C_2(n, \delta, \tau)$, where C_0, $C_1(n, \delta, \tau)$ and $C_2(n, \delta, \tau)$ are defined in Theorem* 1.3.

Moreover, we can prove a vanishing theorem for L^p harmonic 1-forms on complete noncompact hypersurfaces with property $(\mathcal{P}_\varrho)$ similar to Theorem 1.3 except for the condition that the lower bound of K_N depends on δ, p, ϱ. More precisely, we have

Theorem 1.6. *Let N^{n+1} be an $(n+1)$-dimensional Riemannian manifold, and let M^n, $2 \leqslant n \leqslant 6$, be a complete noncompact hypersurface satisfying the weighted Poincaré inequality for some nonnegative function ϱ in N. If the δ-stability inequality* (1.2) *holds on M for some $(n-2)/(2\sqrt{n-1}) < \delta \leqslant 1$, and*

$$K_N > -\frac{4p\delta(n-1) - 2(n-2)\delta - (n-1)\sqrt{n-1}p^2}{\delta p^2(n-1)(2n-2+n\sqrt{n-1})}\varrho,$$

where p satisfies

$$\frac{2\delta}{\sqrt{n-1}}\left(1 - \sqrt{1 - \frac{n-2}{2\delta\sqrt{n-1}}}\right) < p < \frac{2\delta}{\sqrt{n-1}}\left(1 + \sqrt{1 - \frac{n-2}{2\delta\sqrt{n-1}}}\right),$$

then there is no nontrivial L^{2p} harmonic 1*-form on M.*

On the other hand, without the assumption of stability, some vanishing theorems about L^2 harmonic 1-forms have also been obtained. In [26], Yun proved that if $M \hookrightarrow \mathbb{R}^{n+1}$ is a complete minimal hypersurface with sufficiently small total scalar curvature $\|A\|^2_{L^n}$, then there is no nontrivial L^2 harmonic 1-form on M. Later, Seo in [20] proved this result is valid for a complete minimal hypersurface in a hyperbolic space. Thereafter, it turned out that these vanishing theorems hold for more general submanifolds, see [2], [7]. Recently, Cavalcante, Mirandola and Vitório in [3] showed that a complete noncompact submanifold M in a Hadamard manifold N with sectional curvature satisfying $-k^2 \leqslant K_N \leqslant 0$ has no nontrivial L^2 harmonic 1-forms, if the total curvature $\|\Phi\|^2_{L^n}$ is sufficiently small, and with the additional assumption $\lambda_1(M) > (n-1)^2 n^{-1}\Big(k^2 - \inf_M H^2\Big)$ in the case of $K_N \not\equiv 0$. After that, Dung and Seo in [6] proved a similar vanishing theorem for L^2 harmonic 1-forms on complete noncompact submanifolds under the same assumption as in [3] except for the condition that the lower bound of $\lambda_1(M)$ depends on $\|\Phi\|^2_{L^n}$.

In the second part of this paper, motivated by the above results, we prove the following nonexistence result of L^p harmonic 1-forms on a complete noncompact submanifold with property $(\mathcal{P}_\varrho)$, assuming that the total curvature of the submanifold is sufficiently small instead of the assumption of δ-stability. More precisely, We have the following vanishing theorem which is an extension of Theorem 1.2 in [3] and Theorem 1.5 in [4].

Theorem 1.7. *Let M^n be a complete noncompact submanifold with property $(\mathcal{P}_\varrho)$ for some nonnegative function ϱ in a Riemannian manifold N. Assume that*

$$0 \geqslant K_N(x) \geqslant -\frac{n(1-\tau)}{(n-1)^2}\varrho(x) - \gamma \inf_M H^2 \quad \forall\, x \in M$$

for some constants $\tau\colon\ 0 < \tau < 1$ and $\gamma\colon\ 0 \leqslant \gamma < 1$. If there exists a sufficiently small positive constant Λ such that $\|\Phi\|_{L^n} < \Lambda$, then there is no nontrivial L^{2p} harmonic 1-form on M, where p satisfies

$$\frac{n-1-\sqrt{(n-1)^2-n(n-2)(1-\tau)}}{n(1-\tau)} < p < \frac{n-1+\sqrt{(n-1)^2-n(n-2)(1-\tau)}}{n(1-\tau)}.$$

In particular, if we choose $\varrho(x) = \lambda_1(M)$ in Theorem 1.7, we get the following corollary.

Corollary 1.8. *Let N be a Riemannian manifold with $0 \geqslant K_N \geqslant K$, where K is a nonpositive constant. Let M^n be a complete noncompact submanifold in N. In* *the case of* $K_N \not\equiv 0$, *assume further that*

$$\lambda_1(M) \geqslant \frac{(n-1)^2}{(1-\tau)n}\Big(-K - \gamma \inf_M H^2\Big)$$

for some constants $\tau\colon 0 < \tau < 1$ *and* $\gamma\colon 0 \leqslant \gamma < 1$. *If there exists a positive constant* Λ *such that* $\|\Phi\|_{L^n} < \Lambda$, *then there is no nontrivial* L^{2p} *harmonic* 1*-form on* M, *where* p *satisfies*

$$\frac{n-1-\sqrt{(n-1)^2-n(n-2)(1-\tau)}}{n(1-\tau)} < p < \frac{n-1+\sqrt{(n-1)^2-n(n-2)(1-\tau)}}{n(1-\tau)}.$$

If M is a complete minimal submanifold in Theorem 1.7, we also obtain a vanishing result on L^{2p} harmonic 1-forms. In this case, the upper bound of $\|A\|_{L^n}$ has a specific expression which depends on p, n and the lower bound of the sectional curvature of the ambient space.

Theorem 1.9. *Let* M^n *be a complete noncompact minimal submanifold with property* $(\mathcal{P}_\varrho)$ *for some nonnegative function* ϱ *in a Riemannian manifold* N. *Assume that*

$$0 \geqslant K_N(x) \geqslant -\frac{n(1-\tau)}{(n-1)^2}\varrho(x) \quad \forall\, x \in M$$

for some $0 < \tau < 1$, *and*

$$\|A\|^2_{L^n} < n\frac{2(n-1)p - n + 2 - n(1-\tau)p^2}{(n-1)^2p^2S}$$

for some $\big(n-1-\sqrt{(n-1)^2-n(n-2)(1-\tau)}\big)n^{-1}(1-\tau)^{-1} < p < \big(n-1+\sqrt{(n-1)^2-n(n-2)(1-\tau)}\big)n^{-1}(1-\tau)^{-1}$, *where* $S = S(n,2)$ *is the Sobolev constant in Lemma* 2.6. *Then there is no nontrivial* L^{2p} *harmonic* 1*-form on* M.

