

THE EXCEPTIONAL SET FOR DIOPHANTINE INEQUALITY
WITH UNLIKE POWERS OF PRIME VARIABLES

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Abstract. Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are nonzero real numbers, not all negative, $\delta > 0$, \mathcal{V} is a well-spaced set, and the ratio λ_1/λ_2 is algebraic and irrational. Denote by $E(\mathcal{V}, N, \delta)$ the number of $v \in \mathcal{V}$ with $v \leq N$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4 . We show that

$$E(\mathcal{V}, N, \delta) \ll N^{1+2\delta-1/72+\varepsilon}$$

for any $\varepsilon > 0$.

Keywords: Davenport-Heilbronn method; prime variable; exceptional set; Diophantine inequality

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1. INTRODUCTION

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be nonzero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational. Recently, Yang and Li in [14] considered the inequality

$$(1.1) \quad \left| \lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - p - \frac{1}{2} \right| < \frac{1}{2}$$

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and proved that the inequality (1.1) has infinite solutions with prime p and integers x_1, x_2, x_3, x_4 . In [4], Ge and Li replaced the integer variables by prime variables and proved that for any real number η , the inequality

$$(1.2) \quad |\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \left(\max_{1 \leq j \leq 5} p_j^j \right)^{-1/720+\varepsilon}$$

has infinite solutions with primes p_1, p_2, p_3, p_4, p_5 . In this paper, we drop the linear prime variable in (1.2) and consider the exceptional set for the inequality

$$(1.3) \quad |\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v| < v^{-\delta}.$$

First, we recall a definition in [5]. We call a set of positive real numbers \mathcal{V} a *well-spaced set* if there is a $c > 0$ such that

$$u, v \in \mathcal{V}, u \neq v \Rightarrow |u - v| > c.$$

In order to get the full strength of the results quoted here one must also assume that

$$|\{v \in \mathcal{V}: 0 \leq v \leq N\}| \gg N^{1-\varepsilon}.$$

Let $\lambda_1, \lambda_2, \lambda_3$ be nonzero real numbers, not all negative, let \mathcal{V} be a well-spaced set, and let $\delta > 0$. We introduce the notation $E_1(\mathcal{V}, X, \delta)$ to denote the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3 . Harman in [5] showed that if λ_1/λ_2 is irrational and algebraic, then one has

$$E_1(\mathcal{V}, X, \delta) \ll X^{7/8+2\delta+\varepsilon},$$

for any $\varepsilon > 0$. He also proved that $7/8$ can be replaced by $6/7$ using his sieve method.

Recently, under similar conditions, Mu and Lü in [9] proved that for any $\varepsilon > 0$

$$E_2(\mathcal{V}, X, \delta) \ll X^{67/72+2\delta+\varepsilon},$$

where $E_2(\mathcal{V}, X, \delta)$ is the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4 .

In this paper, we consider the inequality (1.3). Denote by $\mathbb{E}(\mathcal{V}, N, \delta)$ the set of $v \in \mathcal{V}$ with $v \leq N$ such that the inequality (1.3) has no solution in primes p_1, p_2, p_3, p_4 . Let $E(\mathcal{V}, N, \delta) = |\mathbb{E}(\mathcal{V}, N, \delta)|$. Using some ingredients from [5] and [8], we establish the following results.

Theorem 1.1. *Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are nonzero real numbers, not all negative, $\delta > 0$, \mathcal{V} is a well-spaced set, the ratio λ_1/λ_2 is algebraic and irrational. Then*

$$(1.4) \quad E(\mathcal{V}, N, \delta) \ll N^{1+2\delta-1/72+\varepsilon}$$

for any $\varepsilon > 0$.

Theorem 1.2. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be nonzero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational. Let \mathcal{V} be a well-spaced set. Let $\delta > 0$. Then there is a sequence $N_j \rightarrow \infty$ such that*

$$(1.5) \quad E(\mathcal{V}, N_j, \delta) \ll N_j^{1+2\delta-1/72+\varepsilon}$$

for any $\varepsilon > 0$. Moreover, if the convergent denominators q_j for λ_1/λ_2 satisfy

$$(1.6) \quad q_{j+1}^{1-\omega} \ll q_j \quad \text{for some } \omega \in [0, 1),$$

then, for all $N \geq 1$,

$$(1.7) \quad E(\mathcal{V}, N, \delta) \ll N^{1+2\delta-(1/6)\chi+\varepsilon}$$

for any $\varepsilon > 0$ with

$$(1.8) \quad \chi = \min \left(\frac{6-6\omega}{29-20\omega}, \frac{1}{12} \right).$$

Remark. In the case of λ_1/λ_2 algebraic, we claim that we can take $\omega = \varepsilon$ in (1.6), where ε is an arbitrarily small positive real number. If not, there are infinitely many q_j satisfying $q_{j+1}^{1-\varepsilon} \gg q_j$. Since q_j is a convergent denominator for λ_1/λ_2 , by the proof of Lemma 4D on page 14 of [11], we have that there are infinitely many q_j satisfying

$$\left\| q_j \frac{\lambda_1}{\lambda_2} \right\| < \frac{1}{q_{j+1}} \ll \frac{1}{q_j^{1/(1-\varepsilon)}}.$$

This is in contradiction with Roth's theorem (Theorem 2A on page 116 of [11]). Thus Theorem 1.1 follows immediately from Theorem 1.2.

Notation. Throughout, the symbols p and p_j denote primes. Nonzero real numbers λ_j , $j = 1, 2, 3, 4$, are given constants, and Vinogradov symbols may depend on λ_j , $j = 1, 2, 3, 4$. The number ε is an arbitrarily small positive real number, but not necessarily the same, we may write $X^{2\varepsilon} \ll X^\varepsilon$. Denote by $\|y\|$ the distance from y to the nearest integer. We write $e(x) = \exp(2\pi ix)$.

