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BOUNDS FOR THE NUMBER OF MEETING EDGES IN GRAPH PARTITIONING

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Abstract. Let G be a weighted hypergraph with edges of size at most 2. Bollobás and Scott conjectured that G admits a bipartition such that each vertex class meets edges of total weight at least $(w_1-\Delta_1)/2+2w_2/3$, where w_i is the total weight of edges of size i and Δ_1 is the maximum weight of an edge of size 1. In this paper, for positive integer weighted hypergraph G (i.e., multi-hypergraph), we show that there exists a bipartition of G such that each vertex class meets edges of total weight at least $(w_0-1)/6+(w_1-\Delta_1)/3+2w_2/3$, where w_0 is the number of edges of size 1. This generalizes a result of Haslegrave. Based on this result, we show that every graph with m edges, except for K_2 and $K_{1,3}$, admits a tripartition such that each vertex class meets at least $\lceil 2m/5\rceil$ edges, which establishes a special case of a more general conjecture of Bollobás and Scott.

Keywords: graph; weighted hypergraph; partition; judicious partition

MSC 2010: 05C35, 05C75

1. Introduction

Let $G=(V,E)$ be a graph. For subsets S and T of V, $e_G(S,T)$ is the number of edges of G with one end in S and the other end in T, and $e_G(S)$ is the number of edges of G with both ends in S. By $d_G(S)$, we mean the number of edges of G meeting S (i.e., containing at least one vertex of S). For a weighted graph (or hypergraph) G with weight function w, denote by $d_G^w(S)$ the total weight of edges of G meeting S. If $S=\{v\}$, then we write $e_G(v,T)$, $d_G(v)$ and $d_G^w(v)$ for $e_G(\{v\},T)$, $d_G(\{v\})$ and $d_G^w(\{v\})$, respectively. When understood, the reference to G in the subscript will be dropped. Additionally, we write $\overline{S}$ for $V\setminus S$, $[t]$ for $\{1,\dots,t\}$ and $\binom{S}{j}$ for the set of all j-element subsets of S.

Classical graph or hypergraph partitioning problems often ask for partitioning the vertex set of a graph or hypergraph into pairwise disjoint subsets that opti-

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mize a single quantity. For example, the well-known Max-Cut problem asks for a maximum bipartite subgraph of a graph, i.e., a bipartition V_1, V_2 of a given graph with m edges maximizing the number of edges between V_1 and V_2. Edwards in [6], [7] proved the essentially best possible result: a bipartite subgraph with at least $m/2 + (\sqrt{2m + 1/4} - 1/2)/4$ edges. An extension of Edwards' bound for partitions into more than two parts was proved in [4].

In practice, one often needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called *Judicious partitioning problems* by Bollobás and Scott in [5]. The *Bottleneck bipartition problem* is a judicious partition problem: Find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{e(V_1), e(V_2)\}$. Bollobás and Scott in [2] showed that every graph with m edges admits a bipartition such that each vertex class spans at most

$$\frac{m}{4} + \frac{\sqrt{2m + \frac{1}{4}} - \frac{1}{2}}{8} \text{ edges.}$$

The bound is tight for the complete graph K_{2n+1}. In the same paper, the authors also extended the result for partitions into more than two parts. For more about judicious partitioning problems, we refer the reader to [1], [8], [9], [11], [12], [13], [14], [15], [18], [19]. For survey articles, see [5], [16].

In this paper, we consider another type of judicious partitioning problems about graphs with requirement on edges as well as on vertices, and such problems are called *mixed partitioning problems*. We follow Bollobás and Scott [5] in using the term "*hypergraph with edges of size at most 2*". Note that a *hypergraph* $G = (V, E)$ consists of a finite set $V := V(G)$ of vertices and a set $E := E(G)$ of edges, where each edge is a subset of V. For each edge $e \in E$, if e contains at most two elements of V, then G is a hypergraph with edges of size at most 2.