2. Some lemmas

Let us recall some useful results which will be used in the proofs of the main theorems. The first two lemmas are the Bochner-Weitzenböck formula and the refined Kato inequality for L^2 harmonic forms.

Lemma 2.1 ([13]). *Given a Riemannian manifold* M^n *for any* 1*-form* ω *on* M^n *we have*

$$\Delta|\omega|^2 = 2|\nabla\omega|^2 + 2\langle\Delta\omega,\omega\rangle + 2\operatorname{Ric}(\omega^\sharp,\omega^\sharp),$$

where $\omega^\sharp$ *is the dual vector field of* ω.

Lemma 2.2 ([1]). *Given a Riemannian manifold M^n for any closed and coclosed k-form ω on M^n we have*

$$|\nabla\omega|^2 \geqslant C_{n,k}\big|\nabla|\omega|\big|^2, \quad \text{where} \quad C_{n,k} = \begin{cases} \dfrac{n-k+1}{n-k}, & 1 \leqslant k \leqslant \dfrac{n}{2}, \\ \dfrac{k+1}{k}, & \dfrac{n}{2} \leqslant k \leqslant n-1. \end{cases}$$

What's more, Shiohama and Xu in [22] proved the following estimate on the Ricci curvature of a submanifold.

Lemma 2.3 ([22]). *Let M be an n-dimensional complete immersed hypersurface in a Riemannian manifold N. If all sectional curvatures of N are bounded pointwise from below by a function k, then*

$$\mathrm{Ric} \geqslant (n-1)(H^2+k) - \frac{n-1}{n}|\Phi|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi|.$$

We should note that in [22], the author assumed that all sectional curvatures of N are bounded below by a constant k. But according to his argument, this assumption was only used at the end of the proof, hence this method can be used to prove the above lemma without any change.

Lemma 2.4. *Let M^n be an n-dimensional orientable submanifold in a Riemannian manifold N. Assuming that H is the mean curvature and A is the second fundamental form of M, we have*

$$(2.1)\quad 2(n-1)H^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H|\sqrt{|A|^2-nH^2} \geqslant \frac{2(n-1)-n\sqrt{n-1}}{2n}|A|^2.$$

Proof. If $|A|=0$, then from $|A|^2-nH^2=|\Phi|^2 \geqslant 0$ we have $H \equiv 0$. Thus the inequality (2.1) is trivial. Now we assume that $|A|>0$. We define $f_n(t)$ on $[0,1/\sqrt{n}]$ by

$$f_n(t) = 2(n-1)t^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}t\sqrt{1-nt^2}.$$

Supposing that c is a constant such that $\min\limits_{[0,1/\sqrt{n}]} f_n(t) \geqslant c$, we have

$$2(n-1)t^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}t\sqrt{1-nt^2} \geqslant c \quad \forall\, t \in \Big[0, \frac{1}{\sqrt{n}}\Big],$$

i.e.

$$n^3(n-1)x^2 - (n-1)(4cn + (n-2)^2)x + c^2 n \geqslant 0,$$

where $x = t^2$ for all $x \in [0, 1/n]$. A simple computation shows that this inequality is equivalent to

$$g_n(c) = 4n^2c^2 - (n-2)^2(n-1) - 8cn(n-1) \geqslant 0.$$

The discriminant of $g_n(c)$ is $\Delta = 16n^4(n-1)$. Thus we get that $c \leqslant (2(n-1) - n\sqrt{n-1})/(2n)$, which completes the proof. □

Moreover, we will need the conditions for the volume of the Riemannian manifold to be infinite.

Lemma 2.5 ([6]). *Let M^n be a complete oriented noncompact immersed hypersurface in a complete Riemannian manifold N^{n+1} with nonnegative sectional curvature. If the δ-stability inequality* (1.2) *holds on M for a constant $\delta\colon\ 0 < \delta \leqslant 1$, then the volume of M is infinite.*

In addition, the following Hoffman-Spruch inequality is also useful.

Lemma 2.6 ([10]). *Let $x\colon M^n \hookrightarrow N$ be an isometric immersion of a complete manifold M in a complete simply connected manifold N with nonpositive sectional curvature. Then for all $1 \leqslant l < n$, the following inequality holds:*

$$\left(\int_M h^{ln/(n-l)}\,\mathrm{d}V\right)^{(n-l)/n} \leqslant S(n,l)\int_M (|\nabla h|^l + (h|H|)^l)\,\mathrm{d}V$$

for all nonnegative C^1-functions $h\colon M^n \to \mathbb{R}$ with compact support, where $S(n,l)^{1/l} = c(n)2l(n-1)/(n-l)$ and $c(n)$ is a positive constant, depending only on n.

The last but most important lemma was proved by Vieira in [24].

Lemma 2.7 ([24]). *Let M be a complete manifold satisfying a weighted Poincaré inequality with a weight function ϱ. Suppose a smooth function u on M satisfies the differential inequality*

$$u\Delta u \geqslant -a\varrho u^2 + b|\nabla u|^2$$

for a constant $0 < a < 1 + b$, and assume

$$\int_M u^2 < \infty.$$

Then the function u is a constant. Moreover, if u is not identically zero, then the volume of M is finite and the weight function ϱ is identically zero.