2. OUTLINE OF THE PROOF AND PRELIMINARY LEMMAS

Suppose that N is some large positive quantity which we will choose later. Let

$$(2.1) \quad \mathbb{E}(\mathfrak{J}) = \mathbb{E}(\mathcal{V}, N, \delta) \cap \mathfrak{J}, \quad E(\mathfrak{J}) = |\mathbb{E}(\mathfrak{J})|,$$

where \mathfrak{J} is any subset of $[0, N]$. By the definition of the well-spaced set \mathcal{V} and the definition of $\mathbb{E}(\mathcal{V}, N, \delta)$, it is easy to show that

$$(2.2) \quad E(\mathcal{V}, N, \delta) = E([N^{71/72}, N]) + E([0, N^{71/72})) \ll E([N^{71/72}, N]) + N^{71/72}.$$

So we just need to estimate $E([N^{71/72}, N])$. We use the standard dyadic argument. Let $N^{71/72} \leq X \leq N$. We will estimate $E([X/2, X])$. So we shall restrict our attention to those v satisfying

$$(2.3) \quad \frac{X}{2} \leq v \leq X.$$

We use the Hardy-Littlewood circle method which was first stated by Davenport-Heilbronn in [3]. Suppose that $\mathcal{I}_j = [(X/16|\lambda_{j-1}|)^{1/j}, (X/|\lambda_{j-1}|)^{1/j}]$ for $j = 2, 3, 4, 5$, and $0 < \tau < 1$. We define

$$(2.4) \quad K(\alpha) = \begin{cases} \left(\frac{\sin \pi \tau \alpha}{\pi \alpha}\right)^2 & \text{if } \alpha \neq 0; \\ \tau^2 & \text{if } \alpha = 0, \end{cases}$$

$$(2.5) \quad S_j(\alpha) = \sum_{p \in \mathcal{I}_j} (\log p) e(\alpha p^j),$$

$$(2.6) \quad F_j(\alpha) = \int_{\mathcal{I}_j} e(\alpha u^j) du, \quad U_j(\alpha) = \sum_{n \in \mathcal{I}_j} e(\alpha n^j),$$

for $j = 2, 3, 4, 5$.

Then we can easily get (also see [12] or [13])

$$(2.7) \quad K(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \quad \int_{-\infty}^{\infty} K(\alpha) e(\alpha x) d\alpha = \max(0, \tau - |x|).$$

For convenience, we define

$$(2.8) \quad \mathcal{I}(v, X; \mathfrak{X}) = \int_{\mathfrak{X}} S_2(\lambda_1 \alpha) S_3(\lambda_2 \alpha) S_4(\lambda_3 \alpha) S_5(\lambda_4 \alpha) K(\alpha) e(-v \alpha) d\alpha,$$

where \mathfrak{X} is a measurable subset of \mathbb{R} . Then by (2.7), we have

$$\begin{aligned} \mathcal{I}(v, X; \mathbb{R}) &= \sum_{p_j \in \mathcal{I}_{j+1}} (\log p_1) \dots (\log p_4) \int_{-\infty}^{\infty} e((\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v) \alpha) K(\alpha) d\alpha \\ &= \sum_{p_j \in \mathcal{I}_{j+1}} (\log p_1) \dots (\log p_4) \max(0, \tau - |\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v|) \\ &\leq (\log X)^4 \sum_{p_j \in \mathcal{I}_{j+1}} \max(0, \tau - |\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v|). \end{aligned}$$

Thus we have

$$(2.9) \quad 0 \leq \mathcal{I}(v, X; \mathbb{R}) \leq \tau (\log X)^4 \mathcal{N}(v, X),$$

where $\mathcal{N}(v, X)$ is the number of solutions to the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v| < \tau, \quad p_j \in \mathcal{I}_{j+1}, \quad j = 1, 2, 3, 4.$$

In order to estimate the integral $\mathcal{I}(v, X; \mathbb{R})$, we divide the region of integration into four parts: the major region \mathfrak{C} , the middle region \mathfrak{D} , the minor region \mathfrak{c} and the trivial region \mathfrak{t} , which are defined as follows

$$(2.10) \quad \begin{aligned} \mathfrak{C} &= \{\alpha: |\alpha| < \varphi\}, & \mathfrak{D} &= \{\alpha: \varphi \leq |\alpha| < \xi\}, \\ \mathfrak{c} &= \{\alpha: \xi \leq |\alpha| < \kappa\}, & \mathfrak{t} &= \{\alpha: |\alpha| \geq \kappa\}, \end{aligned}$$

where $\varphi = X^{-13/15-\varepsilon}$, $\xi = X^{-7/9-\varepsilon}$ and $\kappa = \tau^{-2} X^{7/12+2\varepsilon}$.

We assemble here the important results we need for the analysis of all the regions for α .

Lemma 2.1 ([5], Lemma 3). *Suppose that $(a, q) = 1$ and $|q\alpha - a| \leq q^{-1}$, then for any $\varepsilon > 0$ we have*

$$\sum_{1 \leq p \leq P} (\log p) e(\alpha p^2) \ll P^{1+\varepsilon} (q^{-1} + P^{-1/2} + qP^{-2})^{1/4}.$$

Corollary 2.2 ([5], Corollary 1). *Let $X^{1/2} \geq Y \geq X^{1/2-1/16+\varepsilon}$. If $|S_2(\lambda_1\alpha)| > Y$, then there are coprime integers a, q satisfying*

$$(2.11) \quad 1 \leq q \ll \left(\frac{X^{1/2+\varepsilon}}{Y}\right)^4, \quad |q\lambda_1\alpha - a| \ll \left(\frac{X^{1/2+\varepsilon}}{Y}\right)^4 X^{-1}.$$

Lemma 2.3. *Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a, q) = 1, \quad 1 \leq q \leq P^{3/2} \quad \text{and} \quad |q\alpha - a| < P^{-3/2}.$$

Then one has

$$\sum_{P < p \leq 2P} (\log p)e(p^3\alpha) \ll P^{11/12+\varepsilon} + \frac{P^{1+\varepsilon}}{q^{1/2}(1 + P^3|\alpha - a/q|)^{1/2}}.$$

Proof. This follows from Lemma 8.5 in [15] and Theorem 1.1 in [10] (one can also see Lemma 2.3 in [16]). \square

Corollary 2.4. *Suppose that $X^{1/3} \geq Y \geq X^{1/3-1/36+\varepsilon}$ and $|S_3(\lambda_2\alpha)| \geq Y$. Then there are two coprime integers a, q satisfying*

$$1 \leq q \ll \left(\frac{X^{1/3+\varepsilon}}{Y}\right)^2, \quad |q\lambda_2\alpha - a| \ll \left(\frac{X^{1/3+\varepsilon}}{Y}\right)^2 X^{-1}.$$

Proof. Let $P = X^{1/3}$ and $Q = P^{3/2}$. By Dirichlet's approximation theorem, there exist two coprime integers a, q with $1 \leq q \leq Q$ and $|q\lambda_2\alpha - a| \leq Q^{-1}$. By Lemma 2.3 and the hypothesis $Y \geq X^{1/3-1/36+\varepsilon}$, we have

$$\begin{aligned} X^{1/3-1/36+\varepsilon} \leq Y \leq |S_3(\lambda_2\alpha)| \\ \ll X^{1/3-1/36+\varepsilon/2} + \frac{X^{1/3+\varepsilon/2}}{q^{1/2}(1 + X|\lambda_2\alpha - a/q|)^{1/2}}. \end{aligned}$$