Let G be a weighted hypergraph with edges of size at most 2. Denote by Δ_1 the maximum weight of an edge of size 1 and by w_i the total weight of edges of size i for $i = 1, 2$. Bollobás and Scott in [5] gave the following conjecture.

Conjecture 1.1 (Bollobás and Scott [5]). *Every weighted hypergraph G admits a bipartition such that each vertex class meets edges of total weight at least*

$$\frac{w_1 - \Delta_1}{2} + \frac{2w_2}{3}.$$

Recently, Xu et al. in [17] established a weaker version of the conjecture. For weighted hypergraphs G with weight function $w\colon E \to \mathbb{N}^+$, Haslegrave in [10] confirmed the conjecture for the case $\Delta_1 \leqslant 1$.

Theorem 1.2 (Haslegrave [10]). *For $\Delta_1 \leqslant 1$, the weighted hypergraph G admits a bipartition V_1, V_2 such that for $i = 1, 2$*

$$d^w(V_i) \geqslant \frac{w_1 - \Delta_1}{2} + \frac{2w_2}{3}.$$

By using a different method, we generalize the result of Haslegrave and show

Theorem 1.3. *The weighted hypergraph G has a bipartition V_1, V_2 such that for $i = 1, 2$*

$$d^w(V_i) \geqslant \frac{w_0 - 1}{6} + \frac{w_1 - \Delta_1}{3} + \frac{2w_2}{3},$$

where w_0 is the number of edges of size 1.

Remark. Since the bound of Theorem 1.2 is easy to obtain when $\Delta_1 = 0$, we can always assume that $\Delta_1 \geqslant 1$ in our theorem. Note that $w_0 = w_1$ provided $\Delta_1 = 1$. Thus, our result generalizes Theorem 1.2.

Bollobás and Scott in [5] noted that mixed partitioning problems are useful in proving results about uniform hypergraphs. Particularly, we establish a special case of another conjecture of Bollobás and Scott for graphs based on the $\Delta_1 = 2$ case of Theorem 1.3.

Conjecture 1.4 ([3], [16]). *For every integer $k \geqslant 2$, every graph with m edges has a partition into k sets, each of which meets at least*

$$\frac{2m}{2k-1} \text{ edges.}$$

If true, the complete graph K_{2k-1} shows that the bound should be sharp. Actually, in [16], the author also assumes that $m \geqslant \binom{k}{2}$ to avoid the trivial cases such as K_{k-1}. Ma et al. in [14] solved the conjecture for very large m (in terms of k). In this paper, we confirm the case $k = 3$.

Theorem 1.5. *Let G be a graph with m edges. Suppose that G is not isomorphic to K_2 and $K_{1,3}$ (modulo isolated vertices). Then there exists a tripartition V_1, V_2, V_3 of G such that for $i = 1, 2, 3$*

$$d(V_i) \geqslant \left\lceil \frac{2m}{5} \right\rceil.$$

In this section, we consider the bipartitions of weighted hypergraphs and give the proof of Theorem 1.3. Before proving, we present the following algorithm and lemmas.

Let $G = (V, E(G))$ be a weighted hypergraph with edges of size at most 2 and let $w\colon E \to \mathbb{N}^+$ be its weight function. First, we construct a weighted complete graph $G_1 = (V, E(G_1))$ from G. Let $w^1\colon V \cup E(G_1) \to \mathbb{N}$ be the weight function of G_1 such that for each $v \in V$ and $e \in E(G_1)$

$$w^1(v) = \begin{cases} w(\{v\}) & \text{if } \{v\} \in E(G), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad w^1(e) = \begin{cases} w(e) & \text{if } e \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let Δ_1 be the maximum weight of an edge of size 1 of G . Clearly, by the construction, Δ_1 is also the maximum weight of a vertex in G_1.