3. Proof of the main theorems

P r o o f of Theorem 1.3. Let ω be an L^{2p} harmonic 1-form. Using the Weitzenböck formula and the Kato inequality, we get that

$$|\omega|\Delta|\omega| \geqslant \frac{1}{n-1}|\nabla|\omega||^2 + \operatorname{Ric}(\omega^\sharp,\omega^\sharp). \tag{3.1}$$

Under our hypothesis on the sectional curvature of N, we can estimate the Ricci curvature of M by using Lemma 2.3 and Lemma 2.4:

$$\begin{aligned}
\operatorname{Ric}_M &\geqslant -(n-1)\frac{(1-\tau)\varrho}{(2n-1)(n-1)} + (n-1)H^2 - \frac{n-1}{n}|\Phi|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi| \\
&= -\frac{(1-\tau)\varrho}{2n-1} + 2(n-1)H^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H|\sqrt{|A|^2-nH^2} - \frac{n-1}{n}|A|^2 \\
&\geqslant -\frac{(1-\tau)\varrho}{2n-1} + \frac{2(n-1)-n\sqrt{n-1}}{2n}|A|^2 - \frac{n-1}{n}|A|^2 \\
&= -\frac{(1-\tau)\varrho}{2n-1} - \frac{\sqrt{n-1}}{2}|A|^2.
\end{aligned}$$

Thus equation (3.1) becomes

$$|\omega|\Delta|\omega| \geqslant \frac{1}{n-1}|\nabla|\omega||^2 - \frac{(1-\tau)\varrho}{2n-1}|\omega|^2 - \frac{\sqrt{n-1}}{2}|A|^2|\omega|^2. \tag{3.2}$$

Given any $\alpha > 0$, using (3.2) we have that

$$\begin{aligned}
|\omega|^\alpha\Delta|\omega|^\alpha &= |\omega|^\alpha(\alpha(\alpha-1)|\omega|^{\alpha-2}|\nabla|\omega||^2 + \alpha|\omega|^{\alpha-1}\Delta|\omega|) \\
&= \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2 + \alpha|\omega|^{2\alpha-2}|\omega|\Delta|\omega| \\
&\geqslant \Big(1-\frac{n-2}{(n-1)\alpha}\Big)|\nabla|\omega|^\alpha|^2 - \frac{\alpha\sqrt{n-1}}{2}|A|^2|\omega|^{2\alpha} - \frac{(1-\tau)\varrho\alpha}{2n-1}|\omega|^{2\alpha}.
\end{aligned} \tag{3.3}$$

Given $s > 0$ and a smooth function η with compact support in M, multiplying both sides of the inequality (3.3) by $|\omega|^{2s\alpha}\eta^2$ and integrating over M, we obtain that

$$\begin{aligned}
&\Big(1-\frac{n-2}{(n-1)\alpha}\Big)\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 \\
&\quad\leqslant \int_M |\omega|^{(2s+1)\alpha}\eta^2\Delta|\omega|^\alpha + \frac{\alpha\sqrt{n-1}}{2}\int_M |A|^2|\omega|^{2(s+1)\alpha}\eta^2 \\
&\qquad + \frac{\alpha(1-\tau)}{2n-1}\int_M \varrho|\omega|^{2(s+1)\alpha}\eta^2 \\
&\quad= -(2s+1)\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 - 2\int_M \eta|\omega|^{(2s+1)\alpha}\langle\nabla\eta,\nabla|\omega|^\alpha\rangle \\
&\qquad + \frac{\alpha\sqrt{n-1}}{2}\int_M |A|^2|\omega|^{2(s+1)\alpha}\eta^2 + \frac{\alpha(1-\tau)}{2n-1}\int_M \varrho|\omega|^{2(s+1)\alpha}\eta^2,
\end{aligned}$$

i.e.

$$(3.4)\qquad \Big(2(s+1)-\frac{n-2}{(n-1)\alpha}\Big)\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2$$
$$\leqslant -2\int_M \eta|\omega|^{(2s+1)\alpha}\langle\nabla\eta,\nabla|\omega|^\alpha\rangle+\frac{\alpha\sqrt{n-1}}{2}\int_M |A|^2|\omega|^{2(s+1)\alpha}\eta^2$$
$$+\frac{\alpha(1-\tau)}{2n-1}\int_M \varrho|\omega|^{2(s+1)\alpha}\eta^2.$$

On the other hand, replacing η by $|\omega|^{(s+1)\alpha}\eta$ in (1.2) and applying the lower bound of the sectional curvature of N allows us to conclude that

$$(3.5)\quad \delta\int_M |A|^2|\omega|^{2(s+1)\alpha}\eta^2 \leqslant \int_M |\nabla(|\omega|^{(s+1)\alpha}\eta)|^2+\frac{n\delta(1-\tau)}{(2n-1)(n-1)}\int_M \varrho|\omega|^{2(s+1)\alpha}\eta^2$$
$$=(s+1)^2\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2+\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2$$
$$+2(s+1)\int_M |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^\alpha\rangle$$
$$+\frac{n\delta(1-\tau)}{(2n-1)(n-1)}\int_M \varrho|\omega|^{2(s+1)\alpha}\eta^2.$$

Combining (3.4) with (3.5), we obtain that

$$(3.6)\qquad \Big(2(s+1)-\frac{n-2}{(n-1)\alpha}-\frac{\alpha\sqrt{n-1}}{2}\frac{(s+1)^2}{\delta}\Big)\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2$$
$$\leqslant \frac{\alpha\sqrt{n-1}}{2\delta}\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2+E\int_M \varrho|\omega|^{2(s+1)\alpha}\eta^2$$
$$+\Big(\frac{\alpha\sqrt{n-1}}{2}\frac{2(s+1)}{\delta}-2\Big)\int_M |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^\alpha\rangle,$$

where

$$E=\Big(\frac{n\sqrt{n-1}}{2}+n-1\Big)\frac{\alpha(1-\tau)}{(2n-1)(n-1)}.$$

From the assumption of weighted Poincaré inequality, we obtain that

$$(3.7)\qquad \int_M \varrho(|\omega|^{2(s+1)\alpha}\eta^2)\leqslant \int_M |\nabla(|\omega|^{(s+1)\alpha}\eta)|^2$$
$$=(s+1)^2\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2+\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2$$
$$+2(s+1)\int_M |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^\alpha\rangle.$$

Plugging (3.7) into (3.6) implies that

$$B\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 \leqslant C\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2 + 2D\int_M |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^\alpha\rangle, \tag{3.8}$$

where

$$\begin{aligned} B &= 2(s+1) - \frac{n-2}{(n-1)\alpha} - \frac{\alpha\sqrt{n-1}}{2}\frac{(s+1)^2}{\delta} - E(s+1)^2, \\ C &= \frac{\alpha\sqrt{n-1}}{2\delta} + E, \\ D &= \frac{\alpha\sqrt{n-1}}{2}\frac{(1+s)}{\delta} - 1 + E(s+1). \end{aligned}$$