Thus we have $1 \leq q \ll (X^{1/3+\varepsilon}/Y)^2$, $|q\lambda_2\alpha - a| \ll (X^{1/3+\varepsilon}/Y)^2 X^{-1}$. \square

Lemma 2.5 ([6], Theorem 1). *Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(2.12) \quad (a, q) = 1, \quad |q\alpha - a| < q^{-1}.$$

Let k be a positive integer with $k \geq 2$. Then for any $\varepsilon > 0$, one has

$$(2.13) \quad \sum_{1 \leq p \leq P} (\log p)e(p^k\alpha) \ll P^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{P^{1/2}} + \frac{q}{P^k}\right)^{4^{1-k}}.$$

Corollary 2.6. *Suppose that $X^{-13/15-\varepsilon} \leq |\alpha| \leq X^{-7/9-\varepsilon}$. Then*

$$(2.14) \quad |S_5(\lambda_4\alpha)| \ll X^{1/5-1/2560+\varepsilon}.$$

Proof. Without loss of generality, we need only consider $\lambda_4\alpha > 0$. If we take $q = \lceil |\lambda_4\alpha|^{-1} \rceil$, $a = 1$ in (2.12), where $[x]$ is the integral part of a real number x , then (2.14) follows from (2.13) immediately. \square

Lemma 2.7. *With the previous notation, for $k = 2, 3, 4, 5$ and any positive integer m with $1 \leq m \leq k$, one has*

$$\int_{-\infty}^{\infty} |S_k(\alpha)|^{2^m} K(\alpha) d\alpha \ll \tau X^{(2^m-m)/k+\varepsilon}, \quad \int_{-1}^1 |S_k(\alpha)|^{2^m} d\alpha \ll X^{(2^m-m)/k+\varepsilon}.$$

Proof. These results follow from Hua's lemma, one can also see Lemma 2.5 in [12] for details. \square

Lemma 2.8 ([4], Lemma 4.2). *With the previous notation, one has*

$$\int_{-\infty}^{\infty} |S_2(\lambda_1\alpha)^2 S_4(\lambda_3\alpha)^4| K(\alpha) d\alpha \ll \tau X^{1+\varepsilon}, \quad \int_{-1}^1 |S_2(\lambda_1\alpha)^2 S_4(\lambda_3\alpha)^4| d\alpha \ll X^{1+\varepsilon}.$$

Lemma 2.9. *Suppose that λ_1/λ_2 is an irrational number, and a/q is a continued fraction convergent to λ_1/λ_2 . If a positive integer m satisfies $\|m\lambda_1/\lambda_2\| < 1/2q$, then we have $m \geq q$.*

Proof. We prove by contradiction. Suppose that $m < q$, then by Legendre's law of best approximation for continued fractions (Theorem 5E on page 21 of [11]), we have

$$\left\| q \frac{\lambda_1}{\lambda_2} \right\| < \left\| m \frac{\lambda_1}{\lambda_2} \right\| < \frac{1}{2q}.$$

Since a/q is a continued fraction convergent to λ_1/λ_2 , we have $(a, q) = 1$ and $\lambda_1/\lambda_2 = a/q + \theta/q^2$ with $|\theta| < 1$. Then

$$\left\| q \frac{\lambda_1}{\lambda_2} \right\| = \frac{|\theta|}{q} < \frac{1}{2q},$$

and

$$\frac{1}{2q} > \left\| m \frac{\lambda_1}{\lambda_2} \right\| = \left\| \frac{ma}{q} + \frac{m\theta}{q^2} \right\| \geq \frac{1}{q} - \frac{m|\theta|}{q^2}.$$

Thus we have $|\theta| < 1/2 < m|\theta|/q$; this is in contradiction with the hypothesis $m < q$. \square

3. THE MAJOR REGION \mathfrak{C}

In this section, we evaluate the integral on the major region \mathfrak{C} . The next theorem is our main result.

Theorem 3.1. *We have*

$$(3.1) \quad \mathcal{I}(v, X; \mathfrak{C}) \gg \tau^2 X^{17/60}.$$

First, we need some lemmas prepared to get the assertion of Theorem 3.1. For any real number $r \geq 1$, we set

$$J_r(X, h) = \int_{X/2}^X (\theta((x+h)^{1/r}) - \theta(x^{1/r}) - ((x+h)^{1/r} - x^{1/r}))^2 dx,$$

where $\theta(x) = \sum_{1 \leq p \leq x} \log p$ is the usual Chebyshev function.

Lemma 3.2 ([8], Theorem 2). *For $0 < Y \leq 1/2$ and $k = 2, 3, 4, 5$, one has*

$$\int_{-Y}^Y |S_k(\alpha) - U_k(\alpha)|^2 d\alpha \ll \frac{X^{2/k-2} \log^2 X}{Y} + Y^2 X + Y^2 J_k\left(X, \frac{1}{2Y}\right).$$

Lemma 3.3 ([8], Theorem 3). *Let ε be an arbitrarily small positive constant. For $k = 2, 3, 4, 5$, there exists a positive constant $c_1 = c_1(\varepsilon)$, such that*

$$J_k(X, h) \ll h^2 X^{2/k-1} \exp\left(-c_1 \left(\frac{\log X}{\log \log X}\right)^{1/3}\right)$$

holds uniformly for $X^{1-5/(6k)+\varepsilon} \leq h \leq X$.

From Lemmas 3.2 and 3.3, it is easy to deduce the following corollary.

Corollary 3.4. *For $k = 2, 3, 4, 5$, we have*

$$\int_{|\alpha| \leq X^{-1+5/(6k)-\varepsilon}} |S_k(\alpha) - U_k(\alpha)|^2 d\alpha \ll X^{2/k-1} \exp\left(-c_1 \left(\frac{\log X}{\log \log X}\right)^{1/3}\right).$$

Lemma 3.5. *For $k = 2, 3, 4, 5$, we have*

$$(3.2) \quad F_k(\alpha) \ll \min\left(X^{1/k}, \frac{X^{1/k}}{|\alpha|X}\right),$$

$$(3.3) \quad \int_{-1/2}^{1/2} |F_k(\alpha)|^2 d\alpha \ll X^{2/k-1}.$$

Proof. This result follows from Lemma 1 of [1] (one can also see [5] or [12]). \square

Lemma 3.6. For $k = 2, 3, 4, 5$, we have

$$(3.4) \quad \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} |S_k(\alpha) - F_k(\alpha)|^2 d\alpha \ll X^{2/k-1} \exp\left(-c_1 \left(\frac{\log X}{\log \log X}\right)^{1/3}\right),$$

$$(3.5) \quad \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} |S_k(\alpha)|^2 d\alpha \ll X^{2/k-1}.$$