Now, we construct a graph sequence $\mathcal{G} = (G_i)_{i\geqslant 1}$ consisting of weighted complete graphs $G_i = (V, E(G_i))$ with weight function $w^i\colon V \cup E(G_i) \to \mathbb{N}$ according to the following procedure, which we will call the $\mathcal{G}$ *algorithm*: set $i = 1$ and $s_1 = |\{v \in V\colon w^1(v) > 1\}|$. Repeat the following steps until $s_i \leqslant 1$.

- ▷ Set $s_i = |\{v \in V\colon w^i(v) > 1\}|$. If $s_i = 0$, then stop; otherwise, set $\delta_i = \min\{w^i(v) > 1\colon v \in V\}$ and $\Delta_i = \max\{w^i(v)\colon v \in V\}$.
- ▷ If $s_i = 1$ and $v \in V$ is the unique vertex satisfying $w^i(v) > 1$, then set $w^i(v) = 1$, and stop.
- ▷ If $s_i > 1$, then choose an edge $e = uv$ arbitrarily from G_i satisfying $w^i(u) = \delta_i$ and $w^i(v) = \Delta_i$. Set $w^{i+1}(u) = 1$, $w^{i+1}(v) = \Delta_i - \delta_i + 1$ and $w^{i+1}(e) = w^i(e) + \delta_i - 1$. For each $x \in V\backslash\{u, v\}$ and $f \in \binom{V}{2}\backslash\{e\}$, set $w^{i+1}(v) = w^i(v)$ and $w^{i+1}(f) = w^i(f)$. Increment i.

Let t be the length of the resulting sequence $\mathcal{G}$. Clearly, $1 \leqslant t \leqslant |V|$. For each $S \subseteq V$ and $i \in [t]$, define $\tau^{w^i}(S) = \sum\limits_{v\in S} w^i(v)$. By the construction, we immediately have the following two lemmas.

Lemma 2.1. *For each $S \subseteq V$ and $1 \leqslant i \leqslant j \leqslant t$,*

$$\tau^{w^i}(S) + d^{w^i}(S) \geqslant \tau^{w^j}(S) + d^{w^j}(S).$$

Proof. According to the $\mathcal{G}$ algorithm, for each $v \in V$ we have

$$w^i(v) + d^{w^i}(v) \geqslant w^j(v) + d^{w^j}(v). \tag{1}$$

In fact, the equality holds for each $j \leqslant t-1$ and, if $j = t$, it holds for at least $|V|-1$ vertices. Summing over all $v \in S$ in (1) yields

$$\begin{aligned}\sum_{v\in S} w^i(v) + \sum_{\substack{e\in E(G_i)\\ |e\cap S|=1}} w^i(e) + 2\sum_{\substack{e\in E(G_i)\\ |e\cap S|=2}} w^i(e)\\ \geqslant \sum_{v\in S} w^j(v) + \sum_{\substack{e\in E(G_j)\\ |e\cap S|=1}} w^j(e) + 2\sum_{\substack{e\in E(G_j)\\ |e\cap S|=2}} w^j(e),\end{aligned}$$

which is equivalent to

$$\tau^{w^i}(S) + d^{w^i}(S) \geqslant \tau^{w^j}(S) + d^{w^j}(S) + \sum_{e\in\binom{S}{2}} (w^j(e) - w^i(e)).$$

The inequality follows from the fact that G_i is a complete graph on V for each $i \in [t]$. Note that $w^j(e) \geqslant w^i(e)$ for each $e \in \binom{S}{2}$. Thus, we have

$$\tau^{w^i}(S) + d^{w^i}(S) \geqslant \tau^{w^j}(S) + d^{w^j}(S),$$

as required. $\square$

For each $i \in [t]$, let $w_1^i = \sum\limits_{v\in V} w^i(v)$ and $w_2^i = \sum\limits_{e\in E(G_i)} w^i(e)$. The next lemma shows that G_t has a 'good' judicious partition.