For any $\varepsilon > 0$, using the Cauchy-Schwarz inequality, we can rewrite equation (3.8) as

$$(B - |D|\varepsilon)\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 \leqslant \Big(C + |D|\frac{1}{\varepsilon}\Big)\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2. \tag{3.9}$$

Now if we let $p = (s+1)\alpha$, we see that

$$\begin{aligned} B &= 2(s+1) - \frac{n-2}{(n-1)\alpha} - \frac{\alpha\sqrt{n-1}}{2}\frac{(s+1)^2}{\delta} - E(s+1)^2 \\ &= \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{\sqrt{n-1}}{2\delta}p^2 - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)\frac{(1-\tau)}{(2n-1)(n-1)}p^2\Big\} \\ &= \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{\sqrt{n-1}}{2}\Big[\frac{1}{\delta} + \big(n + 2\sqrt{n-1}\big)\frac{(1-\tau)}{(2n-1)(n-1)}\Big]p^2\Big\}. \end{aligned} \tag{3.10}$$

Let

$$f(p) = -\frac{\sqrt{n-1}}{2}\Big[\frac{1}{\delta} + \big(n + 2\sqrt{n-1}\big)\frac{(1-\tau)}{(2n-1)(n-1)}\Big]p^2 + 2p - \frac{n-2}{n-1},$$

then the discriminant of $f(p)$ is

$$\Delta = 4\Big(1 - \frac{n-2}{2\sqrt{n-1}}\Big[\frac{1}{\delta} + \frac{(n+2\sqrt{n-1})(1-\tau)}{(2n-1)(n-1)}\Big]\Big) > 0, \tag{3.11}$$

which is satisfied under the assumption

$$\frac{n-2}{2\sqrt{n-1} - \frac{(n-2)(n+2\sqrt{n-1})(1-\tau)}{(n-1)(2n-1)}} < \delta.$$

Let

$$g(n)=\frac{n-2}{2\sqrt{n-1}-\frac{(n-2)(n+2\sqrt{n-1})(1-\tau)}{(n-1)(2n-1)}};$$

when $2\leqslant n\leqslant 6$ and $\frac{122-51\sqrt{5}}{12+4\sqrt{5}}<\tau\leqslant 1$, we can see that

$$\begin{aligned}
g(2)&=0<\delta\leqslant 1,\\
g(3)&=\frac{1}{2\sqrt{2}-\frac{1}{10}(3+2\sqrt{2})(1-\tau)}<\delta\leqslant 1,\\
g(4)&=\frac{2}{2\sqrt{3}-\frac{2}{21}(4+2\sqrt{3})(1-\tau)}<\delta\leqslant 1,\\
g(5)&=\frac{3}{4-\frac{3}{4}(1-\tau)}<\delta\leqslant 1,\\
g(6)&=\frac{4}{2\sqrt{5}-\frac{4}{55}(6+2\sqrt{5})(1-\tau)}<\delta\leqslant 1.
\end{aligned}\tag{3.12}$$

Consequently, (3.11) is true under the assumption

$$\frac{n-2}{2\sqrt{n-1}-\frac{(n-2)(n+2\sqrt{n-1})(1-\tau)}{(n-1)(2n-1)}}<\delta\leqslant 1.$$

The condition $C_1<p<C_2$ allows us to conclude that $f(p)>0$, or equivalently $B>0$. Therefore, for a sufficiently small $\varepsilon>0$, we have

$$B-|D|\varepsilon>0.$$

For every $r>0$, let B_r denote the geodesic ball of radius r on M centered at a fixed point and let $\eta\in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases}\eta=1 & \text{on } B_r,\\ \eta=0 & \text{on } M\setminus B_{2r}\end{cases}$$

and $|\nabla\eta|\leqslant 1/r$ on $B_{2r}\setminus B_r$. Then the inequality (3.9) becomes

$$\int_{B_r}|\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\leqslant\frac{F}{r^2}\int_{B_{2r}}|\omega|^{2p},$$

i.e.

$$\int_{B_r}|\nabla|\omega|^p|^2\leqslant\frac{F(s+1)^2}{r^2}\int_{B_{2r}}|\omega|^{2p},$$

where $F>0$ is a constant which depends on n, ε, δ, p, α. Using the fact that $\omega\in L^{2p}(M)$ and letting $r\to\infty$ allows us to conclude that $|\omega|$ is a constant. Consequently, we can get that $\omega \equiv 0$. Otherwise, if $\varrho \equiv 0$, i.e. $K_N \geqslant 0$, from Lemma 2.5 we can conclude that the volume of M is infinite. However, the fact $\omega \in L^{2p}$ infers $\int_M |\omega|^{2p} < \infty$, i.e., the volume of M is finite, which is a contradiction. If $\varrho \not\equiv 0$, from equation (3.7) we deduce that

$$\int_M \varrho|\omega|^{2p} = 0,$$

which implies that $\varrho \equiv 0$. So the space of L^{2p} harmonic 1-forms must be trivial. $\square$

Proof of Theorem 1.6. Let $K_N \geqslant -k\varrho$, where

$$k < \frac{4p\delta(n-1) - 2(n-2)\delta - (n-1)\sqrt{n-1}p^2}{\delta p^2(n-1)(2n-2+n\sqrt{n-1})}.$$

Similarly to the proof of Theorem 1.3, we obtain that

$$\widetilde{B}\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 \leqslant \widetilde{C}\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2 + 2\widetilde{D}\int_M |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta, \nabla|\omega|^\alpha\rangle,$$

where

$$\begin{aligned}
\widetilde{E} &= \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)k\alpha,\\
\widetilde{B} &= 2(s+1) - \frac{n-2}{(n-1)\alpha} - \frac{\alpha\sqrt{n-1}}{2}\frac{(s+1)^2}{\delta} - \widetilde{E}(s+1)^2,\\
\widetilde{C} &= \frac{\alpha\sqrt{n-1}}{2\delta} + \widetilde{E},\\
\widetilde{D} &= \frac{\alpha\sqrt{n-1}}{2}\frac{(s+1)}{\delta} - 1 + \widetilde{E}(s+1).
\end{aligned}$$