Proof. It is easy to show that

$$(3.6) \quad \begin{aligned} \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} |S_k(\alpha) - F_k(\alpha)|^2 d\alpha \\ \ll \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} |S_k(\alpha) - U_k(\alpha)|^2 d\alpha \\ + \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} |U_k(\alpha) - F_k(\alpha)|^2 d\alpha. \end{aligned}$$

By Euler-Maclaurin summation formula, we have

$$(3.7) \quad |U_k(\alpha) - F_k(\alpha)| \ll 1 + |\alpha|X.$$

Then

$$(3.8) \quad \begin{aligned} \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} |U_k(\alpha) - F_k(\alpha)|^2 d\alpha \\ \ll \int_{|\alpha| \leq X^{-1+2/(3k)-\varepsilon}} (1 + |\alpha|X)^2 d\alpha \\ \ll \left(\int_{|\alpha| \leq X^{-1}} + \int_{X^{-1} < |\alpha| \leq X^{-1+2/(3k)-\varepsilon}} \right) (1 + |\alpha|X)^2 d\alpha \\ \ll X^{-1} + X^2 (X^{-1+2/(3k)-\varepsilon})^3 \ll X^{2/k-1-\varepsilon}. \end{aligned}$$

Hence, (3.4) follows immediately from (3.6), (3.8) and Corollary 3.4. Then (3.5) follows from (3.4) and Lemma 3.5. \square

Using the previous lemmas, we prove Theorem 3.1.

Proof of Theorem 3.1. Similarly to Lemma 3.3 in [4], we have

$$\begin{aligned}
& \int_{\mathfrak{C}} |S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha) - F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)|K(\alpha) \, d\alpha \\
& \leq \tau^2 |S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)| \left(\int_{-\varphi}^{\varphi} |S_2(\lambda_1\alpha) - F_2(\lambda_1\alpha)|^2 \, d\alpha \right)^{1/2} \left(\int_{-\varphi}^{\varphi} |S_5(\lambda_4\alpha)|^2 \, d\alpha \right)^{1/2} \\
& \quad + \tau^2 |F_2(\lambda_1\alpha)S_4(\lambda_3\alpha)| \left(\int_{-\varphi}^{\varphi} |S_3(\lambda_2\alpha) - F_3(\lambda_2\alpha)|^2 \, d\alpha \right)^{1/2} \left(\int_{-\varphi}^{\varphi} |S_5(\lambda_4\alpha)|^2 \, d\alpha \right)^{1/2} \\
& \quad + \tau^2 |F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)| \left(\int_{-\varphi}^{\varphi} |S_4(\lambda_3\alpha) - F_4(\lambda_3\alpha)|^2 \, d\alpha \right)^{1/2} \left(\int_{-\varphi}^{\varphi} |S_5(\lambda_4\alpha)|^2 \, d\alpha \right)^{1/2} \\
& \quad + \tau^2 |F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)| \left(\int_{-\varphi}^{\varphi} |S_5(\lambda_4\alpha) - F_5(\lambda_4\alpha)|^2 \, d\alpha \right)^{1/2} \left(\int_{-\varphi}^{\varphi} |F_4(\lambda_3\alpha)|^2 \, d\alpha \right)^{1/2}.
\end{aligned}$$

Note that $\varphi = X^{-13/15-\varepsilon}$. Then, by Lemmas 3.5, 3.6 and the trivial estimates of $S_k(\alpha)$, we easily obtain

$$\begin{aligned}
(3.9) \quad & \int_{\mathfrak{C}} |S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha) \\
& \quad - F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)|K(\alpha) \, d\alpha \\
& \ll \tau^2 X^{17/60} \exp\left(-\frac{1}{2}c_1\left(\frac{\log X}{\log \log X}\right)^{1/3}\right).
\end{aligned}$$

Then, by (2.7) and Lemma 3.5, we have

$$\begin{aligned}
(3.10) \quad & \int_{|\alpha| \geq \varphi} |F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)|K(\alpha) \, d\alpha \\
& \ll \tau^2 X^{77/60-4} \int_{|\alpha| \geq \varphi} |\alpha|^{-4} \, d\alpha \ll \tau^2 X^{17/60-1/4+3\varepsilon}.
\end{aligned}$$

Finally, we will show that

$$(3.11) \quad \int_{-\infty}^{\infty} F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)K(\alpha)e(-v\alpha) \, d\alpha \gg \tau^2 X^{17/60}.$$

Now we prove (3.11) as in Lemma 51 of Davenport, see [3]. We have

$$\begin{aligned}
& \int_{-\infty}^{\infty} F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)K(\alpha)e(-v\alpha) \, d\alpha \\
& = \int_{\mathcal{I}_2} \int_{\mathcal{I}_3} \int_{\mathcal{I}_4} \int_{\mathcal{I}_5} \int_{-\infty}^{\infty} e(\alpha(\lambda_1x_1^2 + \lambda_2x_2^3 + \lambda_3x_3^4 + \lambda_4x_4^5 - v)) \\
& \quad \times K(\alpha) \, d\alpha \, dx_1 \, dx_2 \, dx_3 \, dx_4
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathcal{I}_2} \int_{\mathcal{I}_3} \int_{\mathcal{I}_4} \int_{\mathcal{I}_5} \max(0, \tau - |\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - v|) dx_1 dx_2 dx_3 dx_4 \\
&= \frac{1}{120 \lambda_1^{1/2} \lambda_2^{1/3} \lambda_3^{1/4} \lambda_4^{1/5}} \int_{X/16}^X \int_{X/16}^X \int_{X/16}^X \int_{X/16}^X \max(0, \tau - |y_1 + y_2 + y_3 + y_4 - v|) \\
&\quad \times y_1^{-1/2} y_2^{-2/3} y_3^{-3/4} y_4^{-4/5} dy_1 dy_2 dy_3 dy_4,
\end{aligned}$$

where $y_j = \lambda_j x_j^{j+1}$, $j = 1, 2, 3, 4$. We may take $X/16 \leq y_1, y_2, y_3 \leq X/8$. Note that $X/2 \leq v \leq X$ by the hypothesis (2.3), and $0 < \tau < 1$ (indeed, we will take $\tau = X^{-\delta}$).