Lemma 2.2. *Every weighted graph G_t admits a bipartition V_1, V_2 such that for $j = 1, 2$*

$$\tau^{w^t}(V_j) + d^{w^t}(V_j) \geqslant \frac{w_1^t - 1}{6} + \frac{w_1^1 - \Delta_t}{3} + \frac{2w_2^1}{3}.$$

P r o o f. Note that the difference $w_1^1 - w_1^t$ is the total weight of vertices decreasing in the process of the $\mathcal{G}$ algorithm. Similarly, the difference $w_2^t - w_2^1$ is the total weight of edges increasing in the process of the $\mathcal{G}$ algorithm. If $s_t = 0$, by the construction, we immediately have $w_1^1 - w_1^t = 2(w_2^t - w_2^1)$. If $s_t = 1$, similarly, we have $w_1^1 - w_1^t - (\Delta_t - 1) = 2(w_2^t - w_2^1)$. With help of the preceding two equalities, we conclude

$$w_1^1 - w_1^t - (\Delta_t - 1) \leqslant 2(w_2^t - w_2^1). \tag{2}$$

Now, we view G_t as a weighted hypergraph with edges of size at most 2. Note that $w^t(v) \leqslant 1$ for each $v \in V$ by the construction. Clearly, each edge of size 1 of G_t

has weight at most 1. Thus, by Theorem 1.2, there exists a bipartition V_1, V_2 of G_t such that for $j = 1, 2$

$$\tau^{w^t}(V_j) + d^{w^t}(V_j) \geqslant \frac{w_1^t - 1}{2} + \frac{2w_2^t}{3},$$

which together with (2) implies the desired result. □

We can now complete the proof of our main result.

P r o o f of Theorem 1.3. Note that $d^w(S) = \tau^{w^1}(S) + d^{w^1}(S)$ by the construction of G_1. By Lemma 2.1, for each $S \subseteq V$ we have

$$d^w(S) \geqslant \tau^{w^t}(S) + d^{w^t}(S).$$

It follows from Lemma 2.2 that G admits a bipartition V_1, V_2 such that for $j = 1, 2$

$$d^w(V_j) \geqslant \frac{w_1^t - 1}{6} + \frac{w_1^1 - \Delta_t}{3} + \frac{2w_2^1}{3}. \tag{3}$$

Again, by the construction of G_1, we have $w_1^1 = w_1$ and $w_2^1 = w_2$. In addition, the $\mathcal{G}$ algorithm implies that $w_1^t = w_0$ and $\Delta_t \leqslant \Delta_1$. Now, the result follows immediately from inequality (3). □

3. Tripartitions of graphs

In this section, we consider the tripartitions of graphs and prove Theorem 1.5. First, we introduce some definitions and lemmas.

Let $G = (V, E)$ be a graph. For a partition V_1, V_2, V_3 of G, define the degree of V_1, V_2, V_3 as $d(V_1, V_2, V_3) = \sum_{i=1}^{3} d(V_i)$. We call the partition *optimal* if $d(V_1, V_2, V_3)$ is as large as possible over partitions $V = V_1 \cup V_2 \cup V_3$, and *semi-optimal* if this degree cannot be increased by moving a vertex into V_3. Note that semi-optimality depends on the order of the sets in our partition. We shall always take the last set, V_3, to be the exceptional one. Trivially, every optimal partition is also semi-optimal. In the following, for every semi-optimal partition we show that the degree $d(V_1, V_2, V_3)$ can be lower bounded.

Lemma 3.1. *Let G be a graph with m edges. Suppose that V_1, V_2, V_3 is a semi-optimal partition of G. Then*

$$d(V_1, V_2, V_3) \geqslant 2m - d(V_3).$$

Proof. Since V_1, V_2, V_3 is semi-optimal, for each $v \in V_i$ and $i = 1, 2$ we have

$$e(v, \overline{V_3}) \leqslant e(v, \overline{V_i}). \tag{4}$$

Otherwise, $e(v, \overline{V_3}) > e(v, \overline{V_i})$. Let $X_i = V_i \setminus \{v\}$, $X_{3-i} = V_{3-i}$ and $X_3 = V_3 \cup \{v\}$. Clearly, we have $d(X_i) = d(V_i) - e(v, \overline{V_i})$, $d(X_{3-i}) = d(V_{3-i})$ and $d(X_3) = d(V_3) + e(v, \overline{V_3})$. This implies that $d(X_1, X_2, X_3) > d(V_1, V_2, V_3)$, a contradiction with the choice of V_1, V_2, V_3.