For any $\varepsilon > 0$, applying the Cauchy-Schwarz inequality, we have that

$$(\widetilde{B} - |\widetilde{D}|\varepsilon)\int_M |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 \leqslant \Big(\widetilde{C} + |\widetilde{D}|\frac{1}{\varepsilon}\Big)\int_M |\omega|^{2(s+1)\alpha}|\nabla\eta|^2.$$

Let $p = (s+1)\alpha$, then we have

$$\widetilde{B} = \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{\sqrt{n-1}}{2\delta}p^2 - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\}.$$

Let

$$\widetilde{f}(p) = -(n-1)\sqrt{n-1}p^2 + 4\delta(n-1)p - 2\delta(n-2),$$

then the discriminant of $\widetilde{f}(p)$ is

$$\Delta = 16\delta^2(n-1)^2\Big(1 - \frac{n-2}{2\delta\sqrt{n-1}}\Big) > 0,$$

which is satisfied under the assumption $\frac{1}{2}(n-2)/\sqrt{n-1} < \delta \leqslant 1$. Thus from the conditions on p, we see that $\widetilde{f}(p) > 0$. Moreover, the condition

$$k < \frac{4p\delta(n-1) - 2(n-2)\delta - (n-1)\sqrt{n-1}p^2}{\delta p^2(n-1)(2n-2+n\sqrt{n-1})}$$

allows us to conclude that

$$\begin{aligned}
\widetilde{B} &= \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{\sqrt{n-1}}{2\delta}p^2 - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\} \\
&= \frac{1}{\alpha}\Big\{\frac{4\delta(n-1)p - 2\delta(n-2) - (n-1)\sqrt{n-1}p^2}{2\delta(n-1)} - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\} \\
&= \frac{1}{\alpha}\Big\{\frac{\widetilde{f}(p)}{2\delta(n-1)} - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\} > 0.
\end{aligned}$$

Therefore, for a sufficiently small $\varepsilon > 0$, we have $\widetilde{B} - |\widetilde{D}|\varepsilon > 0$. Using the same argument as before, we complete the proof of Theorem 1.6. □

Proof of Theorem 1.7. Let ω be an L^{2p} harmonic 1-form on M. Using the Weitzenböck formula and the Kato inequality, we get that

$$|\omega|\Delta|\omega| \geqslant \frac{1}{n-1}|\nabla|\omega||^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp). \tag{3.13}$$

Under our hypothesis on the sectional curvature of N, we can estimate the Ricci curvature of M by using Lemma 2.3:

$$\begin{aligned}
\mathrm{Ric}_M(\omega^\sharp, \omega^\sharp) \geqslant&\; -(n-1)\frac{n(1-\tau)}{(n-1)^2}\varrho|\omega|^2 - (n-1)\gamma\inf_M H^2|\omega|^2 + (n-1)H^2|\omega|^2 \\
&- \frac{n-1}{n}|\Phi|^2|\omega|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2 \\
=&\; -\frac{n(1-\tau)}{n-1}\varrho|\omega|^2 - (n-1)\gamma\inf_M H^2|\omega|^2 + (n-1)H^2|\omega|^2 \\
&- \frac{n-1}{n}|\Phi|^2|\omega|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2.
\end{aligned}$$

Plugging this inequality into (3.13) implies that

$$\begin{aligned}
|\omega|\Delta|\omega| \geqslant&\; \frac{1}{n-1}|\nabla|\omega||^2 - \frac{n(1-\tau)}{n-1}\varrho|\omega|^2 - (n-1)\gamma\inf_M H^2|\omega|^2 \\
&+ (n-1)H^2|\omega|^2 - \frac{n-1}{n}|\Phi|^2|\omega|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^2.
\end{aligned} \tag{3.14}$$

Applying (3.14), we get that

$$
\begin{aligned}
(3.15)\qquad |\omega|^p\Delta|\omega|^p &= \frac{p-1}{p}|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega| \\
&\geqslant \Big(1-\frac{n-2}{(n-1)p}\Big)|\nabla|\omega|^p|^2 - \frac{n(1-\tau)}{n-1}p\varrho|\omega|^{2p} \\
&\quad -(n-1)\gamma p\inf_M H^2|\omega|^{2p} + (n-1)pH^2|\omega|^{2p} \\
&\quad -\frac{(n-1)p}{n}|\Phi|^2|\omega|^{2p} - \frac{(n-2)p\sqrt{n(n-1)}}{n}|H||\Phi||\omega|^{2p}.
\end{aligned}
$$

For any $\eta \in C_0^\infty(M)$, multiplying both sides of (3.15) by η^2 and integrating by parts allows us to conclude that

$$
\begin{aligned}
&\Big(1-\frac{n-2}{(n-1)p}\Big)\int_M \eta^2|\nabla|\omega|^p|^2 \\
&\leqslant \int_M \eta^2|\omega|^p\Delta|\omega|^p + \frac{pn(1-\tau)}{n-1}\int_M \varrho\eta^2|\omega|^{2p} + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p} \\
&\quad -(n-1)p\int_M \eta^2H^2|\omega|^{2p} + \frac{(n-1)p}{n}\int_M \eta^2|\Phi|^2|\omega|^{2p} \\
&\quad + \frac{(n-2)p\sqrt{n(n-1)}}{n}\int_M \eta^2|H||\Phi||\omega|^{2p} \\
&= -\int_M \eta^2|\nabla|\omega|^p|^2 - 2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle + \frac{pn(1-\tau)}{n-1}\int_M \varrho\eta^2|\omega|^{2p} \\
&\quad + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p} - (n-1)p\int_M \eta^2H^2|\omega|^{2p} \\
&\quad + \frac{(n-1)p}{n}\int_M \eta^2|\Phi|^2|\omega|^{2p} + \frac{(n-2)p\sqrt{n(n-1)}}{n}\int_M \eta^2|H||\Phi||\omega|^{2p},
\end{aligned}
$$

i.e.

$$
\begin{aligned}
(3.16)\qquad &\Big(2-\frac{n-2}{(n-1)p}\Big)\int_M \eta^2|\nabla|\omega|^p|^2 \\
&\leqslant -2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle + \frac{pn(1-\tau)}{n-1}\int_M \eta^2\varrho|\omega|^{2p} \\
&\quad + \frac{(n-1)p}{n}\int_M \eta^2|\Phi|^2|\omega|^{2p} \\
&\quad + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p} - (n-1)p\int_M \eta^2H^2|\omega|^{2p} \\
&\quad + \frac{(n-2)p\sqrt{n(n-1)}}{n}\int_M \eta^2|H||\Phi||\omega|^{2p}.
\end{aligned}
$$