Then we have

$$v - (y_1 + y_2 + y_3) - \frac{\tau}{2} \geq \frac{X}{2} - \frac{3X}{8} - \frac{\tau}{2} > \frac{X}{16}$$

and

$$v - (y_1 + y_2 + y_3) + \frac{\tau}{2} \leq X - \frac{3X}{16} + \frac{\tau}{2} < X.$$

So we can take $v - (y_1 + y_2 + y_3) - \tau/2 < y_4 < v - (y_1 + y_2 + y_3) + \tau/2$. Thus we have $|y_1 + y_2 + y_3 + y_4 - v| < \tau/2$. Therefore, we obtain that the lower bound of the above integral is

$$\begin{aligned}
&\gg \tau X^{-1/2-2/3-3/4-4/5} \int_{X/16}^{X/8} \int_{X/16}^{X/8} \int_{X/16}^{X/8} \int_{v-(y_1+y_2+y_3)-\tau/2}^{v-(y_1+y_2+y_3)+\tau/2} dy_1 dy_2 dy_3 dy_4 \\
&\gg \tau^2 X^{-163/60} \int_{X/16}^{X/8} \int_{X/16}^{X/8} \int_{X/16}^{X/8} dy_1 dy_2 dy_3 \gg \tau^2 X^{17/60}.
\end{aligned}$$

Combining (3.9), (3.10) and (3.11), Theorem 3.1 follows immediately. \square

4. THE MIDDLE REGION \mathfrak{D}

In this section we show that

$$(4.1) \quad \mathcal{I}(v, X; \mathfrak{D}) = o(\tau^2 X^{17/60}).$$

By Corollary 2.6, for any $\alpha \in \mathfrak{D} = \{\alpha: X^{-13/15-\varepsilon} = \varphi \leq |\alpha| < \xi = X^{-7/9-\varepsilon}\}$ we have

$$(4.2) \quad |S_5(\lambda_4 \alpha)| \ll X^{1/5-1/2560+\varepsilon}.$$

Then by Cauchy's inequality and (2.7) we have

$$\begin{aligned}
(4.3) \quad \mathcal{I}(v, X; \mathfrak{D}) &= \int_{\mathfrak{D}} S_2(\lambda_1 \alpha) S_3(\lambda_2 \alpha) S_4(\lambda_3 \alpha) S_5(\lambda_4 \alpha) K(\alpha) e(-v\alpha) d\alpha \\
&\ll \tau^2 X^{1/4} \max_{\alpha \in \mathfrak{D}} |S_5(\lambda_4 \alpha)| \left(\int_{\mathfrak{D}} |S_2(\lambda_1 \alpha)|^2 d\alpha \right)^{1/2} \left(\int_{\mathfrak{D}} |S_3(\lambda_2 \alpha)|^2 d\alpha \right)^{1/2} \\
&\ll \tau^2 X^{9/20-1/2560+\varepsilon} \left(\int_{\mathfrak{D}} |S_2(\lambda_1 \alpha)|^2 d\alpha \right)^{1/2} \left(\int_{\mathfrak{D}} |S_3(\lambda_2 \alpha)|^2 d\alpha \right)^{1/2}.
\end{aligned}$$

Note that $\xi = X^{-7/9-\varepsilon} = X^{-1+2/(3\cdot 3)-\varepsilon}$. Then by Lemma 3.6, we get

$$(4.4) \quad \int_{\mathfrak{D}} |S_2(\lambda_1\alpha)|^2 d\alpha \ll 1, \quad \int_{\mathfrak{D}} |S_3(\lambda_2\alpha)|^2 d\alpha \ll X^{-1/3}.$$

Hence, (4.1) follows immediately from (4.3) and (4.4).

5. THE TRIVIAL REGION \mathfrak{t}

In this section we show that

$$(5.1) \quad \mathcal{I}(v, X; \mathfrak{t}) = o(\tau^2 X^{17/60}).$$

By Cauchy's inequality and the trivial bounds of $S_4(\lambda_3\alpha)$, $S_5(\lambda_4\alpha)$, we have

$$(5.2) \quad \begin{aligned} \mathcal{I}(v, X; \mathfrak{t}) &= \int_{\mathfrak{t}} S_2(\lambda_1\alpha) S_3(\lambda_2\alpha) S_4(\lambda_3\alpha) S_5(\lambda_4\alpha) K(\alpha) e(-v\alpha) d\alpha \\ &\ll X^{1/4+1/5} \left(\int_{\mathfrak{t}} |S_2(\lambda_1\alpha)|^2 K(\alpha) d\alpha \right)^{1/2} \left(\int_{\mathfrak{t}} |S_3(\lambda_2\alpha)|^2 K(\alpha) d\alpha \right)^{1/2}. \end{aligned}$$

By (2.7), the periodicity of $S_2(\alpha)$ and Lemma 2.7, we have

$$\begin{aligned} \int_{\mathfrak{t}} |S_2(\lambda_1\alpha)|^2 K(\alpha) d\alpha &\ll \int_{\kappa}^{\infty} |S_2(\lambda_1\alpha)|^2 \frac{1}{|\alpha|^2} d\alpha \ll \int_{|\lambda_1|\kappa}^{\infty} |S_2(\alpha)|^2 \frac{1}{|\alpha|^2} d\alpha \\ &\ll \sum_{m=[|\lambda_1|\kappa]}^{\infty} \int_m^{m+1} |S_2(\alpha)|^2 \frac{1}{|\alpha|^2} d\alpha \\ &\ll \sum_{m=[|\lambda_1|\kappa]}^{\infty} \int_m^{m+1} |S_2(\alpha)|^2 \frac{1}{m^2} d\alpha \\ &\ll \int_0^1 |S_2(\alpha)|^2 d\alpha \sum_{m=[|\lambda_1|\kappa]}^{\infty} \frac{1}{m^2} \ll X^{1/2+\varepsilon} \kappa^{-1}. \end{aligned}$$

Similarly, we have

$$(5.3) \quad \int_{\mathfrak{t}} |S_3(\lambda_2\alpha)|^2 K(\alpha) d\alpha \ll X^{1/3+\varepsilon} \kappa^{-1}.$$

Combining (5.2)–(5.4), we have

$$\begin{aligned} \int_{\mathfrak{t}} S_2(\lambda_1\alpha) S_3(\lambda_2\alpha) S_4(\lambda_3\alpha) S_5(\lambda_4\alpha) K(\alpha) e(-v\alpha) d\alpha \\ \ll X^{26/30+\varepsilon} \kappa^{-1} \ll X^{26/30+\varepsilon} \tau^{-2} X^{-7/12-2\varepsilon} \ll \tau^2 X^{17/60-\varepsilon}. \end{aligned}$$

Hence, (5.1) follows immediately.