By (4), for each $v \in V_i$ and $i = 1, 2$ we deduce

$$e(v, V_i) \leqslant e(v, V_3).$$

Summing over all $v \in V_i$ yields $2e(V_i) \leqslant e(V_i, V_3)$, giving that

$$2(e(V_1) + e(V_2)) \leqslant e(V_1, V_3) + e(V_2, V_3) = d(V_3) - e(V_3).$$

This establishes that

$$\sum_{i=1}^{3} e(V_i) \leqslant \frac{d(V_3)}{2} + \frac{e(V_3)}{2} \leqslant d(V_3).$$

Noting that $d(V_1, V_2, V_3) + \sum\limits_{i=1}^{3} e(V_i) = 2m$, we obtain

$$d(V_1, V_2, V_3) = 2m - \sum_{i=1}^{3} e(V_i) \geqslant 2m - d(V_3),$$

as desired. $\square$

Next, we show that the semi-optimality of a partition V_1, V_2, V_3 of G is preserved if we move vertices into V_3.

Lemma 3.2. *Let V_1, V_2, V_3 be a semi-optimal partition of a graph G, and let U_1, U_2, U_3 be another partition of G with $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $U_3 \supseteq V_3$. Then U_1, U_2, U_3 is also semi-optimal.*

Proof. For each $v \in U_i$ and $i = 1, 2$, let $U_i' = U_i \setminus \{v\}$, $U_{3-i}' = U_{3-i}$ and $U_3' = U_3 \cup \{v\}$. Similarly, let $V_i' = V_i \setminus \{v\}$, $V_{3-i}' = V_{3-i}$ and $V_3' = V_3 \cup \{v\}$. Then

$$d(U_i) - d(U_i') = e(v, \overline{U_i}) \geqslant e(v, \overline{V_i}) = d(V_i) - d(V_i')$$

and

$$d(U_3') - d(U_3) = e(v, \overline{U_3}) \leqslant e(v, \overline{V_3}) = d(V_3') - d(V_3).$$

Thus, we have

$$d(U_i') + d(U_3') \leqslant d(U_i) + d(U_3) + (d(V_i') + d(V_3') - d(V_i) - d(V_3)),$$

which is equivalent to

$$d(U_1', U_2', U_3') \leqslant d(U_1, U_2, U_3) + (d(V_1', V_2', V_3') - d(V_1, V_2, V_3)). \tag{5}$$

Since $d(V_1, V_2, V_3)$ cannot be increased by moving a vertex into V_3, we have $d(V_1', V_2', V_3') \leqslant d(V_1, V_2, V_3)$. It follows from (5) that U_1, U_2, U_3 is also a semi-optimal partition of G as claimed. □

Now, we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. Since isolated vertices contribute nothing to the meeting edges, we may assume that G contains no isolated vertices. It is easy to check that the result holds for $m \leqslant 3$, except when G is isomorphic to K_2 or $K_{1,3}$. Assume that $m \geqslant 4$. Let Δ be the maximum degree of G and $l = \lceil 2m/5 \rceil$. We proceed by showing the following several claims.