For any $a > 0$, we have the Cauchy-Schwarz inequality

$$
\begin{aligned}
(3.17)\quad &\frac{(n-2)p\sqrt{n(n-1)}}{n}\int_M \eta^2|H||\Phi||\omega|^{2p}\\
&\leqslant \frac{a(n-2)p\sqrt{n(n-1)}}{2n}\int_M H^2\eta^2|\omega|^{2p} + \frac{(n-2)p\sqrt{n(n-1)}}{2na}\int_M |\Phi|^2\eta^2|\omega|^{2p}.
\end{aligned}
$$

Applying formula (3.17) to (3.16) we get

$$
\begin{aligned}
(3.18)\quad &\Big(2-\frac{n-2}{(n-1)p}\Big)\int_M \eta^2|\nabla|\omega|^p|^2\\
&\leqslant -2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle + \frac{pn(1-\tau)}{n-1}\int_M \eta^2\varrho|\omega|^{2p}\\
&\quad + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p} + C\int_M \eta^2H^2|\omega|^{2p} + B\int_M \eta^2|\Phi|^2|\omega|^{2p},
\end{aligned}
$$

where

$$
\begin{aligned}
(3.19)\quad &B = B(n,a,p) = \frac{(n-1)p}{n} + \frac{(n-2)p\sqrt{n(n-1)}}{2na},\\
&C = C(n,a,p) = -(n-1)p + \frac{a(n-2)p\sqrt{n(n-1)}}{2n}.
\end{aligned}
$$

On the other hand, Lemma 2.6 and the Hölder inequality imply that

$$
\begin{aligned}
(3.20)\quad &\int_M \eta^2|\Phi|^2|\omega|^{2p} \leqslant \Big(\int_M |\Phi|^n\Big)^{2/n}\Big(\int_M (|\omega^p\eta|)^{2n/(n-2)}\Big)^{(n-2)/n}\\
&\leqslant S\|\Phi\|^2_{L^n}\int_M \Big(|\nabla(|\omega^p\eta|)|^2 + \eta^2|\omega|^{2p}H^2\Big)\\
&= S\|\Phi\|^2_{L^n}\Big(\int_M \eta^2|\nabla|\omega|^p|^2 + \int_M |\nabla\eta|^2|\omega|^{2p} + 2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle\Big)\\
&\quad + S\|\Phi\|^2_{L^n}\int_M \eta^2|\omega|^{2p}H^2,
\end{aligned}
$$

where $\|\Phi\|^2_{L^n} = \big(\int_M |\Phi|^n\big)^{2/n}$ and $S = S(n,2)$ is a constant in Lemma 2.6. Plugging (3.20) into (3.18) yields that

$$
\begin{aligned}
(3.21)\quad &\Big(2-\frac{n-2}{(n-1)p} - BS\|\Phi\|^2_{L^n}\Big)\int_M \eta^2|\nabla|\omega|^p|^2\\
&\leqslant 2(BS\|\Phi\|^2_{L^n}-1)\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle + \frac{pn(1-\tau)}{n-1}\int_M \eta^2\varrho|\omega|^{2p}\\
&\quad + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p} + (C+BS\|\Phi\|^2_{L^n})\int_M \eta^2H^2|\omega|^{2p}\\
&\quad + BS\|\Phi\|^2_{L^n}\int_M |\nabla\eta|^2|\omega|^{2p}.
\end{aligned}
$$

The property $(\mathcal{P}_\varrho)$ implies that

$$\int_M \varrho|\omega|^{2p}\eta^2 \leqslant \int_M |\nabla(|\omega|^p\eta)|^2 = \int_M \eta^2|\nabla|\omega|^p|^2 + \int_M |\omega|^{2p}|\nabla\eta|^2 + 2\int_M \eta|\omega|^p\langle\nabla\eta, \nabla|\omega|^p\rangle. \tag{3.22}$$

Combining (3.22) with (3.21), we deduce that

$$\begin{aligned} D\int_M \eta^2|\nabla|\omega|^p|^2 - G\int_M \eta^2H^2|\omega|^{2p} &\leqslant E\int_M |\nabla\eta|^2|\omega|^{2p} + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p} \\ &\quad + 2F\int_M \eta|\omega|^p\langle\nabla\eta, \nabla|\omega|^p\rangle, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} D &= 2 - \frac{n-2}{(n-1)p} - \frac{pn(1-\tau)}{n-1} - BS\|\Phi\|^2_{L^n}, \\ E &= \frac{pn(1-\tau)}{n-1} + BS\|\Phi\|^2_{L^n}, \\ F &= \frac{pn(1-\tau)}{n-1} + BS\|\Phi\|^2_{L^n} - 1, \\ G &= C + BS\|\Phi\|^2_{L^n}. \end{aligned} \tag{3.24}$$

For any $\varepsilon > 0$, applying the Cauchy-Schwarz inequality again, we see that

$$\begin{aligned} (D - |F|\varepsilon)\int_M \eta^2|\nabla|\omega|^p|^2 - G\int_M \eta^2H^2|\omega|^{2p} \\ \leqslant \left(E + |F|\frac{1}{\varepsilon}\right)\int_M |\nabla\eta|^2|\omega|^{2p} + (n-1)\gamma p\inf_M H^2\int_M \eta^2|\omega|^{2p}. \end{aligned} \tag{3.25}$$

Choose $0 < b < 1/2$, $a = a(b) > 0$ and $\Lambda = \Lambda(b) > 0$ satisfying

$$\begin{cases} \dfrac{a(n-2)p\sqrt{n(n-1)}}{2n} < (n-1)bp, \\ SB\Lambda^2 < (n-1)bp. \end{cases} \tag{3.26}$$

Now we let

$$\begin{aligned} \overline{D} &= 2 - \frac{n-2}{(n-1)p} - \frac{pn(1-\tau)}{n-1} - BS\Lambda^2, \\ \overline{E} &= \frac{pn(1-\tau)}{n-1} + BS\Lambda^2, \\ \overline{F} &= \frac{pn(1-\tau)}{n-1} + BS\Lambda^2 - 1, \\ \overline{G} &= C + BS\Lambda^2. \end{aligned} \tag{3.27}$$