6. THE MINOR REGION \mathfrak{c}

Our argument is similar to that used in [2], [5]. First we fix our parameter N . Let a/q be a continued fraction convergent to λ_1/λ_2 and put

$$(6.1) \quad N = q^{(72/71) \cdot (18/13)}.$$

In this section, we take

$$(6.2) \quad \sigma = \frac{1}{12}.$$

Let $\tilde{\mathfrak{c}} = \tilde{\mathfrak{c}}_1 \cup \tilde{\mathfrak{c}}_2$ and $\hat{\mathfrak{c}} = \mathfrak{c} \setminus \tilde{\mathfrak{c}}$, where

$$\begin{aligned} \tilde{\mathfrak{c}}_1 &= \{\alpha \in \mathfrak{c}: |S_2(\lambda_1\alpha)| < X^{1/2-(2/3)\sigma+2\varepsilon}\}, \\ \tilde{\mathfrak{c}}_2 &= \{\alpha \in \mathfrak{c}: |S_3(\lambda_2\alpha)| < X^{1/3-(1/3)\sigma+2\varepsilon}\}. \end{aligned}$$

Lemma 6.1. *We have*

$$(6.3) \quad \int_{\tilde{\mathfrak{c}}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j\alpha) \right|^2 K(\alpha) \, d\alpha \ll \tau X^{47/30-1/72+2\varepsilon}.$$

Proof. By Hölder's inequality and Lemmas 2.7, 2.8, we have

$$\begin{aligned} & \int_{\tilde{\mathfrak{c}}_1} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j\alpha) \right|^2 K(\alpha) \, d\alpha \\ & \ll \left(\max_{\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^{1/4} \right) \left(\int_{\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^4 K(\alpha) \, d\alpha \right)^{3/16} \\ & \quad \times \left(\int_{\tilde{\mathfrak{c}}_1} |S_3(\lambda_2\alpha)|^8 K(\alpha) \, d\alpha \right)^{1/4} \\ & \quad \times \left(\int_{\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^2 |S_4(\lambda_3\alpha)|^4 K(\alpha) \, d\alpha \right)^{1/2} \\ & \quad \times \left(\int_{\tilde{\mathfrak{c}}_1} |S_5(\lambda_4\alpha)|^{32} K(\alpha) \, d\alpha \right)^{1/16} \\ & \ll \tau X^{173/120+\varepsilon} \max_{\alpha \in \tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^{1/4} \\ & \ll \tau X^{47/30-(1/6)\sigma+2\varepsilon} = \tau X^{47/30-1/72+2\varepsilon}, \end{aligned}$$

and

$$\begin{aligned}
& \int_{\tilde{c}_2} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \\
& \ll \left(\max_{\alpha \in \tilde{c}_2} |S_3(\lambda_2 \alpha)|^{1/2} \right) \left(\int_{\tilde{c}_2} |S_2(\lambda_1 \alpha)|^4 |K(\alpha)| \, d\alpha \right)^{1/4} \left(\int_{\tilde{c}_2} |S_3(\lambda_2 \alpha)|^8 |K(\alpha)| \, d\alpha \right)^{3/16} \\
& \quad \times \left(\int_{\tilde{c}_2} |S_2(\lambda_1 \alpha)|^2 |S_4(\lambda_3 \alpha)|^4 |K(\alpha)| \, d\alpha \right)^{1/2} \left(\int_{\tilde{c}_2} |S_5(\lambda_4 \alpha)|^{32} |K(\alpha)| \, d\alpha \right)^{1/16} \\
& \ll \tau X^{7/5+\varepsilon} \max_{\alpha \in \tilde{c}_2} |S_3(\lambda_2 \alpha)|^{1/2} \ll \tau X^{47/30-(1/6)\sigma+2\varepsilon} = \tau X^{47/30-1/72+2\varepsilon}.
\end{aligned}$$

Then Lemma 6.1 follows immediately. \square

Lemma 6.2. *We have*

$$(6.4) \quad \int_{\hat{c}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \ll \tau X^{47/30+1/9-(71/72)\cdot(13/18)+5\varepsilon}.$$

Proof. We divide the region \hat{c} into disjoint sets $A(Y_1, Y_2, u)$, where

$$A(Y_1, Y_2, u) = \{\alpha \in \hat{c}: Y_1 \leq |S_2(\lambda_1 \alpha)| < 2Y_1, Y_2 \leq |S_3(\lambda_2 \alpha)| < 2Y_2, u \leq |\alpha| < 2u\}$$

and $Y_1 = X^{1/2-(2/3)\sigma+2\varepsilon} 2^{l_1}$, $Y_2 = X^{1/3-(1/3)\sigma+2\varepsilon} 2^{l_2}$, $u = \xi 2^r$ for some positive integers l_1, l_2, r . Thus by Corollaries 2.2 and 2.4, there are integers a_1, q_1, a_2, q_2 satisfying

$$(6.5) \quad \begin{aligned} & (a_1, q_1) = 1, \quad (a_2, q_2) = 1, \\ & 1 \leq q_1 \ll \left(\frac{X^{1/2+\varepsilon}}{Y_1} \right)^4, \quad |q_1 \lambda_1 \alpha - a_1| \ll \left(\frac{X^{1/2+\varepsilon}}{Y_1} \right)^4 X^{-1}, \end{aligned}$$

$$(6.6) \quad 1 \leq q_2 \ll \left(\frac{X^{1/3+\varepsilon}}{Y_2} \right)^2, \quad |q_2 \lambda_2 \alpha - a_2| \ll \left(\frac{X^{1/3+\varepsilon}}{Y_2} \right)^2 X^{-1}.$$

Here we have $a_1 a_2 \neq 0$. If $a_1 = 0$, by (6.5) we have

$$(6.7) \quad |\alpha| \ll |q_1 \lambda_1 \alpha - a_1| \ll \left(\frac{X^{1/2+\varepsilon}}{Y_1} \right)^4 X^{-1} \ll X^{-7/9-4\varepsilon}.$$

This contradicts $|\alpha| \geq \xi = X^{-7/9-\varepsilon}$, thus we have $a_1 \neq 0$. Similarly, we also have $a_2 \neq 0$.