Claim 1. $\Delta < l$. Otherwise, let v be a vertex in G with degree $\Delta \geqslant l$. Consider the graph H_1 induced by $V \setminus \{v\}$. We view H_1 as a weighted hypergraph with m edges, of which Δ have size 1 and $m - \Delta$ have size 2. Let w be the weight function of H_1. For each $f \in E(H)$, we define $w(f) = 1$. Now, we use Theorem 1.2 setting $\Delta_1 = 1$, $w_1 = \Delta$ and $w_2 = m - \Delta$. Thus, there exists a bipartition U_1, U_2 of H_1 such that for $i = 1, 2$

$$d(U_i) \geqslant \frac{w_1 - 1}{2} + \frac{2w_2}{3} = \frac{2m}{3} - \frac{\Delta}{6} - \frac{1}{2} > l - 1.$$

The last inequality holds because $\Delta \leqslant m$ and $m \geqslant 4$. By the integrality of $d(U_i)$, we have $d(U_i) \geqslant l$. Set $U_3 = \{v\}$. Clearly, U_1, U_2, U_3 will do for our tripartition. This completes the proof of Claim 1.

Let V_1, V_2, V_3 be an optimal partition of G, ordered so that $d(V_1) \geqslant d(V_2) \geqslant d(V_3)$. If $d(V_3) \geqslant l$, we are done. Suppose that $d(V_3) \leqslant l-1$.

Claim 2. $d(V_2) \geqslant l$. Let H_2 be the graph induced by $\overline{V_3}$. By the maximality of $d(V_1, V_2, V_3)$, we know that

$$(*) \qquad V_1, V_2 \textit{ is a bipartition of } H_2 \textit{ minimizing } e_{H_2}(V_1) + e_{H_2}(V_2).$$

Otherwise, let V_1', V_2' be another bipartition of H_2 such that

$$e_{H_2}(V_1') + e_{H_2}(V_2') < e_{H_2}(V_1) + e_{H_2}(V_2).$$

Note that $d(V_1, V_2, V_3) = 2m - \sum_{i=1}^{3} e(V_i)$ and $e_{H_2}(S) = e(S)$ for each $S \subseteq \overline{V_3}$. Clearly, V_1', V_2', V_3 is another partition of G satisfying $d(V_1', V_2', V_3) > d(V_1, V_2, V_3)$, contradicting the choice of V_1, V_2, V_3.

For each $v \in V_i$ and $i = 1, 2$, it follows from $(*)$ that

$$e(v, V_i) \leqslant e(v, V_{3-i}).$$

Summing over all $v \in V_i$ gives that $2e(V_i) \leqslant e(V_1, V_2)$. Observing that $e(\overline{V_3}) = e(V_i) + e(V_{3-i}) + e(V_1, V_2)$, we have

$$3e(V_i) + e(V_{3-i}) \leqslant e(\overline{V_3}).$$

It follows that $e(V_i) \leqslant e(\overline{V_3})/3$ for $i = 1, 2$. Note that $d(V_2) \geqslant e(\overline{V_3}) - e(V_1)$ and $d(V_3) \leqslant l - 1$ by our assumption. Thus,

$$d(V_2) \geqslant \frac{2e(\overline{V_3})}{3} = \frac{2(m - d(V_3))}{3} > l - 1,$$

i.e., $d(V_2) \geqslant l$ by integrality, completing the proof of Claim 2.

For $i = 1, 2$, let X_i be a minimal subset of V_i satisfying $d(X_i) \geqslant l$, and $X_3 = V \setminus (X_1 \cup X_2)$. If $d(X_3) \geqslant l$, then X_1, X_2, X_3 is a suitable tripartition. Suppose that $d(X_3) \leqslant l - 1$. Without loss of generality, we may assume that $d(X_1) \geqslant d(X_2)$.