Assume that the total curvature satisfies $\|\Phi\|_{L^n} < \Lambda$. Plugging the above choices in (3.25), we obtain that

$$(\overline{D} - |\overline{F}|\varepsilon) \int_M \eta^2 |\nabla |\omega|^p|^2 - \overline{G} \int_M \eta^2 H^2 |\omega|^{2p} \leqslant \left(\overline{E} + |\overline{F}|\frac{1}{\varepsilon}\right) \int_M |\nabla \eta|^2 |\omega|^{2p} + (n-1)\gamma p \inf_M H^2 \int_M \eta^2 |\omega|^{2p}. \tag{3.28}$$

Combining equations (3.19), (3.26) with (3.27), we get that

$$-\overline{G} = (n-1)p - \frac{a(n-2)p\sqrt{n(n-1)}}{2n} - BS\Lambda^2 > (n-1)p(1-2b) > 0. \tag{3.29}$$

Thus equation (3.28) becomes

$$(\overline{D} - |\overline{F}|\varepsilon) \int_M \eta^2 |\nabla |\omega|^p|^2 - (\overline{G} + (n-1)\gamma p) \inf_M H^2 \int_M \eta^2 |\omega|^{2p} \leqslant \left(\overline{E} + |\overline{F}|\frac{1}{\varepsilon}\right) \int_M |\nabla \eta|^2 |\omega|^{2p}. \tag{3.30}$$

Then we can choose b sufficiently small as to satisfy that

$$-(\overline{G} + (n-1)\gamma p) > (n-1)p(1-\gamma-2b) > 0.$$

Now we let $f(p) = -n(1-\tau)p^2 + 2(n-1)p - n + 2$. After a simple computation, we have that the discriminant of $f(p)$ is

$$\Delta = 4[(n-1)^2 - n(n-2)(1-\tau)] > 0.$$

Consequently, the condition on p implies that $f(p) > 0$. Choosing sufficiently small $\varepsilon > 0$, $b > 0$, we deduce that

$$\begin{aligned}
\overline{D} - |\overline{F}|\varepsilon &= 2 - \frac{n-2}{(n-1)p} - \frac{pn(1-\tau)}{n-1} - BS\Lambda^2 - |\overline{F}|\varepsilon \\
&= \frac{2(n-1)p - n + 2 - n(1-\tau)p^2}{(n-1)p} - BS\Lambda^2 - |\overline{F}|\varepsilon \\
&= \frac{f(p)}{(n-1)p} - BS\Lambda^2 - |\overline{F}|\varepsilon \\
&> \frac{f(p)}{(n-1)p} - (n-1)bp - |\overline{F}|\varepsilon > 0.
\end{aligned}$$

For every $r > 0$, let B_r denote the geodesic ball of radius r on M centered at a fixed point and let $\eta \in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases} \eta = 1 & \text{on } B_r, \\ \eta = 0 & \text{on } M \setminus B_{2r} \end{cases}$$

and $|\nabla\eta| \leqslant 1/r$ on $B_{2r} \setminus B_r$. Using (3.30) with η and the fact that $\omega \in L^{2p}$ while letting $r \to \infty$, we conclude

$$\big|\nabla|\omega|^p\big|^2 = \inf_M H^2|\omega|^{2p} = 0,$$

which implies $|\omega|$ is a constant. If $|\omega| \neq 0$, then $\inf\limits_M H^2 \equiv 0$. Using equation (3.28) with η and taking $r \to \infty$ implies $H^2|\omega|^{2p} \equiv 0$, i.e. $H^2 \equiv 0$. Applying the same way, from equation (3.20) and the fact that $\|\Phi\| < \Lambda$, we obtain that $|\Phi|^2 \equiv 0$. Consequently, equation (3.15) becomes

$$\big|\omega|^p\Delta|\omega\big|^p \geqslant \Big(1 - \frac{n-2}{(n-1)p}\Big)|\nabla|\omega|^p|^2 - \frac{n(1-\tau)p}{n-1}\varrho|\omega|^{2p},$$

where we have used the Cauchy-Schwarz inequality. From Lemma 2.7, the fact that $f(p) > 0$ and $\omega \in L^{2p}(M)$, we deduce that $\varrho \equiv 0$ and the volume of M is finite. Thus the condition on K_N becomes $K_N \geqslant 0$. The conclusion $H^2 = |\Phi|^2 \equiv 0$ implies that M is totally geodesic in N. Thus M has nonnegative Ricci curvature, which gives the conclusion that the volume of M is infinite, see [25], which is a contradiction. So the space of L^{2p} harmonic 1-forms must be trivial. □

P r o o f of Theorem 1.9. Let ω be an L^{2p} harmonic 1-form on M. Using the Weitzenböck formula and the Kato inequality, we get that

$$|\omega|\Delta|\omega| \geqslant \frac{1}{n-1}|\nabla|\omega||^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp). \tag{3.31}$$

Under our hypothesis on the sectional curvature of N, we can estimate the Ricci curvature of M by using Lemma 2.3:

$$\mathrm{Ric}_M \geqslant -\frac{n(1-\tau)}{n-1}\varrho - \frac{n-1}{n}|\Phi|^2.$$

The minimality of M implies $|\Phi|^2 = |A|^2$. Thus inequality (3.31) becomes

$$|\omega|\Delta|\omega| \geqslant \frac{1}{n-1}|\nabla|\omega||^2 - \frac{n(1-\tau)}{n-1}\varrho|\omega|^2 - \frac{n-1}{n}|A|^2|\omega|^2. \tag{3.32}$$

Applying formula (3.32) yields that

$$\begin{aligned} |\omega|^p\Delta|\omega|^p &= \frac{p-1}{p}|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega| \\ &\geqslant \Big(1 - \frac{n-2}{(n-1)p}\Big)|\nabla|\omega|^p|^2 - \frac{(n-1)p}{n}|A|^2|\omega|^{2p} \\ &\quad - \frac{pn(1-\tau)}{n-1}\varrho|\omega|^{2p}. \end{aligned} \tag{3.33}$$