Note that $|\alpha| \geq \xi = X^{-7/9-\varepsilon}$, hence we have

$$(6.8) \quad \left| \frac{a_1}{\alpha} \right| \ll \left| \frac{a_1}{\lambda_1 \alpha} \right| \ll q_1 + \left(\frac{X^{1/2+\varepsilon}}{Y_1} \right)^4 X^{-1} \frac{1}{|\lambda_1 \alpha|} \ll q_1 + X^{-\varepsilon} \ll q_1,$$

$$(6.9) \quad \left| \frac{a_2}{\alpha} \right| \ll \left| \frac{a_2}{\lambda_2 \alpha} \right| \ll q_2 + \left(\frac{X^{1/3+\varepsilon}}{Y_2} \right)^2 X^{-1} \frac{1}{|\lambda_2 \alpha|} \ll q_2 + X^{-1/6-\varepsilon} \ll q_2.$$

We further divide the set $A(Y_1, Y_2, u)$ into sets $A(Y_1, Y_2, u, Q_1, Q_2)$ with α satisfying (6.5) and (6.6), where $Q_j \leq q_j < 2Q_j$ on each set. Then we have

$$(6.10) \quad \left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| = \left| \frac{a_1(a_2 - q_2 \lambda_2 \alpha) + a_2(q_1 \lambda_1 \alpha - a_1)}{\lambda_2 \alpha} \right| \\ \ll Q_1 \left(\frac{X^{1/3+\varepsilon}}{Y_2} \right)^2 X^{-1} + Q_2 \left(\frac{X^{1/2+\varepsilon}}{Y_1} \right)^4 X^{-1} \\ \ll \left(\frac{X^{1/2+\varepsilon}}{Y_1} \right)^4 \left(\frac{X^{1/3+\varepsilon}}{Y_2} \right)^2 X^{-1} \ll X^{-1+(10/3)\sigma-\varepsilon} \\ \ll X^{-13/18-\varepsilon} \ll N^{-(71/72) \cdot (13/18)-\varepsilon},$$

since $N^{71/72} \leq X \leq N$. Also, by (6.9) and $\alpha \in A(Y_1, Y_2, u, Q_1, Q_2)$, we have

$$(6.11) \quad |a_2 q_1| \ll |q_1 q_2 \alpha| \ll u Q_1 Q_2.$$

Since $N^{(71/72) \cdot (13/18)} = q$ is sufficiently large, where a/q is a continued fraction convergent to λ_1/λ_2 , then by (6.10) we have

$$(6.12) \quad \left\| a_2 q_1 \frac{\lambda_1}{\lambda_2} \right\| < \frac{1}{4q}.$$

Then by Lemma 2.9, we have $|a_2 q_1| \geq q$. Suppose that $|a_2 q_1|$ only takes L distinct values, then we have $L \leq |\mathfrak{T}|$, where

$$(6.13) \quad \mathfrak{T} = \left\{ m \in \mathbb{N} : q \leq m \ll u Q_1 Q_2, \left\| m \frac{\lambda_1}{\lambda_2} \right\| < \frac{1}{4q} \right\}.$$

For any two positive integers $m_1, m_2 \in \mathfrak{T}$, we have

$$\left\| |m_1 - m_2| \frac{\lambda_1}{\lambda_2} \right\| \leq \left\| m_1 \frac{\lambda_1}{\lambda_2} \right\| + \left\| m_2 \frac{\lambda_1}{\lambda_2} \right\| < \frac{1}{2q}.$$

Then by Lemma 2.9, we have $|m_1 - m_2| \geq q$. Thus by the pigeon-hole principle, we have

$$(6.14) \quad L \leq |\mathfrak{T}| \ll \frac{u Q_1 Q_2}{q}.$$

Each value of $|a_2q_1|$ corresponds to much less than X^ε values of a_2, q_1 , by the well-known bound on the divisor function. For every fixed a_2 and q_1 , by (6.10), the value of $|a_1q_2|$ is the integral part of $a_2q_1\lambda_1/\lambda_2$, so there are much less than X^ε values of a_1, q_2 . Therefore, by (6.5) and (6.6) we get that the length of $A(Y_1, Y_2, u, Q_1, Q_2)$ is

$$\begin{aligned} &\ll LX^{2\varepsilon} \min\left(\frac{1}{Q_1X}\left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^4, \frac{1}{Q_2X}\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2\right) \\ &\ll \frac{X^{2\varepsilon}uQ_1Q_2}{q} \frac{1}{Q_1^{1/2}Q_2^{1/2}X} \left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^2 \frac{X^{1/3+\varepsilon}}{Y_2} \\ &\ll \frac{X^{1/3+5\varepsilon}uQ_1^{1/2}Q_2^{1/2}}{qY_1^2Y_2} \ll \frac{X^{5/3+8\varepsilon}u}{qY_1^4Y_2^2}. \end{aligned}$$

Here we have used $Q_1 \ll (X^{1/2+\varepsilon}/Y_1)^4$, $Q_2 \ll (X^{1/3+\varepsilon}/Y_2)^2$ and the inequality $\min(M, N) \leq M^{1/2}N^{1/2}$ for any positive numbers M, N .

Now we evaluate the integral over the set $A(Y_1, Y_2, u, Q_1, Q_2)$. By (2.7) and the trivial bounds of $S_4(\lambda_3\alpha), S_5(\lambda_4\alpha)$, we have

$$\begin{aligned} (6.15) \quad &\int_{A(Y_1, Y_2, u, Q_1, Q_2)} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j\alpha) \right|^2 K(\alpha) \, d\alpha \\ &\ll \min(\tau^2, u^{-2}) Y_1^2 Y_2^2 X^{2/4} X^{2/5} \int_{A(Y_1, Y_2, u, Q_1, Q_2)} d\alpha \\ &\ll \tau u^{-1} Y_1^2 Y_2^2 X^{9/10} \frac{X^{5/3+8\varepsilon}u}{qY_1^4Y_2^2} \ll \frac{\tau X^{77/30+8\varepsilon}}{qY_1^2} \\ &\ll \frac{\tau X^{47/30+(4/3)\sigma+4\varepsilon}}{q} \ll \tau X^{47/30+1/9+4\varepsilon} N^{-(71/72)\cdot(13/18)} \\ &\ll \tau X^{47/30+1/9-(71/72)\cdot(13/18)+4\varepsilon}. \end{aligned}$$

Now we sum over all possible values of Y_1, Y_2, u, Q_1, Q_2 , and we obtain that

$$(6.16) \quad \int_{\hat{\mathfrak{c}}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j\alpha) \right|^2 K(\alpha) \, d\alpha \ll \tau X^{47/30+1/9-(71/72)\cdot(13/18)+5\varepsilon}.$$

□

7. THE PROOF OF THEOREM 1.2

We take $\tau = X^{-\delta}$. Let \mathcal{V} be a well-spaced set. Then by (2.8) and (2.9), we have $\mathcal{I}(v, X; \mathbb{R}) = 0$ for every $v \in \mathbb{E}([X/2, X])$, where $\mathbb{E}([X/2, X])$ is defined by (2.1). Hence we have

$$(7.1) \quad \begin{aligned} & \sum_{v \in \mathbb{E}([X/2, X])} \mathcal{I}(v, X; \mathbb{R}) \\ &= \sum_{v \in \mathbb{E}([X/2, X])} (\mathcal{I}(v, X; \mathfrak{C}) + \mathcal{I}(v, X; \mathfrak{D}) + \mathcal{I}(v, X; \mathfrak{c}) + \mathcal{I}(v, X; \mathfrak{t})) = 0. \end{aligned}$$