Claim 3. $|X_1| = 2$. Since V_1, V_2, V_3 is an optimal partition of G, by Lemma 3.2, X_1, X_2, X_3 is a semi-optimal partition of G. It follows from Lemma 3.1 that

$$d(X_1) + d(X_2) \geqslant 2(m - d(X_3)) \geqslant 2(m - l + 1).$$

Clearly, we have $d(X_1) \geqslant m - l + 1$. By the minimality of X_1, for each vertex $x \in X_1$ there are at least $m - 2(l-1)$ edges meeting X_1 only at x. Since otherwise

$$d(X_1 \setminus \{x\}) = d(X_1) - e(x, \overline{X_1}) \geqslant (m - l + 1) - (m - 2(l-1) - 1) = l,$$

a contradiction. Note that $m-2(l-1) \geqslant l/2$. Hence, any two vertices of X_1 meet at least l edges, and so two vertices cannot be a proper subset of X_1. Thus, $|X_1| \leqslant 2$. Since $d(X_1) \geqslant l$, by Claim 1 we have $|X_1| = 2$. Thus, we complete the proof of Claim 3.

Let $X_1 = \{x_1, x_2\}$ and $\theta = |N(x_1) \cap N(x_2)|$. Since $\Delta < l$, we may assume that

$$d(x_1) + d(x_2) = 2(l-1) - r, \tag{6}$$

where $r \geqslant 0$ is an integer. Write $e = x_1x_2$ and define the indicator variable $\mathbf{1}_e = 1$ if and only if $e \in E(G)$, otherwise $\mathbf{1}_e = 0$. Now, we may write

$$\theta = l - 1 - \mathbf{1}_e - s, \tag{7}$$

where $s \geqslant 0$ is an integer. Let $\mathbf{g} = (m, r, s, \mathbf{1}_e)$, $\mathbf{g}_1 = (6, 0, 0, 1)$, $\mathbf{g}_2 = (8, 0, 0, 1)$, $\mathbf{g}_3 = (8, 0, 0, 0)$ and $\mathbf{g}_4 = (8, 0, 1, 1)$.

Claim 4. $\mathbf{g} \in \{\mathbf{g}_i \colon 1 \leqslant i \leqslant 4\}$. Consider the graph H_3 induced by $X_2 \cup X_3$ and view H_3 as a weighted hypergraph with weight function w. Let $N_1 = N(x_1) \cup N(x_2)$ and $N_2 = N(x_1) \cap N(x_2)$. For each $x \in N_1 \setminus \{x_1, x_2\}$, let $\{x\}$ be the edge of size 1 of H_3. If also $x \in N_2$, define $w(\{x\}) = 2$; otherwise, set $w(\{x\}) = 1$. For each edge f of G contained in $X_2 \cup X_3$, let f be the edge of size 2 of H_3 and define $w(f) = 1$. Now, we apply Theorem 1.3 setting $\Delta_1 = 2$, $w_1 = d(x_1) + d(x_2) - 2 \cdot \mathbf{1}_e$, $w_2 = m - w_1 - \mathbf{1}_e$ and $w_0 = w_1 - \theta$. Thus, there exists a bipartition X_2', X_3' of H_3 such that for $i = 2, 3$

$$d(X_i') \geqslant \frac{w_0 - 1}{6} + \frac{w_1 - \Delta_1}{3} + \frac{2w_2}{3} = \frac{2m}{3} - \frac{l}{2} + \frac{r + s - 2 - \mathbf{1}_e}{6}.$$

The last equality follows from (6) and (7). In the following, we aim at showing that X_1, X_2', X_3' is a suitable tripartition of G. By integrality, it suffices to show that

$$\frac{2m}{3} - \frac{l}{2} + \frac{r + s - 2 - \mathbf{1}_e}{6} > l - 1,$$

which is equivalent to proving that

$$4m + r + s + 3 \geqslant 9l + \mathbf{1}_e. \tag{8}$$

Clearly, if $r + s \geqslant 1 + \mathbf{1}_e$, (8) follows immediately from the fact $m \geqslant 4$. Note that, if $r \geqslant 1$, we have $\min\{d(x_1), d(x_2)\} \leqslant l-2$ by (6). Since $\theta \leqslant \min\{d(x_1), d(x_2)\} - \mathbf{1}_e$, by (7) we know that $s \geqslant 1$ provided $r \geqslant 1$. Thus, we may assume that $r = 0$ and $s \leqslant 1$. Now, it is easy to check that (8) holds except when $\mathbf{g} = \mathbf{g}_i$, where $i = 1, 2, 3, 4$. This completes the proof of Claim 4.