For any $\eta \in C_0^\infty(M)$, multiplying both sides of (3.33) by η^2 and integrating by parts, we obtain

$$\begin{aligned}\Big(1-\frac{n-2}{(n-1)p}\Big)\int_M \eta^2|\nabla|\omega|^p|^2 &\leqslant \int_M \eta^2|\omega|^p\Delta|\omega|^p + \frac{p(n-1)}{n}\int_M \eta^2|A|^2|\omega|^{2p}\\ &\quad + \frac{pn(1-\tau)}{n-1}\int_M \eta^2\varrho|\omega|^{2p}\\ &= -\int_M \eta^2|\nabla|\omega|^p|^2 - 2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle\\ &\quad + \frac{p(n-1)}{n}\int_M \eta^2|A|^2|\omega|^{2p} + \frac{pn(1-\tau)}{n-1}\int_M \eta^2\varrho|\omega|^{2p},\end{aligned}$$

i.e.

$$\begin{aligned}\Big(2-\frac{n-2}{(n-1)p}\Big)\int_M \eta^2|\nabla|\omega|^p|^2 &\leqslant -2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle\\ &\quad + \frac{p(n-1)}{n}\int_M \eta^2|A|^2|\omega|^{2p} + \frac{pn(1-\tau)}{n-1}\int_M \eta^2\varrho|\omega|^{2p}.\end{aligned} \tag{3.34}$$

The property $(\mathcal{P}_\varrho)$ implies that

$$\begin{aligned}\int_M \varrho|\omega|^{2p}\eta^2 &\leqslant \int_M |\nabla(|\omega|^p\eta)|^2\\ &= \int_M \eta^2|\nabla|\omega|^p|^2 + \int_M |\omega|^{2p}|\nabla\eta|^2 + 2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle.\end{aligned} \tag{3.35}$$

Combining (3.34) with (3.35), we deduce that

$$\begin{aligned}&\Big(2-\frac{n-2}{(n-1)p}-\frac{pn(1-\tau)}{n-1}\Big)\int_M \eta^2|\nabla|\omega|^p|^2\\ &\quad\leqslant \frac{p(n-1)}{n}\int_M \eta^2|A|^2|\omega|^{2p} + 2\Big(\frac{pn(1-\tau)}{n-1}-1\Big)\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle\\ &\quad\quad + \frac{pn(1-\tau)}{n-1}\int_M |\nabla\eta|^2|\omega|^{2p}.\end{aligned} \tag{3.36}$$

On the other hand, Lemma 2.6 and the Hölder inequality imply that

$$\begin{aligned}\int_M \eta^2|A|^2|\omega|^{2p} &\leqslant \Big(\int_M |A|^n\Big)^{2/n}\Big(\int_M (|\omega^p\eta|)^{2n/(n-2)}\Big)^{(n-2)/n}\\ &\leqslant S\|A\|_{L^n}^2\int_M |\nabla(|\omega^p\eta|)|^2\\ &\leqslant S\|A\|_{L^n}^2\Big(\int_M \eta^2|\nabla|\omega|^p|^2 + \int_M |\nabla\eta|^2|\omega|^{2p} + 2\int_M \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle\Big),\end{aligned} \tag{3.37}$$

where $\|A\|_{L^n}^2 = \left(\int_M |A|^n\right)^{2/n}$, $S = S(n,2)$. Plugging (3.37) into (3.36) gives

$$B\int_M \eta^2|\nabla|\omega|^p|^2 \leqslant C\int_M |\nabla\eta|^2|\omega|^{2p} + 2D\int_M \eta|\omega|^p\langle\nabla\eta, \nabla|\omega|^p\rangle,$$

where

$$\begin{aligned} B &= 2 - \frac{n-2}{(n-1)p} - \frac{pn(1-\tau)}{n-1} - \frac{p(n-1)}{n}S\|A\|_{L^n}^2, \\ C &= \frac{pn(1-\tau)}{n-1} + \frac{p(n-1)}{n}S\|A\|_{L^n}^2, \\ D &= \frac{pn(1-\tau)}{n-1} - 1 + \frac{p(n-1)}{n}S\|A\|_{L^n}^2. \end{aligned}$$

For any $\varepsilon > 0$, applying the Cauchy-Schwarz inequality we see that

$$(B - |D|\varepsilon)\int_M \eta^2|\nabla|\omega|^p|^2 \leqslant \left(C + |D|\frac{1}{\varepsilon}\right)\int_M |\nabla\eta|^2|\omega|^{2p}. \tag{3.38}$$

Now we let $f(p) = -n(1-\tau)p^2 + 2(n-1)p - n + 2$. After a simple computation, we find that the discriminant of $f(p)$ is

$$\Delta = 4[(n-1)^2 - n(n-2)(1-\tau)] > 0.$$

Consequently, the condition on p implies that $f(p) > 0$. Since

$$\begin{aligned} B &= 2 - \frac{n-2}{(n-1)p} - \frac{pn(1-\tau)}{n-1} - \frac{p(n-1)}{n}S\|A\|_{L^n}^2 \\ &= \frac{2(n-1)p - n + 2 - n(1-\tau)p^2}{(n-1)p} - \frac{p(n-1)}{n}S\|A\|_{L^n}^2 \\ &= \frac{f(p)}{(n-1)p} - \frac{p(n-1)}{n}S\|A\|_{L^n}^2, \end{aligned}$$

the conditions on p and $\|A\|_{L^n}^2$ allow us to conclude that $B > 0$. Choosing a sufficiently small $\varepsilon > 0$, we deduce that

$$B - |D|\varepsilon > 0.$$

For every $r > 0$, let B_r denote the geodesic ball of radius r on M centered at a fixed point and let $\eta \in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases} \eta = 1 & \text{on } B_r, \\ \eta = 0 & \text{on } M \setminus B_{2r} \end{cases}$$

and $|\nabla\eta| \leqslant 1/r$ on $B_{2r} \setminus B_r$. Using (3.38) with η we have

$$\int_{B_r} |\nabla|\omega|^p|^2 \leqslant C(n,p,\varepsilon)\frac{1}{r^2}\int_{B_{2r}\setminus B_r} |\omega|^{2p}.$$

Letting $r \to \infty$ and using the fact that $\omega \in L^{2p}$, we conclude $|\omega|$ is a constant. The same argument as before shows that $\omega \equiv 0$. □

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