This together with (3.1), (4.1) and (5.1) yields

$$(7.2) \quad \left| \sum_{v \in \mathbb{E}([X/2, X])} \int_{\mathfrak{c}} \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) e(-v\alpha) K(\alpha) \, d\alpha \right| \gg \tau^2 X^{17/60} E\left(\left[\frac{X}{2}, X\right]\right).$$

By Cauchy's inequality and (2.7) we get

$$(7.3) \quad \begin{aligned} & \left| \sum_{v \in \mathbb{E}([X/2, X])} \int_{\mathfrak{c}} \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) e(-v\alpha) K(\alpha) \, d\alpha \right| \\ & \ll \left(\int_{-\infty}^{\infty} \left| \sum_{v \in \mathbb{E}([X/2, X])} e(-v\alpha) \right|^2 K(\alpha) \, d\alpha \right)^{1/2} \\ & \quad \times \left(\int_{\mathfrak{c}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \right)^{1/2} \\ & = \left(\sum_{v_1, v_2 \in \mathbb{E}([X/2, X])} \int_{-\infty}^{\infty} e((v_1 - v_2)\alpha) K(\alpha) \, d\alpha \right)^{1/2} \\ & \quad \times \left(\int_{\mathfrak{c}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \right)^{1/2} \\ & = \left(\sum_{v_1, v_2 \in \mathbb{E}([X/2, X])} \max(0, \tau - |v_1 - v_2|) \right)^{1/2} \\ & \quad \times \left(\int_{\mathfrak{c}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \right)^{1/2} \\ & \ll \tau^{1/2} \left(E\left(\left[\frac{X}{2}, X\right]\right) \right)^{1/2} \left(\int_{\mathfrak{c}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \right)^{1/2}. \end{aligned}$$

Here we have used that for every $v_1, v_2 \in \mathbb{E}([X/2, X]) \subset \mathcal{V}$, if $v_1 \neq v_2$, then there is a constant c such that $|v_1 - v_2| > c$, since \mathcal{V} is a well-spaced set. Note that $\tau = X^{-\delta}$, $\delta > 0$ and X is a sufficiently large quantity. Thus we have

$$\sum_{v_1, v_2 \in \mathbb{E}([X/2, X])} \max(0, \tau - |v_1 - v_2|) = \tau E\left(\left[\frac{X}{2}, X\right]\right).$$

Combining (7.2) and (7.3), we obtain

$$(7.4) \quad E\left(\left[\frac{X}{2}, X\right]\right) \ll \tau^{-3} X^{-17/30} \int_c \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) d\alpha.$$

Now we begin to prove the first part of Theorem 1.2. Note that a/q is convergent to λ_1/λ_2 , $N = q^{(72/71) \cdot (18/13)}$ by (6.1), and $N^{71/72} \leq X \leq N$. By (7.4), Lemmas 6.1 and 6.2, we have

$$(7.5) \quad E([X/2, X]) \ll \tau^{-3} X^{-17/30} \tau X^{47/30 - 1/72 + 2\varepsilon} \ll X^{1+2\delta - 1/72 + 2\varepsilon}.$$

Thus, by (2.1) and (7.5), we conclude that

$$(7.6) \quad \begin{aligned} E(\mathcal{V}, N, \delta) &\ll E([N^{71/72}, N]) + N^{71/72} \\ &\ll \sum_{l=1}^{[(1/72) \log_2 N] + 1} E([2^{-l} N, 2^{1-l} N]) + N^{71/72} \\ &\ll \sum_{l=1}^{[(1/72) \log_2 N] + 1} (2^{1-l} N)^{1+2\delta - 1/72 + 2\varepsilon} + N^{71/72} \\ &\ll N^{1+2\delta - 1/72 + 3\varepsilon}. \end{aligned}$$

Obviously, there are infinitely many q we could have taken since λ_1/λ_2 is irrational, and this gives the sequence $N_j \rightarrow \infty$. This completes the proof of the first part of Theorem 1.2.

Next, we begin to prove the second part of Theorem 1.2. Now, if the convergent denominators for λ_1/λ_2 satisfy (1.6), then we can modify our work in Section 6. Now let N be a sufficiently large number, $N^{71/72} \leq X \leq N$, and assume that

$$(7.7) \quad \sigma = \chi,$$

where χ is given by (1.8).

Then the expression corresponding to (6.3) in Lemma 6.1 is

$$(7.8) \quad \int_{\tilde{\mathfrak{c}}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \ll \tau X^{47/30 - (1/6)\chi + \varepsilon}.$$

Now we modify our argument in Lemma 6.2, argument obtaining

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_2 q_1 \right| \ll X^{-1 + (10/3)\chi - \varepsilon}.$$

However, we know from (1.6) that there is a convergent a/q to λ_1/λ_2 with

$$X^{(1-\omega)(1-(10/3)\chi)} \ll q \ll X^{1-(10/3)\chi}.$$

Indeed, if not, there would be two convergent denominators q_j and q_{j+1} satisfying

$$q_j \ll X^{(1-\omega)(1-(10/3)\chi)}, \quad q_{j+1} \gg X^{1-(10/3)\chi}.$$

Thus we have $q_j \ll q_{j+1}^{1-\omega}$, which is in contradiction with (1.6).

Then, the expression corresponding to (6.15) is

$$\begin{aligned} \int_{A(Y_1, Y_2, u, Q_1, Q_2)} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \\ \ll \frac{\tau X^{47/30 + (4/3)\chi + 4\varepsilon}}{q} \ll \tau X^{47/30 + (4/3)\chi + 4\varepsilon - (1-\omega)(1-(10/3)\chi)} \\ \ll \tau X^{47/30 - (1/6)\chi + 4\varepsilon - (1-\omega) + (1/6)\chi(29-20\omega)} \ll \tau X^{47/30 - (1/6)\chi + 4\varepsilon} \end{aligned}$$

by our choice of χ . Thus the expression corresponding to (6.4) in Lemma 6.2 is

$$(7.9) \quad \int_{\hat{\mathfrak{c}}} \left| \prod_{j=1}^4 S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha) \, d\alpha \ll \tau X^{47/30 - (1/6)\chi + 5\varepsilon}.$$

Recall $\mathfrak{c} = \tilde{\mathfrak{c}} \cup \hat{\mathfrak{c}}$. Combining (7.4), (7.8) and (7.9), the second part of Theorem 1.2 follows immediately by modifying our argument in (7.6). \square

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