Since $\Delta < l$ and $r = 0$, we have $d(x_1) = d(x_2) = l-1$ by (6). Therefore, $\Delta = l-1$. Let G_i be the graph G satisfying $\mathbf{g} = \mathbf{g}_i$ for each $i = 1,2,3,4$. Note that G_i contains at least $\lceil 2E(G_i)/\Delta(G_i)\rceil$ vertices.

Claim 5. For each $1 \leqslant i \leqslant 4$, G_i admits a tripartition such that each vertex class meets at least l edges.

If $\mathbf{g} = \mathbf{g}_1$, then $l = 3$, $\theta = 1$ and $d(x_1) = d(x_2) = 2$. Suppose that $N(x_1) = \{x_2, x_3\}$ and $N(x_2) = \{x_1, x_3\}$. Since G_1 contains at least 6 vertices, assume that $\{x_1, \ldots, x_6\} \subseteq V(G_1)$. Let Z_1, Z_2, Z_3 be a partition of G_1 satisfying $\{x_1, x_4\} \subseteq Z_1$, $\{x_2, x_5\} \subseteq Z_2$ and $\{x_3, x_6\} \subseteq Z_3$. Clearly, Z_1, Z_2, Z_3 will do for our tripartition.

If $\mathbf{g} = \mathbf{g}_i$ for $i = 2,3$, then $l = 4$ and $d(x_1) = d(x_2) = 3$. Moreover, if $\mathbf{g} = \mathbf{g}_2$, then $\theta = 2$. Set $N(x_1) = \{x_2, x_3, x_4\}$ and $N(x_2) = \{x_1, x_3, x_4\}$. If $\mathbf{g} = \mathbf{g}_3$, then $\theta = 3$. Set $N(x_1) = N(x_2) = \{x_3, x_4, x_5\}$. Again, G_i contains at least 6 vertices, say $\{x_1, \ldots, x_6\} \subseteq V(G_i)$. Let Z_1, Z_2, Z_3 be a partition of G_i satisfying $\{x_1, x_6\} \subseteq Z_1$, $\{x_2, x_5\} \subseteq Z_2$ and $\{x_3, x_4\} \subseteq Z_3$. In either case, Z_1, Z_2, Z_3 will do for our tripartition.

If $\mathbf{g} = \mathbf{g}_4$, then $l = 4$, $\theta = 1$ and $d(x_1) = d(x_2) = 3$. Let $N(x_1) = \{x_2, x_3, x_4\}$ and $N(x_2) = \{x_1, x_3, x_5\}$. Suppose that G_4 contains at least 7 vertices, say $\{x_1, \ldots, x_7\} \subseteq V(G_4)$. Let Z_1, Z_2, Z_3 be a partition of G_4 satisfying $\{x_1, x_6\} \subseteq Z_1$, $\{x_2, x_7\} \subseteq Z_2$ and $\{x_3, x_4, x_5\} \subseteq Z_3$. Clearly, Z_1, Z_2, Z_3 is a desired tripartition. Thus, G_4 contains exactly 6 vertices, say $x_1, \ldots, x_6$. Note that $\sum\limits_{i=1}^{6} d(x_i) = 2m = 16$. If $d(x_6) = 1$, then $d(x_i) = 3$ for each $1 \leqslant i \leqslant 5$. Clearly, $\{x_1, x_6\}$, $\{x_2, x_5\}$ and $\{x_3, x_4\}$ is a desired tripartition. Hence, $d(x_6) \geqslant 2$. Now, $\{x_1, x_5\}$, $\{x_2, x_4\}$ and $\{x_3, x_6\}$ will do for our tripartition as required.

Thus, we complete the proof of Theorem 1.5. $\square$

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